Answers to the problems suggested in Practice Sheet 3

6.1

2: $R=\infty$, interval of convergence: $(-\infty,\infty)$

15: $R_{x=0} \geq 5$, $R_{x=1} \geq 4$

22: $y_1 = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots$, $y_2 = x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \cdots$

24: $y_1 = 1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 + \cdots$, $y_2 = x$

30: $y = 2 - x - 2x^2 - \frac{1}{3}x^3 + \frac{5}{12}x^4 + \cdots$

32: $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$

6.2

6: irregular singular point: $x=5$, regular singular point: $x=0$

8: regular singular points: $x=0, \pm i$

12: for the (only) regular singular point, write: $x^2y'' + x\overbrace{(x+3)}^{p(x)}y' + \overbrace{7x^3}^{q(x)}y = 0$

14: indicial roots: 0 and 0; expect Frobenius method to yield a single series solution.

20: general solution on $(0,\infty)$:

$$y = C_1 x^{2/3}\left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \cdots\right) + C_2 x^{1/3}\left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots\right)$$

22: general solution on $(0,\infty)$:

$$y = C_1 x^{-2/3}\left(1 - \frac{3}{4}x^2 + \frac{9}{128}x^4 + \cdots\right) + C_2 x^{2/3}\left(1 - \frac{3}{20}x^2 + \frac{9}{1280}x^4 + \cdots\right)$$

24: general solution on $(0,\infty)$:

$$y = C_1 x^{-1}\left(1 + 2x - 2x^2 + \frac{4}{9}x^3 + \cdots\right) + C_2 x^{1/2}\left(1 - \frac{2}{5}x + \frac{2}{35}x^2 - \frac{4}{935}x^3 + \cdots\right)$$

25: general solution on $(0,\infty)$:

$$y = C_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n} + C_2 \frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n} = \frac{1}{x}\left[c_1 \sinh(x) + c_2 \cosh(x)\right]$$

7.1

8: $\mathcal{L}(f(t)) = \dfrac{2e^{-s}}{s^2}$

26: $\dfrac{48}{s^4} - \dfrac{24}{s^3} + \dfrac{6}{s^2} - \dfrac{1}{s}$

30: $\dfrac{1}{s-2} - \dfrac{2}{s} + \dfrac{1}{s+2}$

36: $\dfrac{1}{2s} + \dfrac{1}{2(s+2)}$

37: $\dfrac{2}{s^2+16}$

$\boxed{7.2}$

4: $4t - \frac{2}{3}t^3 + \frac{1}{120}t^5$

14: $\frac{1}{2}\sin\left(\frac{t}{2}\right)$

18: $-\frac{1}{4} + \frac{5}{4}e^{4t}$

28: $\frac{1}{6\sqrt{3}}\sinh(\sqrt{3}\,t) - \frac{1}{6\sqrt{3}}\sin(\sqrt{3}\,t)$

34: $y = \frac{1}{13}e^{t} - \frac{1}{13}\cos(5t) + \frac{5}{13}\sin(5t)$

38: $y = \frac{1}{10}e^{t} - \frac{1}{30}\sin(3t) - \frac{1}{10}\cos(3t)$

40: $y = \frac{13}{60}e^{t} - \frac{13}{20}e^{-t} + \frac{16}{39}e^{-2t} + \frac{3}{130}\cos(3t) - \frac{1}{65}\sin(3t)$

$\boxed{7.3}$

6: $\frac{2}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{s-2}$

16: $2e^{-3t}\cos(5t) - \frac{1}{5}e^{-3t}\sin(5t)$

20: $te^{-2t} - t^2e^{-2t} + \frac{1}{6}t^3e^{-2t}$

30: $y = \frac{7}{25} + \frac{1}{5}t - \frac{7}{25}e^{t}\cos(2t) + \frac{51}{25}e^{t}\sin(2t)$

42: $\frac{s\,e^{-\pi s/2}}{s^2+1}$

48: $\left(-t+1+e^{t-2}\right)\mathcal{U}(t-2)$

58: $\frac{-s\,e^{-3\pi s/2}}{s^2+1}$

63: $y = \left(5 - 5e^{-(t-1)}\right)\mathcal{U}(t-1)$

68: $y = \left[\frac{1}{6} - \frac{1}{2}e^{2(t-1)} + \frac{1}{3}e^{3(t-1)}\right]\mathcal{U}(t-1) - e^{2t} + e^{3t}$

$\boxed{7.4}$

2: $\frac{6}{(s-1)^4}$

8: $\frac{(s+3)^2 - 9}{\left((s+3)^2+9\right)^2}$

10: $y = e^{t}\sin t - te^{t}\cos t$

12: $y = \cos t - \frac{1}{2}\sin t - \frac{1}{2}t\cos t$

17: $y = \frac{2}{3}t^3 + \frac{c}{2}t^2$

20: $\frac{2}{s^3(s-1)^2}$

26: $\frac{2}{(s^2+1)^2}$

30: $\frac{3s+1}{s^2(s+1)^3}$

40: $f(t) = e^{-t}$

44: $f(t) = \frac{1}{2}t - \frac{1}{12}t^3$

52: $\frac{1-e^{-s}}{s^2(1-e^{-2s})}$

$\boxed{7.5}$

2: $y = 2e^{-t} + e^{-(t-1)}\mathcal{U}(t-1)$

6: $y = \cos t + \sin t\left[\mathcal{U}(t-2\pi) + \mathcal{U}(t-4\pi)\right]$

8: $y = \frac{3}{4}e^{2t} - \frac{3}{4} - \frac{1}{2}t + \left[\frac{1}{2}e^{2(t-2)} - \frac{1}{2}\right]\mathcal{U}(t-2)$

10: $(t-1)e^{-(t-1)}\mathcal{U}(t-1)$

$\boxed{7.6}$

4: $x = \frac{1}{3} - \frac{1}{3}e^{t} - \frac{1}{3}te^{t}$, $y = -\frac{1}{3} + \frac{1}{3}e^{t} + \frac{4}{3}te^{t}$

6: $x = -\sqrt{3}\,e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$, $y = e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)$

11: $x = 1 + t + \frac{1}{2}t^2 - e^{-t}$, $y = \frac{1}{3}te^{-t} + \frac{1}{3}e^{-t} - \frac{1}{3}$

$\boxed{8.1}$

12: ✓ 20: Yes $[W(X_1, X_2, X_3) = -84e^{-t} \neq 0]$

24: ✓ 26: $X_1 = \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{\sqrt{2}t}$, $X_2 = \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix} e^{-\sqrt{2}t}$: fundamental set of solutions of homogeneous eq.

$X_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: particular solution

$\boxed{8.2}$

8: $X = c_1 \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -7 \\ 3 \\ 5 \end{pmatrix} e^{5t} + c_3 \begin{pmatrix} -7 \\ 5 \\ 5 \end{pmatrix} e^{7t}$

13: $X = \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t/2} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^{t/2}$

24: $X = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{-t}$

30: $X = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} e^{t} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} e^{t}$

34: $X = c_1 \begin{pmatrix} \sin t - \cos t \\ 2\cos t \end{pmatrix} + c_2 \begin{pmatrix} -\cos t - \sin t \\ 2\sin t \end{pmatrix}$

40: $X = c_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} -2\cos(2t) + \sin(2t) \\ 3\sin(2t) \\ 2\cos(2t) \end{pmatrix} e^{t} + c_3 \begin{pmatrix} -\cos(2t) - 2\sin(2t) \\ -3\cos(2t) \\ 2\sin(2t) \end{pmatrix} e^{t}$

46: $X = -2 \begin{pmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix} e^{5t} + 5 \begin{pmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix} e^{5t}$