Math 170, Section 2, Test 1

September 18, 2008

Name:

Pledged

Honor code: I have neither given nor received help on this test.

ATTEMPT FIVE QUESTIONS. THERE ARE SIX QUESTIONS IN TOTAL.

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1. **(20pts) Find the general solution, in the form $y = f(x)$, of the differential equation**

$$(2x+4)\frac{dy}{dx} - x^2y^2 - x^2 + 4y^2 + 4 = 0.$$

Rewrite the equation as : $2(x+2)\dfrac{dy}{dx} = x^2y^2 + x^2 - 4y^2 - 4$

$2(x+2)\dfrac{dy}{dx} = (x^2-4)(y^2+1)$, thus $2\dfrac{dy}{dx} = (x-2)(y^2+1)$.

Separate the variables : $2\dfrac{dy}{y^2+1} = (x-2)\,dx$,

and integrate : $2 \times \text{Arctan}(y) = \dfrac{x^2}{2} - 2x + C$

We have : $\text{Arctan}(y) = \dfrac{x^2}{4} - x + \dfrac{C}{2}$

that is : $\boxed{y = \tan\left(\dfrac{x^2}{4} - x + D\right)}$, D arbitrary constant

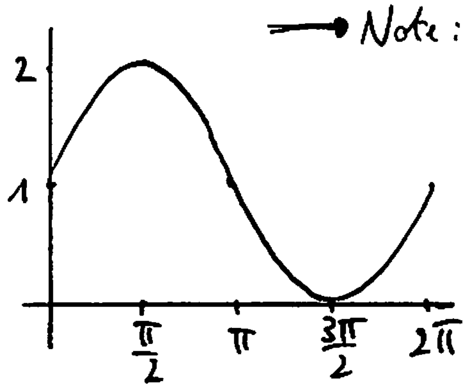
2. **(20pts) Consider the function**

$$f(x) = 1 + \sin x, \qquad x \in [0, 2\pi].$$

Sketch the graph of this function.
Find the coordinates of the centroid of the region delimited by

$$x = 0, \quad x = \pi, \quad y = 0, \quad y = f(x).$$

You may find it helpful to use the identity $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$.



→ Note: this is for the region delimited by: $0 \le x \le 2\pi, 0 \le y \le f(x)$

The area of the region is:

$$S = \int_0^{2\pi} (1+\sin x)\, dx = \left[x - \cos x\right]_0^{2\pi}$$

$$= (2\pi - 1) - (0-1), \quad \text{that is } \underline{S = 2\pi}$$

For the region of the question: $S = \pi + 2$

The coordinates of the centroid are:

$$x_c = \frac{1}{S}\int_0^{2\pi} x \times (1+\sin x)\, dx = \frac{1}{2\pi}\left\{\int_0^{2\pi} x\,dx + \int_0^{2\pi} x \sin x \, dx\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{x^2}{2}\right]_0^{2\pi} + \left[x \times (-\cos x)\right]_0^{2\pi} - \int_0^{2\pi} 1 \times (-\cos x)\, dx\right\} = \frac{1}{2\pi}\left\{2\pi^2 - 2\pi + \underbrace{\left[\sin x\right]_0^{2\pi}}_{=0}\right\}$$

$\underline{x_c = \pi - 1}$ For the region of the question: $x_c = \frac{\pi}{2}$ (symmetry)

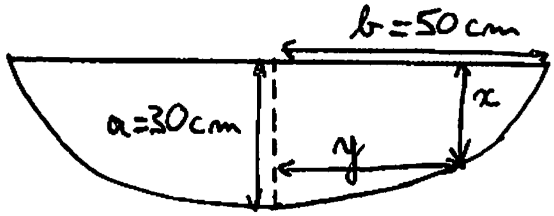
$$y_c = \frac{1}{S}\int_0^{2\pi} \frac{1}{2}(1+\sin x)^2\, dx = \frac{1}{4\pi}\int_0^{2\pi}\left(1 + 2\sin x + \sin^2 x\right) dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\left(1 + 2\sin x + \frac{1-\cos(2x)}{2}\right) dx = \frac{1}{4\pi}\left\{\int_0^{2\pi}\left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos(2x)\right) dx\right\}$$

$$= \frac{1}{4\pi}\left[\frac{3}{2}x + 2(-\cos x) - \frac{1}{4}\sin(2x)\right]_0^{2\pi} = \frac{1}{4\pi}\left(\frac{3}{2}(2\pi - 0) + 0 + 0\right)$$

$\underline{y_c = \frac{3}{4}}$ For the region of the question: $y_c = \dfrac{3\pi + 8}{4(\pi+2)}$

3. **(20pts) A trough is filled with water of density $\rho = 1000\,\mathrm{kg.m^{-3}}$. The end of the trough are semi-ellipses whose dimensions are indicated on the figure.**



With the notations of this figure, what is the relation between x and y?

(a) $\frac{y}{b} = \sqrt{1 - \left(\frac{x}{a}\right)^2}$

(b) $\frac{y}{a} = \sqrt{1 - \left(\frac{x}{b}\right)^2}$

(c) $\frac{y}{b} = \sqrt{1 + \left(\frac{x}{a}\right)^2}$

Answer: (a)

[(b) and (c) are not consistent with $y = 0$ when $x = a$]

Find the hydrostatic force on one end of the trough. For numerical purposes, we will make the approximation $g = 10\,\mathrm{N.kg^{-1}}$.

The pressure at depth x is: $P = \rho g x$.

A thin rectangle of width $2y$ and height dx has an area of:

$$dS = 2y\,dx = 2b\sqrt{1 - \frac{x^2}{a^2}}\,dx.$$

Thus, the force on this rectangle is:

$$dF = P \times dS = 2\rho g b\, x\sqrt{1 - \frac{x^2}{a^2}}\,dx.$$

The total force on the end of the trough is:

$$F = \int_0^a dF = 2\rho g b \int_0^a x\sqrt{1 - \frac{x^2}{a^2}}\,dx = 2\rho g b\left[-\frac{a^2}{3}\left(1 - \frac{x^2}{a^2}\right)^{3/2}\right]_0^a$$

$$= 2\rho g b\left[0 + \frac{a^2}{3} \times 1\right], \text{ hence } \boxed{F = \frac{2}{3}\rho g a^2 b}$$

Numerically:

$$F \approx \frac{2}{3} \times 10^3 \times 10 \times 9 \times 10^{-2} \times 5 \times 10^{-1} = \frac{2 \times 9 \times 5 \times 10}{3}, \quad \boxed{F \approx 300\,N}$$

4. **(20pts) Determine the area of the surface obtained by the rotation about the x-axis of the curve**

$$y = \cosh(x), \qquad x \in [0,1].$$

The area of this surface is given by:

$$A = \int_0^1 2\pi \cosh(x)\sqrt{1+\sinh^2(x)}\,dx$$

In view of: $1+\sinh^2(x) = \cosh^2(x)$ and of: $\cosh(x) > 0$, we obtain:

$$A = 2\pi \int_0^1 \cosh^2(x)\,dx$$

$$= 2\pi \int_0^1 \left(\frac{e^{x}+e^{-x}}{2}\right)^2 dx = 2\pi \int_0^1 \left(\frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4}\right) dx$$

$$= 2\pi \left\{ \left[\frac{e^{2x}}{8}\right]_0^1 + \frac{1}{2} + \left[\frac{-e^{-2x}}{8}\right]_0^1 \right\}$$

$$= 2\pi \left\{ \frac{e^2}{8} - \frac{1}{8} + \frac{1}{2} - \frac{e^{-2}}{8} + \frac{1}{8} \right\}$$

$$\boxed{A = \pi \left\{ \frac{e^2}{4} + 1 - \frac{e^{-2}}{4} \right\}}$$

5. **(20pts) Find the equation of the curve passing through the point $(-1,-1)$ whose slope at each point (x,y) is given by $\left(\frac{2y+3}{4x+5}\right)^2$. Give your answer in the form $y = f(x)$.**

The question translates into the initial-value problem:

$$\frac{dy}{dx} = \frac{(2y+3)^2}{(4x+5)^2} \quad \text{subject to } y(-1) = -1.$$

Separate the variables: $$\frac{dy}{(2y+3)^2} = \frac{dx}{(4x+5)^2}$$

Integrate: $$-\frac{1}{2}\frac{1}{2y+3} = -\frac{1}{4}\frac{1}{4x+5} + C$$

Initial condition: $$-\frac{1}{2}\frac{1}{-2+3} = -\frac{1}{4}\frac{1}{-4+5} + C \quad \text{, thus } C = -\frac{1}{4}$$

We get: $$-\frac{1}{2}\frac{1}{2y+3} = -\frac{1}{4}\left(\frac{1}{4x+5} + 1\right) = -\frac{1}{4}\frac{4x+6}{4x+5}$$

$$\frac{1}{2y+3} = \frac{2x+3}{4x+5}$$

$$2y+3 = \frac{4x+5}{2x+3}$$

$$2y = \frac{4x+5-3(2x+3)}{2x+3} = \frac{-2x-4}{2x+3}$$

$$\boxed{y = -\frac{x+2}{2x+3}}$$

6. **(20pts) Determine the arc length of the curve $y = f(x)$ between the points $(1, -6/5)$ and $(8, 33/5)$, where**

$$f(x) = \frac{3}{10}x^{5/3} - \frac{3}{2}x^{1/3}.$$

The arc length is given by:

$$L = \int_1^8 \sqrt{1 + \left(\frac{1}{2}x^{2/3} - \frac{1}{2}x^{-2/3}\right)^2}\, dx$$

$$= \int_1^8 \sqrt{1 + \frac{1}{4}x^{4/3} - \frac{1}{2} + \frac{1}{4}x^{-4/3}}\, dx$$

$$= \int_1^8 \sqrt{\left(\frac{1}{2}x^{2/3} + \frac{1}{2}x^{-2/3}\right)^2}\, dx = \int_1^8 \left(\frac{1}{2}x^{\frac{2}{3}} + \frac{1}{2}x^{-\frac{2}{3}}\right) dx$$

$$= \left[\frac{3}{10}x^{\frac{5}{3}} + \frac{3}{2}x^{\frac{1}{3}}\right]_1^8 = \frac{3}{10}\left(2^5 - 1\right) + \frac{3}{2}\left(2 - 1\right)$$

$$= \frac{3}{10}\,31 + \frac{3}{2} = \frac{3}{10}\left(31 + 5\right) = \frac{3 \times 36}{10} = \frac{3 \times 18}{5}$$

$$\boxed{L = \frac{54}{5}}$$