Math 170, Section 2, Test 2

October 9, 2008

Name:

Pledged

Honor code: I have neither given nor received help on this test.

ATTEMPT 5 QUESTIONS. THERE ARE 6 QUESTIONS IN TOTAL.

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1. (20pts)

   (a) The Cartesian coordinates (x, y) of a point with polar coordinates (r, θ) are given by

   $$x = \underline{r \cos \theta}, \quad y = \underline{r \sin \theta}.$$

   (b) Let F_1 and F_2 be two points separated by a distance $2c$.
   The ellipse of foci $F1$ and F_2 and of major axis $2a$ is the set of points P for which

   $$\underline{|PF_1| + |PF_2| = 2a}.$$

   The hyperbola of foci F_1 and F_2 and of eccentricity e is the set of points P for which

   $$\underline{|PF_1| - |PF_2| = \pm 2c/e}.$$

   (c) If $f(\theta + 3\pi) = f(\theta)$ for all θ, then the curve of polar equation $r = f(\theta)$ is symmetric about the pole.

2. **(20pts) Identify the conic of Cartesian equation**

$$2y^2 - 3x^2 - 4y + 12x + 8 = 0.$$

Find the vertices and the foci.

Observe that $2y^2 - 4y = 2\left((y-1)^2 - 1\right)$,

$-3x^2 + 12x = -3\left((x-2)^2 - 4\right)$.

The equation becomes : $2(y-1)^2 - 2 - 3(x-2)^2 + 12 + 8 = 0$,

$3(x-2)^2 - 2(y-1)^2 = 18$, $\dfrac{(x-2)^2}{6} - \dfrac{(y-1)^2}{9} = 1$

We identify $a = \sqrt{6}$ and $b = 3$, so that $c = \sqrt{15}$.

We recognize an hyperbola ,

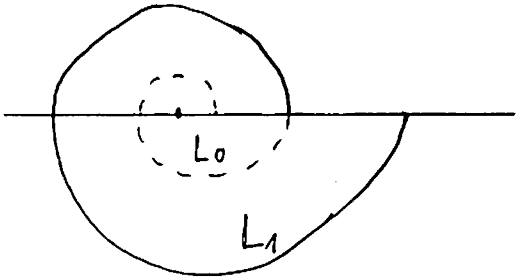
with vertices $(2 \pm \sqrt{6}, 1)$, that is $(2-\sqrt{6}, 1)$ and $(2+\sqrt{6}, 1)$

and foci $(2 \pm \sqrt{15}, 1)$, that is $(2-\sqrt{15}, 1)$ and $(2+\sqrt{15}, 1)$

3. (20pts) We consider, for some constant c, the spiral of polar equation

$$r = e^{c\theta}, \qquad \theta \geq 0.$$

Find the lengths L_0 and L_1 of the portions of the spiral defined by $0 \leq \theta \leq 2\pi$ and by $2\pi \leq \theta \leq 4\pi$, respectively. How to choose the constant c so that $L_1 = e^{\pi/4} L_0$?



Note that $\frac{dr}{d\theta} = c\, e^{c\theta}$. We calculate

$$L_0 = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{e^{2c\theta} + c^2 e^{2c\theta}}\, d\theta$$

$$= \sqrt{1+c^2} \int_0^{2\pi} e^{c\theta}\, d\theta = \frac{\sqrt{1+c^2}}{c} \left. e^{c\theta} \right]_0^{2\pi}$$

$$\underline{L_0 = \frac{\sqrt{1+c^2}}{c} \left(e^{c2\pi} - 1\right)}.$$

To calculate L_1, we just have to change the bounds:

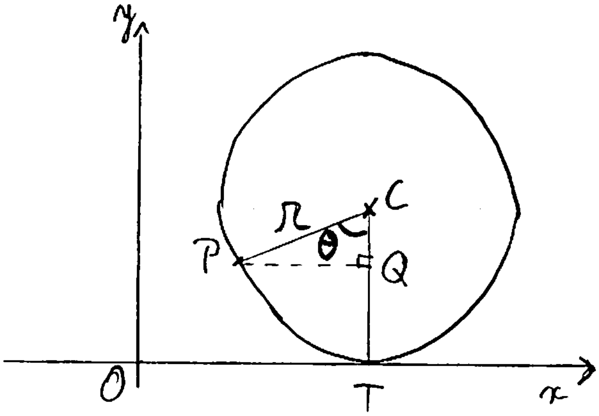
$$L_1 = \int_{2\pi}^{4\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \dots = \frac{\sqrt{1+c^2}}{c} \left. e^{c\theta} \right]_{2\pi}^{4\pi} = \frac{\sqrt{1+c^2}}{c} \left(e^{c4\pi} - e^{c2\pi}\right)$$

$$\underline{L_1 = \frac{\sqrt{1+c^2}}{c} e^{c2\pi} \left(e^{c2\pi} - 1\right)}$$

We observe that $L_1 = e^{c2\pi} L_0$, so to have $L_1 = e^{\pi/4} L_0$,

we need $c2\pi = \frac{\pi}{4}$, that is $\underline{c = \frac{1}{8}}$.

4. **(20pts) The cycloid is the curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line. Let θ is the parameter shown in the figure. Derive parametric equations of the form $x = f(\theta)$ and $y = g(\theta)$ for the cycloid. Calculate d^2y/dx^2 and determine the concavity of an arch of the cycloid, for example when $0 \leq \theta \leq 2\pi$.**



We have: $|OT| = r\theta$, $|CT| = r$, $|PQ| = r\sin\theta$, $|CQ| = r\cos\theta$.

We get: $x = |OT| - |PQ|$, $|y| = |CT| - |CQ|$,

thus: $\underline{x = r(\theta - \sin\theta)}$, $\underline{y = r(1 - \cos\theta)}$.

We calculate: $\dfrac{dx}{d\theta} = r(1-\cos\theta)$, $\dfrac{dy}{d\theta} = r(\sin\theta)$,

so that $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\sin\theta}{1-\cos\theta}$.

$$\text{Then: } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\left(\frac{dy}{dx}\right)/d\theta}{dx/d\theta} = \frac{\cos\theta(1-\cos\theta) - \sin^2\theta}{(1-\cos\theta)^2} \times \frac{1}{r(1-\cos\theta)}$$

$$= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{r(1-\cos\theta)^3} = \frac{\cos\theta - 1}{r(1-\cos\theta)^3}$$

$$\boxed{\frac{d^2y}{dx^2} = -\frac{1}{r(1-\cos\theta)^2}}$$

Since $\dfrac{d^2y}{dx^2} < 0$, we conclude that the arch of the cycloid, $0 \leq \theta \leq 2\pi$, is <u>concave down</u>.

5. **(20pts) Identify the conic of polar equation**

$$r = \frac{4}{2+\cos(\theta)}.$$

State the values of r at the two vertices, find the distance between the two vertices, determine the eccentricity, and give the Cartesian equation of the directrix.

We have: $r = \dfrac{2}{1+\frac{1}{2}\cos\theta} = \dfrac{\frac{1}{2}\times 4}{1+\frac{1}{2}\cos\theta}$.

We recognize the polar equation of an ellipse,

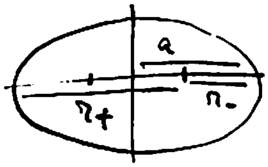
with $\underline{e = 1/2}$ and $d = 4$. The equation of the

directrix is $\underline{x = 4.}$

At the vertices ($\theta = 0$ and $\theta = \pi$), r takes on the values:

$$\underline{r_- = \frac{4}{2+1} = \frac{4}{3}} \quad \text{and} \quad \underline{r_+ = \frac{4}{2-1} = 4}$$

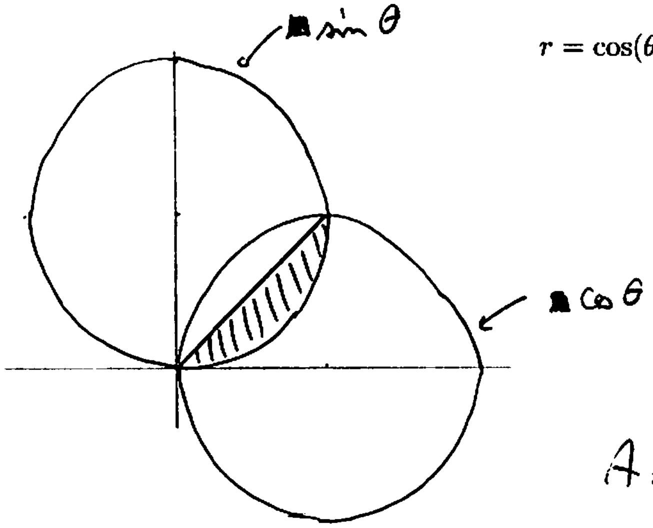
We get $2a = r_- + r_+ = \dfrac{16}{3}$



distance between vertices $= \dfrac{16}{3}$

6. **(20pts) Find the area of the region that lies inside both polar curves**

$$r = \cos(\theta) \qquad \text{and} \qquad r = \sin(\theta).$$



The two curves intersect for $\theta = \frac{\pi}{4}$.

Note that the required area is twice the area shown on the figure, that is

$$A = 2 \times \int_0^{\frac{\pi}{4}} \frac{1}{2} (\sin\theta)^2 \, d\theta = \int_0^{\frac{\pi}{4}} \frac{1-\cos(2\theta)}{2} \, d\theta$$

$$A = \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4} \right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{8} - \frac{1}{4} \right) - (0)$$

$$\boxed{A = \frac{\pi}{8} - \frac{1}{4}}$$