Math 170, Section 2, Test 4

November 20, 2008

Name:

Pledged

Honor code: I have neither given nor received help on this test.

ATTEMPT ALL FIVE QUESTIONS.

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1. (30pts) This question does not require any justification.

   (a) If $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} =$ <u>1/4</u>, then the radius of convergence of the power series $\sum c_n x^n$ is equal to 4.

   (b) Give an example of a power series whose interval of convergence is $[-1, 1)$.

   $$\sum_{n\geq 1} \frac{1}{n} x^n$$

   (c) $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = \cos(\pi) = -1.$$

   (d) According to Taylor's inequality, if $|f^{(n+1)}(x)| \leq M$ for $|x-a| < d$, then the remainder $R_n(x)$ of the Taylor series satisfies

   $$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \qquad \text{for } |x-a| < d.$$

   In particular, if M and d can be chosen independently of n, we have

   $$\lim_{n\to\infty} R_n(x) = 0 \qquad \text{for } |x-a| < d.$$

   which means that f equals <u>the sum of its Taylor series</u> on the interval $|x-a| < d$.

   (e) $$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

2. (15pts) Find the 4-th degree Taylor polynomial of the function $f(x) = \dfrac{(x^2+1)\sin(x)}{e^x}$.

$$f(x) = (1+x^2)(\sin x)(e^{-x})$$
$$= (1+x^2)\left(x - \frac{x^3}{6} + \cdots\right)\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)$$
$$= \left(x - \frac{x^3}{6} + x^3 + \cdots\right)\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)$$
$$= \left(x + \frac{5}{6}x^3 + \cdots\right)\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)$$
$$= x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{5}{6}x^3 - \frac{5}{6}x^4 + \cdots$$
$$= x - x^2 + \frac{4}{3}x^3 - x^4 + \cdots$$

The fourth-degree Taylor polynomial of f is:

$$\underline{T_4(x) = x - x^2 + \frac{4}{3}x^3 - x^4}$$

3. (20pts) Determine the radius of convergence of the function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}\,(n!)^2}\, x^{2n}.$$

Find the power series representations of the functions $x^2 J_0''(x)$ and $x J_0'(x)$ and state their radii of convergence. Establish that

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0.$$

For any x, we have: $\left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)} \left((n+1)!\right)^2} \times \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right| = \frac{|x^2|}{4(n+1)^2} \xrightarrow[n\to\infty]{} 0$,

therefore $\sum \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$ converges for all x. Radius of convergence $= \infty$

By the term-by-term differentiation theorem, we get, for all x:

$J_0'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2n}{2^{2n}(n!)^2} x^{2n-1}$, so $x J_0'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2n}{2^{2n}(n!)^2} x^{2n}$

$J_0''(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2n(2n-1)}{2^{2n}(n!)^2} x^{2n-2}$, so $x^2 J_0''(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2n(2n-1)}{2^{2n}(n!)^2} x^{2n}$

Note that the power series representations of $xJ_0'(x)$ and $x^2J_0''(x)$ can be started at $n=1$, since the terms corresponding to $n=0$ are equal to zero. We deduce:

$$x^2 J_0''(x) + x J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n \left[2n + 2n(2n-1)\right]}{2^{2n}(n!)^2} x^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n \left[(2n)^2\right]}{2^{2n}(n!)^2} x^2$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2(n-1)}\left((n-1)!\right)^2} x^{2(n-1)} x^2 \underset{k=n-1}{=} \left(\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^{2k}(k!)^2} x^{2k} \right) \times x^2$$

$$= -J_0(x) \times x^2$$

We conclude $x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0$

4. **(15pts) Determine the interval of convergence for the series**

$$\sum \frac{n}{2n^3-1}(x-4)^n.$$

For any x, we have: $\left| \frac{(n+1)(x-4)^{n+1}}{2(n+1)^3-1} \times \frac{2n^3-1}{n(x-4)^n} \right| = \frac{(n+1)(2n^3-1)}{(2(n+1)^3-1)n} |x-4|$

By the ratio test: $\xrightarrow[n\to\infty]{} |x-4|$.

$$\sum \frac{n}{2n^3-1}(x-4)^n \begin{cases} \text{converges} & \text{if } |x-4|<1 \\ \text{diverges} & \text{if } |x-4|>1 \end{cases}$$

If $x-4=-1$, that is if $x=3$, we have:

$\sum \frac{n}{2n^3-1}(-1)^n$ converges by the alternating series test, since

$\frac{n}{2n^3-1} > 0$ (because $n \geq 1$ and $2n^3-1 \geq 2-1>0$), $\lim_{n\to\infty} \frac{n}{2n^3-1} = 0$,

and $\left\{\frac{n}{2n^3-1}\right\}_{n\geq 1}$ is decreasing (because $f(x) = \frac{x}{2x^3-1}$ is decreasing on $[1,+\infty)$, since $f'(x) = \frac{2x^3-1-6x^2 x}{(2x^3-1)^2} = \frac{-3x^3-1}{(2x^3-1)^2} < 0$).

If $x-4=1$, that is if $x=5$, we have:

$\sum \frac{n}{2n^3-1} 1^n = \sum \frac{n}{2n^3-1}$ converges by the limit comparison test, since

$$\frac{\frac{n}{2n^3-1}}{\frac{1}{n^2}} = \frac{n^3}{2n^3-1} \xrightarrow[n\to\infty]{} \frac{1}{2} \in (0,+\infty)$$ and $\sum \frac{1}{n^2}$ converges.

Finally, we conclude that the interval of convergence is $\underline{[3,5]}$.

5. **(20pts) Find a power series representation and determine the radius of convergence for the functions**

- $f(x) = \dfrac{x}{x+4}$,
- $g(x) = \ln(5-x)$.

$\circ\ f(x) = \dfrac{x}{4} \times \dfrac{1}{1+\frac{x}{4}} = \dfrac{x}{4} \sum_{n=0}^{\infty} \left(-\dfrac{x}{4}\right)^n = \dfrac{x}{4} \sum_{n=0}^{\infty} \dfrac{(-1)^n}{4^n} x^n = \underline{\sum_{n=0}^{\infty} \dfrac{(-1)^n}{4^{n+1}} x^{n+1}}$

valid for $\left|\frac{x}{4}\right| < 1$, that is $|x| < 4$. Radius of convergence $= 4$

$\circ\ g'(x) = \dfrac{-1}{5-x} = -\dfrac{1}{5} \dfrac{1}{1-\frac{x}{5}} = -\dfrac{1}{5} \sum_{n=0}^{\infty} \left(\dfrac{x}{5}\right)^n = -\sum_{n=0}^{\infty} \dfrac{x^n}{5^{n+1}}$,

valid for $\left|\frac{x}{5}\right| < 1$, i.e. $|x| < 5$. Radius of convergence of $g' = 5$,

so the radius of convergence of g is also 5. By term-by-term

integration, we obtain:

$$g(x) = C - \sum_{n=0}^{\infty} \frac{1}{5^{n+1}} \frac{x^{n+1}}{n+1}$$

Looking at the value $x=0$: $g(0) = C$, i.e. $\ln(5) = C$

Thus: $$\underline{g(x) = \ln 5 - \sum_{n=0}^{\infty} \frac{1}{5^{n+1}\cdot(n+1)} x^{n+1}}$$