Math 170, Fall 2008, Section 2
Assignment 4

Honor code: You should neither give nor receive help on this assignment.
Organize your work in a concise way and show every step of your argument. Write legibly.

1. **(7pts) Use a power series to approximate the definite integral $\int_0^{0.4} \ln(1+x^4)\,dx$ to six decimal places.**

2. **(6pts) For a fixed number $k \neq -1$, we consider the sequence $\{u_n\}_{n\geq 0}$ defined by**

$$u_n = \int_0^1 x^k (\ln x)^n \, dx.$$

   (a) **Use integration by parts to find a relation between u_n and u_{n-1}.**

   (b) **Write a proof by induction to show that**

$$\int_0^1 x^k (\ln x)^n \, dx = \frac{(-1)^n \, n!}{(k+1)^{n+1}}, \qquad n \geq 0.$$

   (c) **Find a series [not a power series!] representation for the function x^x. [*Hint: think exponential.*]**

   (d) **Use term-by-term integration [do not worry about justifying it] to obtain**

$$\int_0^1 x^x \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}}.$$

3. **(7pts) We want to know if it makes sense to consider the arc-length of the curve $y = f(x)$ between the points $(0,0)$ and $(1,-1)$, where**

$$f(x) = \begin{cases} x\cos\left(\dfrac{\pi}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

   (a) **Prove that f is continuous on $(-\infty, \infty)$.**

   (b) **Draw a rough sketch for the graph of f.**

   (c) **For an integer $n \geq 1$, we denote by ℓ_n the length of the curve $y = f(x)$ between the points $\left(\dfrac{1}{n+1}, f\left(\dfrac{1}{n+1}\right)\right)$ and $\left(\dfrac{1}{n}, f\left(\dfrac{1}{n}\right)\right)$. Write down, but do not evaluate, an integral expression for ℓ_n.**

(d) By considering the line joining the points $\left(\frac{1}{n+1}, f\left(\frac{1}{n+1}\right)\right)$ and $\left(\frac{1}{n}, f\left(\frac{1}{n}\right)\right)$, give a lower bound for ℓ_n, that is find a positive number u_n such that $\ell_n \geq u_n$. [Remember that $\cos(\pi k) = (-1)^k$.]

(e) Does the series $\sum u_n$ converge?

(f) Does the the arc-length of the curve $y = f(x)$ between the points $\left(\frac{1}{n+1}, f\left(\frac{1}{n+1}\right)\right)$ and $(1, f(1))$ have a finite limit as $n \to \infty$?

(g) Conclude. Why is there no contradiction with the arc-length formula that usually yields a finite length?

1/ Recall that

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \qquad \text{for } |x|<1$$

[integration of $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ ...], therefore

$$\ln(1+x^4) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^{4k} \qquad \text{for } |x^4|<1, \text{ i.e. } |x|<1.$$

In particular, using term-by-term integration, we get

$$\int_0^{0.4} \ln(1+x^4)\,dx = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_0^{0.4} x^{4k}\,dx = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{0.4^{4k+1}}{k(4k+1)}$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} b_k, \quad \text{where } b_k = \frac{0.4^{4k+1}}{k(4k+1)}.$$

Note that:
* $b_n > 0$
* $\lim_{n\to\infty} b_n = 0$, since $\sum b_n$ converges by the ratio test
* $b_{n+1} \le b_n$, since $\begin{cases} \frac{1}{n+1} \le \frac{1}{n} \\ \frac{1}{4(n+1)+1} \le \frac{1}{4n+1} \\ 0.4^{4(n+1)+1} \le 0.4^{4n+1} \end{cases}$

Thus, we can apply the Alternating Series Estimation Theorem to get

$$\left| \sum_{k=1}^{\infty} (-1)^{k-1} b_k - \sum_{k=1}^{n} (-1)^{k-1} b_k \right| \le b_{n+1}, \quad \text{that is}$$

$$\left| \int_0^{0.4} \ln(1+x^4)\,dx - \sum_{k=1}^{n} (-1)^{k-1} \frac{0.4^{4k+1}}{k(4k+1)} \right| \le \frac{0.4^{4n+5}}{(n+1)(4n+5)}$$

Taking $n=2$, the right-hand side is already $\approx 1.72\times 10^{-7}$, so that

$$\left| \int_0^{0.4} \ln(1+x^4)\,dx - \left\{ \frac{0.4^5}{5} - \frac{0.4^9}{18} \right\} \right| < 10^{-6},$$

and the approximation $\int_0^{0.4} \ln(1+x^4)\,dx \approx 0.002033$ is exact up to 6 decimal places.

2/ Let $u_m = \int_0^1 x^k (\ln x)^m \, dx$ for some $k \neq -1$.

(a) $$u_m \underset{(IbP)}{=} \left. \frac{x^{k+1}}{k+1} (\ln x)^m \right|_0^1 - \frac{m}{k+1} \int_0^1 x^{k+1} \times (\ln x)^{m-1} \times \frac{1}{x} \, dx$$

$$= 0 - \frac{m}{k+1} \int_0^1 x^k (\ln x)^{m-1} \, dx$$

$$\boxed{u_m = -\frac{m}{k+1} u_{m-1}}$$

(b) Let us prove by induction on $m \geq 0$ that

$$\boxed{u_m = \int_0^1 x^k (\ln x)^m \, dx = \frac{(-1)^m \, m!}{(k+1)^{m+1}}}.$$

Base case $m = 0$

$$u_0 = \int_0^1 x^k \, dx = \frac{1}{k+1}$$

$$\frac{(-1)^0 \, 0!}{(k+1)^{0+1}} = \frac{1}{k+1}$$

, that is: $u_0 = \dfrac{(-1)^0 \, 0!}{(k+1)^{0+1}}$

Let us now assume that, for some $m \geq 1$, we have

$$u_{m-1} = \frac{(-1)^{m-1} (m-1)!}{(k+1)^m},$$

and let us prove that

$$u_m = \frac{(-1)^m \, m!}{(k+1)^{m+1}}.$$

According to (a) and the induction hypothesis, we get:

$$u_m = \frac{-m}{k+1} u_{m-1} = \frac{-m}{k+1} \times \frac{(-1)^{m-1} (m-1)!}{(k+1)^m} = \frac{(-1)^m \, m!}{(k+1)^m},$$ as expected

The proof by induction is now complete.

(c) For the function x^x, we have:

$$x^x = \exp(x \ln x) = \sum_{n=0}^{\infty} \frac{1}{n!} (x \ln x)^n$$

$$\underline{\left| x^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n (\ln x)^n \right.}$$

[not justified!]

(d) Using term-by-term integration, we obtain

$$\int_0^1 x^x \, dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 x^n (\ln x)^n \, dx .$$

Now specializing the result of (b) to $k=n$, we have

$$\int_0^1 x^n (\ln x)^n \, dx = \frac{(-1)^n \, n!}{(n+1)^{n+1}} .$$

We finally conclude that

$$\underline{\left| \int_0^1 x^x \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}} \right.} .$$

3/ Consider the function

$$f(x) = \begin{cases} x \cos\left(\frac{\pi}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

(a) As a product of continuous functions on $(-\infty, 0)$ and on $(0, +\infty)$, the function f is continuous on $(-\infty, 0)$ and on $(0, +\infty)$. To prove that it is continuous on $(-\infty, \infty)$, it is therefore enough to prove continuity of f at 0, i.e. to prove that $\lim_{x \to 0} x \cos\left(\frac{\pi}{x}\right) = 0$.

Observe that, for $x > 0$, we have

$$-x \leq x \cos\left(\frac{\pi}{x}\right) \leq x ,$$

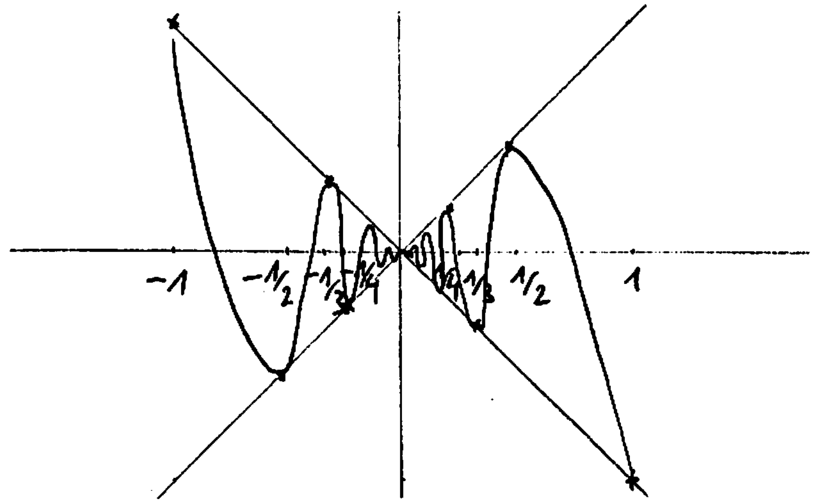
and for $x<0$, we have

$$x \le x\cos\left(\frac{\pi}{x}\right) \le -x .$$

By the Squeeze Theorem, we get: $\lim\limits_{x\to 0^+} x\cos\left(\frac{\pi}{x}\right) = \lim\limits_{x\to 0^-} x\cos\left(\frac{\pi}{x}\right) = 0$,

which is what we needed.

(b) Rough sketch for the graph of f:



(c) Let l_n be the length of the curve $y=f(x)$ between $\left(\frac{1}{n+1}, f\left(\frac{1}{n+1}\right)\right)$ and $\left(\frac{1}{n}, f\left(\frac{1}{n}\right)\right)$

We have: $$l_n = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \sqrt{1+f'(x)^2}\; dx$$

Note that, for $x\neq 0$, we have:

$$f'(x) = 1\times\cos\left(\frac{\pi}{x}\right) + x\times\left(\frac{-\pi}{x^2}\right)\times\left(-\sin\left(\frac{\pi}{x}\right)\right) = \cos\left(\frac{\pi}{x}\right) + \frac{\pi}{x}\sin\left(\frac{\pi}{x}\right)$$

Thus: $$\boxed{l_n = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \sqrt{1+\left(\cos\left(\frac{\pi}{x}\right)+\frac{\pi}{x}\sin\left(\frac{\pi}{x}\right)\right)^2}\; dx}$$

(d) The length l_n is larger than the distance from

$$\left(\frac{1}{n+1}, f(n+1)\right) = \left(\frac{1}{n+1}, \frac{(-1)^{n+1}}{n+1}\right) \quad \text{to} \quad \left(\frac{1}{n}, f\left(\frac{1}{n}\right)\right) = \left(\frac{1}{n}, \frac{(-1)^n}{n}\right).$$

This translates into:

$$l_n \geq \sqrt{\left(\frac{1}{n}-\frac{1}{n+1}\right)^2+\left(\frac{(-1)^n}{n}-\frac{(-1)^{n+1}}{n+1}\right)^2} = \sqrt{\left(\frac{1}{n}-\frac{1}{n+1}\right)^2+\left(\frac{1}{n}+\frac{1}{n+1}\right)^2}$$

$$l_n \geq \sqrt{\frac{1}{n^2}+\frac{1}{(n+1)^2}-\frac{2}{n(n+1)}+\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{2}{n(n+1)}}$$

$$\boxed{l_n \geq u_n = \sqrt{\frac{2}{n^2}+\frac{2}{(n+1)^2}}}$$

(e) Observe that $u_n \geq \sqrt{\frac{2}{(n+1)^2}+\frac{2}{(n+1)^2}} = \frac{2}{n+1}$,

so that $\underline{\sum u_n \text{ diverges}}$

(f) The arc-length of the curve $y=f(x)$ between $\left(\frac{1}{n+1}, f(n+1)\right)$ and $(1, f(1))$

is: $L_n = l_n + l_{n-1} + \dots + l_1 = \sum_{k=0}^{n} l_k$.

Since $l_k \geq u_k$ and $\sum u_k$ diverges, we can conclude that

$\underline{L_n \text{ does not have a limit as } n\to\infty}$.

(g) Thus, it makes no sense to consider the arc-length of the curve $y=f(x)$ between $(0, f(0)) = (0, 0)$ and $(1, f(1)) = (1, -1)$.

This is not in contradiction with a possible formula of the form $\int_0^1 \sqrt{1+f'(x)^2}\, dx$,

because such a formula is only valid if $\underline{f' \text{ is continuous on } [0,1]}$