Math 170, Section 2, Test 3

November 4, 2008

Name:

Pledged

Honor code: I have neither given nor received help on this test.

ATTEMPT QUESTIONS 1, 2, 3 AND TWO QUESTIONS AMONG QUESTIONS 4, 5, 6. THE TOTAL NUMBER OF POINTS IS 110. ANYTHING ABOVE 100 IS EXTRA CREDIT.

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1. (35pts) This question does not require any justification.

   (a) The series $\sum u_k$ is called absolutely convergent if $\sum |u_k|$ converges.

   (b) Is the series $\sum[1/2^n + 1/n]$ convergent or divergent? divergent

   (c) $$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

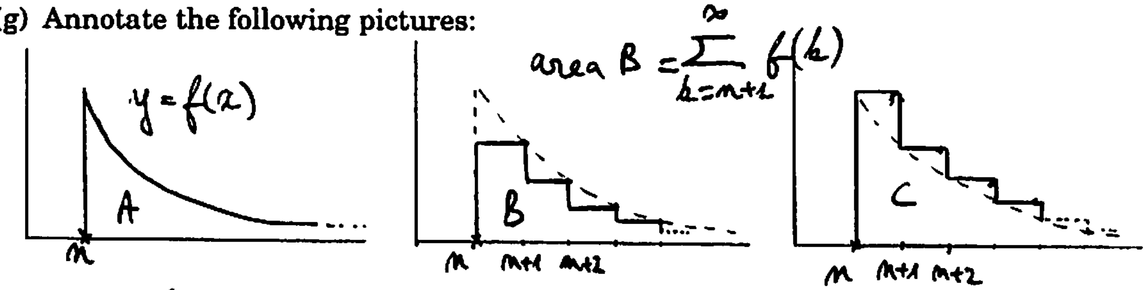
   (d) One can state that a bounded sequence is convergent if it is also monotonic.

   (e) An example of a convergent, but not absolutely convergent, series is $\sum (-1)^n/n$.

   (f) The integral test says that, given a function f that is continuous, decreasing, and positive on the interval $[1, +\infty)$, the series $\sum f(k)$ is convergent if and only if $\int_1^\infty f(x)\,dx$ is convergent.

   (g) Annotate the following pictures:

area $B = \sum_{k=n+1}^{\infty} f(k)$



area $A = \int_n^\infty f(x)\,dx$

area $C = \sum_{k=n}^{\infty} f(k)$

2. **(15pts) We consider the sequence $\{a_n\}_{n\geq 1}$ defined by**

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right), \quad n \geq 1.$$

We admit that $a_n > 1$ for all $n \geq 1$. Is the sequence $\{a_n\}_{n\geq 1}$ monotonic? Is it convergent? If yes, find its limit.

For $n \geq 1$, we have

$$\frac{a_{n+1}}{a_n} = \frac{1}{2}\left(1 + \frac{1}{a_n^2}\right) \underset{a_n > 1}{<} \frac{1}{2}(1+1) = 1, \text{ that is: } a_{n+1} < a_n.$$

Thus, the sequence $\{a_n\}$ is <u>decreasing</u>. It is also bounded below by 1, so by the monotonic sequence theorem, the sequence $\{a_n\}$ is <u>convergent</u>. Let L be the value of its limit. The relations

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right), \qquad a_n > 1,$$

give, at the limit $n \to \infty$:

$$L = \frac{1}{2}\left(L + \frac{1}{L}\right), \qquad L \geq 1$$

$$\frac{1}{2}L = \frac{1}{2}\frac{1}{L}, \qquad L \geq 1$$

$$L^2 = 1, \qquad L \geq 1.$$

This implies $L = 1$, that is $\underline{\lim_{n\to\infty} a_n = 1.}$

3. **(40pts) Determine if the following series are convergent or divergent. Show your work.**

(a) $\sum(-1)^n e^{-n}$

This is a geometric series $\Sigma\left(-e^{-1}\right)^n$, where $|-e^{-1}|=\frac{1}{e}<1$.

The series is convergent.

(b) $\sum \sin(1/n)$

For $n \geqslant 1$, we have $0<\frac{1}{n}\leq 1<\frac{\pi}{2}$, so that $\sin\left(\frac{1}{n}\right)>0$.

The series $\Sigma \sin(1/n)$ and $\Sigma 1/n$ have positive terms, and $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n}=1$. Because $\Sigma 1/n$ diverges, the limit comparison test implies that $\Sigma \sin(1/n)$ diverges.

(c) $\sum \frac{n^2}{2^n}$

We have $\left|\frac{(n+1)^2/2^{n+1}}{n^2/2^n}\right| = \frac{(n+1)^2}{2^{n+1}}\,\frac{2^n}{n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \longrightarrow \frac{1}{2}<1$

We deduce from the ratio test that $\Sigma \frac{n^2}{2^n}$ is convergent.

(d) $\sum \frac{n^3}{n^3+2n^2+1}$

$$\lim_{n\to\infty} \frac{n^3}{n^3+2n^2+1} = 1 \neq 0$$

Thus, the series $\Sigma \frac{n^3}{n^3+2n^2+1}$ diverges.

(e) $\sum \frac{n\sqrt{n}}{n^3+2n^2+1}$

Consider the series $\sum \frac{n\sqrt{n}}{n^3+2n^2+1}$ and $\sum \frac{1}{n^{3/2}}$, whose terms are positive.

Since $\frac{n\sqrt{n}}{n^3+2n^2+1} \times n^{3/2} = \frac{n^3}{n^3+2n^2+1} \xrightarrow[n\to\infty]{} 1$, and since $\sum \frac{1}{n^{3/2}}$ converges, the limit comparison test implies that $\sum \frac{n\sqrt{n}}{n^3+2n^2+1}$ converges.

(f) $\sum \left(\frac{\arctan(n)}{2}\right)^n$

$$\sqrt[n]{\left|\left(\frac{\arctan(n)}{2}\right)^n\right|} = \frac{\arctan(n)}{2} \xrightarrow[n\to\infty]{} \frac{\pi/2}{2} = \frac{\pi}{4} < 1$$

By the root test, the series $\sum \left(\frac{\arctan(n)}{2}\right)^n$ is convergent.

(g) $\sum \frac{\cos(3n)}{3^n}$

We have $\left|\frac{\cos(3n)}{3^n}\right| \leq \frac{1}{3^n}$, so, by the comparison test, the series $\sum \frac{\cos(3n)}{3^n}$ is absolutely convergent, hence convergent.

(h) $\sum \frac{\ln(n)}{n}$

The series $\sum_{n\geq 2} \frac{\ln(n)}{n}$ and $\sum_{n\geq 2} \frac{\ln(2)}{n}$ have positive terms, and the latter diverges. What's more, for $n \geq 2$, we have $\frac{\ln(n)}{n} \geq \frac{\ln(2)}{n}$. Thus, by the comparison test, the series $\sum \frac{\ln(n)}{n}$ diverges.

4. **(10pts) Express the number $0.\overline{73} = 0.73737373737373\ldots$ as a ratio of two integers.**

$$0.\overline{73} = 0.73 + \frac{0.73}{100} + \frac{0.73}{10000} + \frac{0.73}{1000000} + \cdots$$

$$= 0.73 \times \left(1 + \frac{1}{100} + \frac{1}{(100)^2} + \frac{1}{(100)^3} + \cdots\right) = 0.73 \times \frac{1}{1 - \frac{1}{100}}$$

$$= \frac{0.73 \times 100}{100 - 1}$$

$$\boxed{0.\overline{73} = \frac{73}{99}}$$

5. **(10pts) Determine whether the series $\sum \frac{\pi^n}{4^{n-1}}$ is convergent. If it is, find the value of the infinite sum $\sum_{k=1}^{\infty} \frac{\pi^n}{4^{n-1}}$.**

Observe that $\sum \frac{\pi^n}{4^{n-1}} = \pi \sum \left(\frac{\pi}{4}\right)^{n-1}$. The series is therefore convergent, as a geometric series, because $\frac{\pi}{4} < 1$. We have:

$$\sum_{n=1}^{\infty} \frac{\pi^n}{4^{n-1}} = \pi \times \sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^{n-1} = \pi \times \frac{1}{1 - \frac{\pi}{4}} = \boxed{\frac{4\pi}{4-\pi}}$$

6. **(10pts) Find an integer n for which the partial sum $\sum_{k=1}^{n} \frac{1}{k^2}$ approximates the infinite sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$ within an error of 10^{-2}.**

We have: $\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=n+1}^{\infty} \frac{1}{k^2} \le \int_n^{\infty} \frac{1}{x^2}\,dx$ (see 1.(g))

$$= -\frac{1}{x}\Big|_n^{\infty} = \frac{1}{n}.$$

Thus, we will have $\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \le 10^{-2}$ as soon as $\frac{1}{n} \le 10^{-2}$,

or $n \geq 100$. We may take $n = 100.$