Math 170, Fall 2008, Section 2
Assignment 3

Honor code: You should neither give nor receive help on this assignment.
Organize your work in a concise way and show every step of your argument. Write legibly.

1. (11pts)

   Determine whether the following series are convergent or divergent. Show all your work. In particular, verify all the hypotheses of any theorem, and name these theorems.

   (a) $\sum \frac{1000^n}{n!}$

   (b) $\sum \frac{1}{n + \sin^2(n)}$

   (c) $\sum \frac{\sin(1/n)}{n}$

   (d) $\sum \frac{e^{-n^2}}{n^n}$

   (e) $\sum \frac{1}{n \ln n}$

   (f) $\sum \frac{1}{n(\ln n)^2}$

   (g) $\sum \frac{\cos(n)}{2^n}$

   (h) $\sum \frac{(-1)^n}{n \ln n}$

   (i) $\sum \frac{(-1)^n}{\arctan n}$

   (j) $\sum \frac{(-1)^n}{(\arctan n)^n}$

   (k) $\sum a_n,$ where $a_1 = 1, \quad a_{n+1} = n \tan(2/n) a_n$

2. (3pts) Determine the minimal value of the integer n for which the partial sum $\sum_{k=1}^{n} \frac{1}{k^4}$ approximates the sum $\sum_{k=1}^{\infty} \frac{1}{k^4}$ within an error of 10^{-6}. You may use 9.72×10^{-7} instead.

3. (6pts) Let r be a number that satisfies $|r| < 1$. We consider the sequence $\{u_n\}_{n\geq 0}$ defined by

$$u_n = (n+1)r^n, \qquad n \geq 0.$$

(a) Explain why the series $\sum u_n$ converges. What is the value of $\lim_{n\to\infty} u_n$?

(b) What is the value of the partial sum $V_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$?

(c) For $k \geq 1$, express r^k in terms of V_k and V_{k-1}. Using this, find an expression for the the partial sum $U_n = \sum_{k=0}^{n} u_k$. You will need to separate the sum in two and to make an appropriate change of summation index in one of the two resulting sums.

(d) Deduce the value of the infinite sum

$$\sum_{k=0}^{\infty}(k+1)r^k = 1 + 2r + 3r^2 + 4r^3 + \cdots.$$

(e) Compare with the formal differentiation, with respect to r, of the formula

$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

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1/ (a) $\sum \frac{1000^n}{n!}$

$$\frac{1000^{n+1}}{(n+1)!} \times \frac{n!}{1000} = \frac{1000}{n+1} \xrightarrow[n\to\infty]{} 0 < 1.$$

By the ratio test, we conclude that $\sum \frac{1000^n}{n!}$ is absolutely convergent, hence convergent. [in fact, absolutely convergence and convergence are the same in this case, because the series has positive terms]

(b) $\sum \frac{1}{n+\sin^2(n)}$

For each $n \geq 1$, $n+\sin^2(n) \leq n+1 \leq 2n$, so $\frac{1}{n+\sin^2(n)} \geq \frac{1}{2n}$.

Moreover, the series $\sum \frac{1}{n+\sin^2(n)}$ and $\sum \frac{1}{2n}$ have positive terms.

Because $\sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$ is known to be divergent [harmonic series], the comparison test allows to conclude that $\sum \frac{1}{n+\sin^2(n)}$ is divergent.

(c) $\sum \frac{\sin(1/n)}{n}$

Observe that $\sum \frac{|\sin(1/n)|}{n}$ and $\sum \frac{1}{n^2}$ are series with positive terms, and that $\frac{\frac{|\sin(1/n)|}{n}}{\frac{1}{n^2}} = \frac{\sin(1/n)}{1/n} \xrightarrow[n\to\infty]{} 1$.

Since the series $\sum \frac{1}{n^2}$ converges [p series], the limit comparison test implies that $\sum \frac{|\sin(1/n)|}{n}$ converges. This means that $\sum \frac{\sin(1/n)}{n}$ is absolutely convergent, hence convergent.

(d) $\sum \frac{e^{-n^2}}{n^n}$

$$\sqrt[n]{\left|\frac{e^{-n^2}}{n^n}\right|} = \left(\frac{e^{-n^2}}{n^n}\right)^{1/n} = \frac{e^{-n^2/n}}{n} = \frac{e^{-n}}{n} \xrightarrow[n\to\infty]{} 0 < 1.$$

By the <u>root test</u>, the series $\sum \frac{e^{-n^2}}{n^n}$ is <u>absolutely convergent</u>, hence <u>convergent</u>.

(e) $\sum \frac{1}{n \ln n}$

Consider the function f defined on $[2, +\infty)$ by $f(x) = \frac{1}{x \ln x}$

* f is a continuous function, as a product of two continuous functions $\left[\frac{1}{x} \text{ and } \frac{1}{\ln x}\right]$
* f is a positive function, since, for $x \geq 2$, we have $x > 0$ and $\ln x > 0$
* f is a decreasing function, as a product of two positive decreasing functions $\left[\frac{1}{x} \text{ and } \frac{1}{\ln x}\right]$.

Observe that $\int_2^t f(x)\,dx = \int_2^t \frac{1/x}{\ln x}\,dx = \ln(\ln x)\Big|_2^t = \ln\left(\frac{\ln t}{\ln 2}\right)$

$$\xrightarrow[t\to\infty]{} +\infty.$$

Since $\int_2^\infty f(x)\,dx$ is divergent, the <u>integral test</u> allows to conclude that $\sum f(n) = \sum \frac{1}{n \ln n}$ is <u>divergent.</u>

(f) $\sum \frac{1}{n(\ln n)^2}$

We now introduce the function f defined on $[2, +\infty)$ by $f(x) = \frac{1}{x(\ln x)^2}$.
For the same reasons as above, f is continuous, positive, and decreasing on $[2, +\infty)$.

We have $\int_2^t f(x)\,dx = \int_2^t \frac{1/x}{(\ln x)^2}\,dx = -\ln(x)^{-1}\Big|_2^t = \frac{-1}{\ln t} + \frac{1}{\ln 2} \xrightarrow[t\to\infty]{} \frac{1}{\ln 2}$.

Since $\int_2^\infty f(x)\,dx$ is convergent, we conclude, by the <u>integral test</u>, that $\sum f(n) = \sum \frac{1}{n(\ln n)^2}$ is <u>convergent.</u>

(g) $\sum \frac{\cos(n)}{2^n}$

The series $\sum \frac{|\cos(n)|}{2^n}$ and $\sum \frac{1}{2^n}$ have positive terms, and, for all $n \geq 1$, $\frac{|\cos(n)|}{2^n} \leq \frac{1}{2^n}$.

Since the series $\sum \frac{1}{2^n}$ is convergent [geometric series], the <u>comparison test</u> implies that $\sum \frac{|\cos(n)|}{2^n}$ is convergent.

Since absolute convergence implies convergence, we can conclude that $\sum \frac{\cos(n)}{2^n}$ is <u>convergent.</u>

(h) $\sum \frac{(-1)^n}{n \ln n}$

We have, for all $n \geq 2$, $\frac{1}{n\ln n} > 0$, and $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$.

Furthermore, we have, for all $n \geq 2$, $\frac{1}{(n+1)\ln(n+1)} \leq \frac{1}{n \ln n}$,

because the function $\frac{1}{x \ln x}$ is decreasing on $[2, +\infty)$ (see (e)).

Therefore, by the <u>alternating series test</u>, we can conclude that

$\sum \frac{(-1)^n}{n \ln n}$ is <u>convergent.</u>

(i) $\sum \frac{(-1)^n}{\arctan n}$

Since $\arctan n \xrightarrow[n\to\infty]{} \frac{\pi}{2}$, we get $\frac{(-1)^n}{\arctan n} \underset{n\to\infty}{\not\to} 0$.

Thus, the series $\sum \frac{(-1)^n}{\arctan n}$ <u>diverges.</u> (<u>test for divergence</u>)

(j) $\sum \frac{(-1)^n}{(\arctan n)^n}$

$$\sqrt[n]{\left|\frac{(-1)^n}{(\arctan n)^n}\right|} = \left(\frac{1}{(\arctan n)^n}\right)^{1/n} = \frac{1}{\arctan n} \xrightarrow[n\to\infty]{} \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} < 1.$$

By the <u>root test</u>, we conclude that $\sum \frac{(-1)^n}{(\arctan n)^n}$ is absolutely convergent, hence <u>convergent</u>.

(k) $\sum a_n$, where $a_1 = 1$, $a_{n+1} = n \tan(2/n)\, a_n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|n \tan(2/n)\right| = 2\left|\frac{\tan(2/n)}{2/n}\right| = 2\left|\frac{1}{\cos(2/n)}\right|\left|\frac{\sin(2/n)}{2/n}\right|$$

$$\xrightarrow[n\to\infty]{} 2 \times 1 \times 1 = 2 > 1.$$

The <u>ratio test</u> implies that the series $\sum a_n$ is <u>divergent</u>.

2/ Let us consider the convergent series $\sum \frac{1}{k^4}$. We want to compare the remainder $R_n = \sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{n} \frac{1}{k^4} = \sum_{k=n+1}^{\infty} \frac{1}{k^4}$ with the error 9.72×10^{-7} (or 1×10^{-6}).

Because the function $\frac{1}{x^4}$ is continuous, positive, and decreasing on $[1, +\infty)$, we can write, for any $n \geq 1$,

$$\int_{n+1}^{\infty} \frac{1}{x^4}\,dx \leq R_n \leq \int_{n}^{\infty} \frac{1}{x^4}\,dx \text{ , i.e. } -\frac{1}{3x^3}\Big|_{n+1}^{\infty} \leq R_n \leq -\frac{1}{3x^3}\Big|_{n}^{\infty},$$

that is
$$\frac{1}{3(n+1)^3} \leq R_n \leq \frac{1}{3n^3}.$$

Observe that: $\frac{1}{3n^3} < 9.72 \times 10^{-7} \iff n > \left(\frac{10^7}{3 \times 9.72}\right)^{1/3} \iff n > 69.99.. \iff n \geq 70.$

We get: $R_{70} \leq \frac{1}{3 \times (70)^3} < 9.72 \times 10^{-7}$,

and: $R_{68} \geq \frac{1}{3 \times (69)^3} > 9.72 \times 10^{-7}$.

This shows that the smallest integer n for which $R_n < 9.72 \times 10^{-7}$ is either 69 or 70. Let us prove that $R_{69} > 9.72 . 10^{-7}$, so that this smallest integer cannot be 69.

The bound $R_{69} \geq \frac{1}{3(70)^3} \simeq 9.718 . 10^{-7}$ is not enough, but we can remark that $R_{69} = \frac{1}{70^4} + R_{70} \geq \frac{1}{70^4} + \frac{1}{3(71)^3} \simeq 9.729 \times 10^{-7} > 9.72 \times 10^{-7}$.

We may conclude that the smallest n for which $R_n < 9.72 \times 10^{-7}$ is $\underline{\underline{n = 70}}$.

Note: for the error 1×10^{-6}, the smallest n is 69. This is a bit more subtle to establish. See the complement on page 8/8.

3/ We consider the sequence $\{u_n\}_{n\geq 0}$ defined by

$$u_n = (n+1)\, r^n \,, \quad n \geq 0, \qquad \text{where} \quad |r| < 1 \quad \text{is fixed.}$$

(a) We have $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)\, r^{n+1}}{n \quad r^n}\right| = \frac{n+1}{n}|r| \xrightarrow[n\infty]{} |r| < 1$.

By the ratio test, $\sum u_n$ is absolutely convergent, hence convergent.
This implies in particular that $\lim_{n\to\infty} u_n = 0$.

(b) For the partial geometric sum $V_n = 1 + r + r^2 + \dots + r^n$, we have

$$V_n = \frac{1 - r^{n+1}}{1-r}.$$

[remember that: $(1-r)V_n = (1-r)(1+r+r^2+\dots+r^n) = \begin{array}{l} 1 + r + \dots + r^n \\ -r - \dots - r^n - r^{n+1} \end{array} = 1 - r^{n+1}$]

(c) For $k \geq 1$, we have $r^k = V_k - V_{k-1}$. We obtain:

$$U_n = \sum_{k=0}^{n} u_k = u_0 + \sum_{k=1}^{n} u_k = 1 + \sum_{k=1}^{n} (k+1)\, r^k = 1 + \sum_{k=1}^{n} (k+1)(V_k - V_{k-1})$$

$$= 1 + \sum_{k=1}^{n} (k+1) V_k - \sum_{k=1}^{n} (k+1) V_{k-1} = (0+1)\times V_0 + \sum_{k=1}^{n} (k+1) V_k - \sum_{k=1}^{n} (k+1) V_{k-1}$$

$$= \sum_{k=0}^{n} (k+1) V_k - \sum_{k=1}^{n} (k+1) V_{k-1} \underset{[l=k+1]}{=} \sum_{l=1}^{n+1} l\, V_{l-1} - \sum_{k=1}^{n} (k+1) V_{k-1}$$

$$= (n+1) V_n + \sum_{k=1}^{n} k\, V_{k-1} - \sum_{k=1}^{n} (k+1) V_{k-1} = (n+1) V_n + \sum_{k=1}^{n} [k - (k+1)]\, V_{k-1}$$

$$= (n+1) V_n - \sum_{k=1}^{n} V_{k-1} = (n+1) \frac{1 - r^{n+1}}{1-r} - \sum_{k=1}^{n} \frac{1 - r^k}{1-r}$$

$$= \frac{1}{1-r}\left[(n+1) - (n+1)\, r^{n+1} - \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} r^k \right]$$

$$= \frac{1}{1-r}\left[(n+1) - (n+1)\, r^{n+1} - n + \sum_{k=0}^{n} r^k - 1 \right]$$

$$U_n = \frac{1}{1-r}\left[-(n+1)r^{n+1} + \frac{1-r^{n+1}}{1-r}\right].$$

We have established, $\boxed{U_n = \frac{1-r^{n+1}}{(1-r)^2} - \frac{(n+1)r^{n+1}}{1-r}}$

(d) Because $r^{n+1} \xrightarrow[n\to\infty]{} 0$ and $(n+1)r^{n+1} \xrightarrow[n\to\infty]{} 0$ (see (a)), we get:

$\lim_{n\to\infty} U_n = \frac{1}{(1-r)^2}$, that is to say $\sum_{k=0}^{\infty} u_k = \frac{1}{(1-r)^2}$,

in other words: $\boxed{\sum_{k=0}^{\infty} (k+1) r^k = 1 + 2r + 3r^2 + \cdots = \frac{1}{(1-r)^2}}$.

(e) Let us recall the value of the geometric series $\sum_{k=0}^{\infty} r^k$,

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

By formally differentiating with respect to r, we obtain:

$$0 + 1 + 2r + 3r^2 + \cdots = -\frac{-1}{(1-r)^2},$$

that is: $1 + 2r + 3r^2 + \cdots = \frac{1}{(1-r)^2}$.

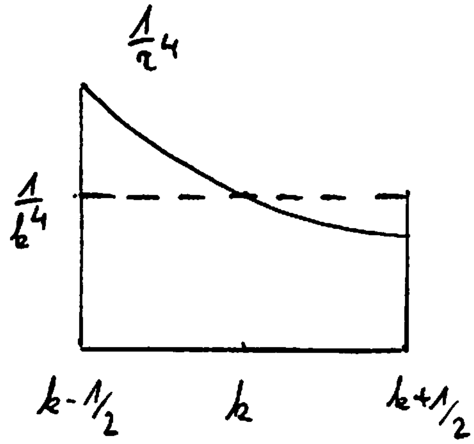
this is simply the formula obtained in (d).

2/ Complement.

If we replace the error 9.72×10^{-7} of Question 2/ by the error 1×10^{-6}, we still have: $R_{70} \leq \dfrac{1}{3\times 70^3} < 1\times 10^{-6}$ and $R_{68} \geq \dfrac{1}{3\times (68)^3} \simeq 1.06.10^{-6} > 1\times 10^{-6}$.

So the smallest n for which $R_n < 1\times 10^{-6}$ is either 69 or 70.

We will show that it is 69 by proving that $R_{69} < 1\times 10^{-6}$ using a refined estimation of R_n. We start with the following picture:



By comparison of areas, we have:

$$\frac{1}{k^4} \leq \int_{k-1/2}^{k+1/2} \frac{1}{x^4}\,dx$$

Summing for k from $n+1$ to infinity we get

$$\sum_{k=n+1}^{\infty} \frac{1}{k^4} \leq \sum_{k=n+1}^{\infty} \int_{k-1/2}^{k+1/2} \frac{1}{x^4}\,dx = \int_{n+1/2}^{\infty} \frac{1}{x^4}\,dx.$$

Thus: $$R_n \leq \frac{1}{3(n+\frac{1}{2})^3}.$$

In particular, we obtain $R_{69} \leq \dfrac{1}{3(69.5)^3} \simeq 9.92\times 10^{-7} < 1\times 10^{-6}$,

as announced.