Math 226 / CS 255, Spring 2007
Test 2

1. (50 pts) State the trapezoidal and Simpson's rules for integrating a function f on the interval $[0,1]$. After having partitioned the interval $[0,1]$ into 2^{n-1} subintervals of equal length, we apply the trapezoidal rule on each of these subintervals to obtain a composite trapezoidal rule. Write down the associated quadrature formula.

   How are the entries $R(n,k)$ of the Romberg array

$$\begin{array}{cccccc} R(1,1) & & & & & \\ R(2,1) & R(2,2) & & & & \\ R(3,1) & R(3,2) & R(3,3) & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ R(M+1,1) & R(M+1,2) & R(M+1,3) & \cdots & R(M+1,M+1) \end{array}$$

   calculated? Give the expression for $R(n,1)$, and derive the value of the entry $R(2,2)$. What do you recognize?

2. (20 pts) The error term for the approximation

$$f'(x) \approx \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}$$

   may be written as $c\,h^k f^{(n)}(x) + \mathcal{O}(h^{k+1})$. Identify c, k, and n.

3. (30 pts) Find a nonzero polynomial which is orthogonal to the space $\mathcal{P}_2$ on the interval $[-1,1]$ with respect to the weight function $1+x^2$. You should exploit symmetries in order to reduce the amount of calculations.

4. (40 pts) We integrate a function f on the interval $[a, b]$ using the composite trapezoidal rule based on n subintervals of equal length $h = \dfrac{b-a}{n}$. Among the following error terms, which is the correct one?

   - $\dfrac{1}{6}\,(b-a)^3\,f''(\xi)$,
   - $-\dfrac{1}{6}\,(b-a)^3\,h\,f'(\xi)$,
   - $-\dfrac{1}{12}\,(b-a)\,h^2\,f''(\xi)$,
   - $\dfrac{1}{6}\,(b-a)\,h\,f'(\xi)$.

   We want to estimate the integral $\displaystyle\int_0^{\pi} \sin x\; dx$ with an error $\leq 10^{-12}$. How large should n be taken? Will the estimate be too big or too small?

5. (20 pts) How many function evaluations are needed to construct the Romberg array with N rows and N columns?

6. (40 pts) The Legendre polynomial of degree 3 is of the form

$$P_3(x) = \frac{5}{2}x^3 - c\,x.$$

   Find the value of the coefficient c.

   The error estimate for the quadrature formula

$$\int_{-1}^{1} f(x)dx \approx \frac{5}{9}\,f\big(-\sqrt{3/5}\big) + \frac{8}{9}\,f\big(0\big) + \frac{5}{9}\,f\big(\sqrt{3/5}\big)$$

   involves a term $f^{(k)}(\xi)$ for some $\xi \in (-1, 1)$. What is the value of k?