Math 170, Fall 2008, Section 2
Assignment 2

Honor code: You should neither give nor receive help on this assignment.
Organize your work in a concise way and show every step of your argument. Write legibly.

Note: the total number of points is 23. Anything above 20 is extra credit.

1. (a) (1pt) Find the polar equation of the circle C_1 whose center has Cartesian coordinates $C = (1/2, 0)$ and passing through the origin $O = (0, 0)$. Show all your work.
   (b) (2pts) For a point P on the circle, the line of equation $x = 1$ intersects the line through O and P at a point Q. Let M be the point on this line that satisfies $|OM| = |PQ|$. Find the polar equation of the curve C_2 described by the point M when the point P describes the circle C_1. The curve C_2 is called a cissoid.
   (c) (1pt) Find the two points of intersection of C_1 and C_2, other than the pole. Give the Cartesian coordinates, as well as polar coordinates, of these two points A and B .
   (d) (1pt) What is the value of the area delimited by the curve C_2, the ray through O and A, and the ray through O and B? Represent this area on a picture.
   (e) (2pts) Find the area of the region that lies between the curves C_1 and C_2.

2. The Earth travels in an elliptical orbit for which the Sun is a focus. The closest point to the Sun on the orbit, called perihelion, is reached in early January, when the distance Earth-Sun is $r_- = 147$ million km. The farthest point from the Sun on the orbit, called aphelion, is reached in early July, when the distance Earth-Sun is $r_+ = 152$ million km.
   (a) (1pt) What is the eccentricity of the elliptical orbit?
   (b) (1pt) Give a polar equation for the trajectory of the Earth. Make sure to explain where you take the pole and the polar axis.
   (c) (2pts) Find the area swept out by the segment Earth-Sun during one year. Note that polar coordinates are not necessarily the most appropriate.
   (d) (1pt) Set up, but do not evaluate, an integral for the length of the elliptical orbit.

3. Consider the curve of polar equation

$$r = \frac{\sin^2(\theta)}{\cos(\theta)}, \qquad -\pi/2 \leq \theta \leq 0.$$

Yes, this is the lower half of the cissoid C_2.

(a) (1pt) Find an expression for the slope of the tangent line as a function of θ. Simplify your answer as much as you can.

(b) (1pt) Find the horizontal and vertical tangent lines.

(c) (2pts) What is the concavity of the curve? Start by simplifying the expression for d^2y/dx^2 as much as possible. Be cautious when performing this calculation, since it may become a lengthy one.

4. (a) (2pts) Find a Cartesian equation for the ellipse of foci $(1,-1)$ and $(1,1)$ and major axis of length 4.

(b) (2pts) Find the Cartesian equation of the hyperbola of foci $(-1,1)$ and $(1,1)$ and asymptote $y = -x + 1$.

5. Consider the curve of polar equation

$$r = \frac{\sin^2(\theta)}{\cos(\theta)}, \qquad 0 \leq \theta \leq \pi/4.$$

Yes, this is the part of the upper half of the cissoid C_2 that lies inside the circle C_1.

(a) (1pt) Give a Cartesian equation of the form $y = f(x)$ for this curve. *[Hint: Start by expressing the quantities y^2, x^3, and $x - 1$ as functions of θ, then obtain a relation between these quantities where θ does not appear.]*

(b) (1pt) The parametric equations

$$x(t) = \frac{t^2}{1+t^2}, \qquad y(t) = \frac{t^3}{1+t^2}, \qquad 0 \leq t \leq t_{max}.$$

also represent the given curve. Determine the appropriate value of t_{max}. Verify that $x(t)$ and $y(t)$ satisfy the Cartesian equation you have found in (a).

(c) (1pt) Using the latter parametric representation, set up, but do not evaluate, an integral for the surface area of the region obtained by rotating the given curve about the y-axis.

Math 170, Assignment 2 : Solutions

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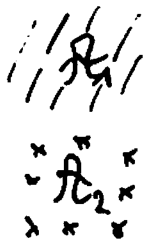

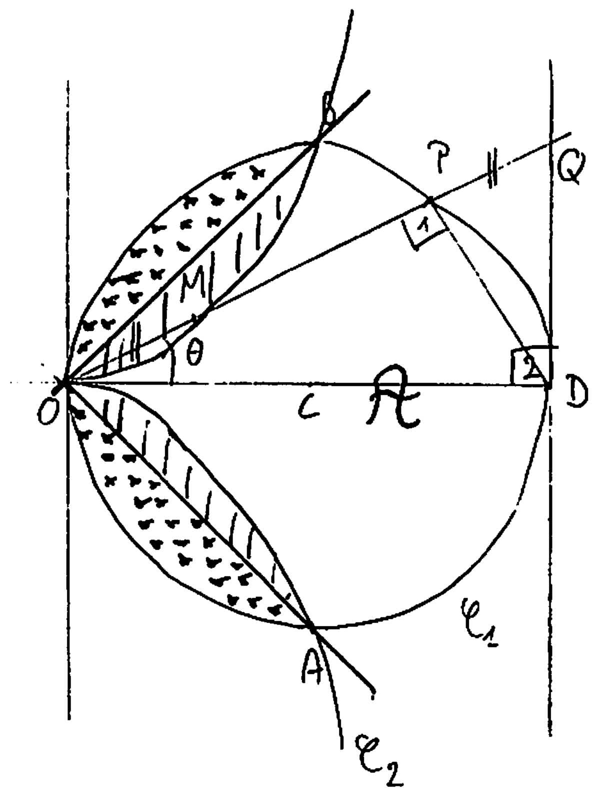

(a) By looking at the right triangle $\boxed{1}$, we see that

$$\cos\theta = \frac{|OP|}{|OD|}, \quad \text{with } |OD| = 1, \quad \text{and } |OP| = r$$

Hence, the equation of the circle $\mathcal{C}_1$ is: $\underline{r = \cos\theta}$, $\underline{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}$

(b) Looking at the right triangle $\boxed{2}$, we see that

$$\cos\theta = \frac{|OD|}{|OQ|}, \quad \text{thus } |OQ| = \frac{1}{\cos\theta}.$$

It follows that: $|OM| = |PQ| = |OQ| - |OP| = \frac{1}{\cos\theta} - \cos\theta$.

Thus the equation of the cissoid $\mathcal{C}_2$ is: $\underline{r = \frac{1}{\cos\theta} - \cos\theta}$

Note that it can also be written as:

$$r = \frac{1 - \cos^2\theta}{\cos\theta}, \quad \text{that is} \quad r = \frac{\sin^2\theta}{\cos\theta}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

(c) To find the points of intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$, we need to solve:

$$\cos\theta = \frac{\sin^2\theta}{\cos\theta},$$

[we need not worry about negative r's, as they are positive in this case]

that is $\cos^2\theta = \sin^2\theta$, so $\cos^2\theta = \frac{1}{2}$.

With $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, this gives $\cos\theta = \frac{1}{\sqrt{2}}$, and $\theta = \pm\frac{\pi}{4}$.

The points of intersection are therefore:

$A = \left(\frac{1}{\sqrt{2}}, -\frac{\pi}{4}\right)$ (polar), $A = \left(\frac{1}{2}, -\frac{1}{2}\right)$ (Cartesian)

$B = \left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$ (polar), $B = \left(\frac{1}{2}, \frac{1}{2}\right)$ (Cartesian)

(d) The area of the region delimited by $\mathcal{C}_2$ and the rays (OA) and (OB) is (see picture)

$$\mathcal{A}_1 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{1}{\cos\theta} - \cos\theta\right)^2 d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\pi/4} \left(\frac{1}{\cos^2\theta} - 2 + \cos^2\theta\right) d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sec^2\theta - 2 + \frac{\cos 2\theta + 1}{2}\right) d\theta$$

$$= \frac{1}{2}\left[\tan\theta - 2\theta + \frac{\sin(2\theta)}{4} + \frac{1}{2}\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \left[\tan\theta - \frac{3}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{4}}$$

$$= \left(1 - \frac{3}{2}\frac{\pi}{4} + \frac{1}{4}\right) - 0 = \frac{5}{4} - \frac{3}{8}\pi$$

$$\mathcal{A}_1 = \frac{5}{4} - \frac{3}{8}\pi$$

(Note: is it positive? that is: $10 - 3\pi > 0$? Yes)

(e) Let $\mathcal{A}$ be the area between the curves $\mathcal{C}_1$ and $\mathcal{C}_2$. We have:

$$\mathcal{A} = \underbrace{\frac{\pi}{4}}_{\text{area of the circle } \mathcal{C}_1} - \mathcal{A}_1 - \mathcal{A}_2$$

Notice that $\mathcal{A}_2$ is twice the area of the region delimited by $\mathcal{C}_2$ and the rays $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$. Thus,

$$\mathcal{A}_2 = 2 \times \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} (\cos\theta)^2 \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos(2\theta)+1}{2} \, d\theta = \left[\frac{\sin(2\theta)}{4} + \frac{\theta}{2} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(0 + \frac{\pi}{4}\right) - \left(\frac{1}{4} + \frac{\pi}{8}\right), \quad \text{that is} \quad \mathcal{A}_2 = \frac{\pi}{8} - \frac{1}{4}$$

(Note: it is positive)

Finally, we obtain

$$\mathcal{A} = \frac{\pi}{4} - \left(\frac{5}{4} - \frac{3}{8}\pi\right) - \left(\frac{\pi}{8} - \frac{1}{4}\right), \quad \mathcal{A} = \frac{\pi}{2} - 1$$

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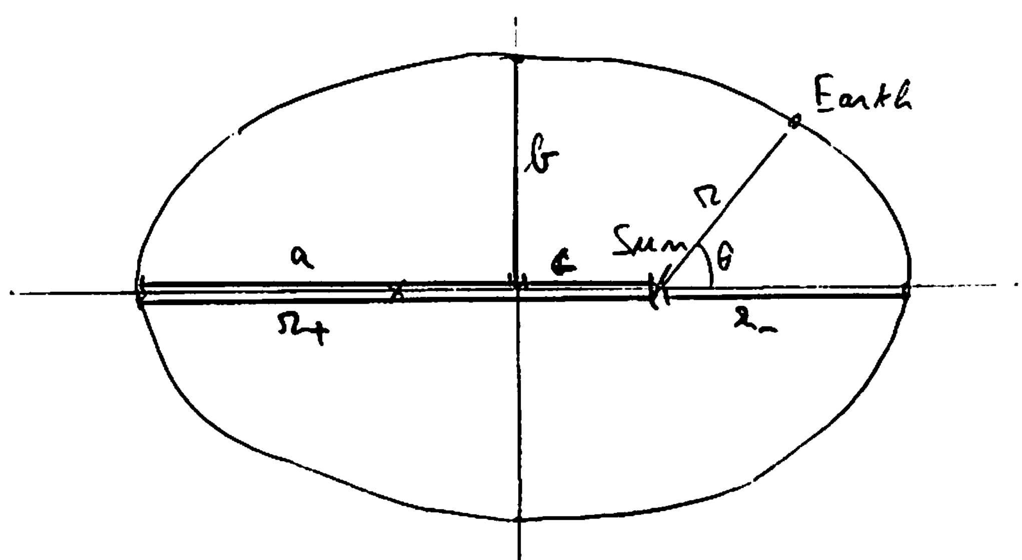


(a) We have (see picture):

$$r_+ + r_- = 2a$$
$$r_+ - r_- = 2c$$

Therefore, the eccentricity $e = \frac{c}{a}$ is equal to

$$e = \frac{r_+ - r_-}{r_+ + r_-} \qquad e = \frac{152-147}{152+147} = \frac{5}{299}, \quad e \approx 0.0167$$

(b) We chose the pole at the location of the Sun and the polar axis as the line perihelion – aphelion. The polar equation of the trajectory is then:

$$r = \frac{K}{1+e\cos\theta} \quad \text{for some constant } K \text{ to be determined.}$$

Note that $r_- = \frac{K}{1+e}$, so that $K = (1+e)\, r_-$.

We finally get: $\boxed{r = \frac{1+e}{1+e\cos\theta}\, r_-}$

(c) The area swept by the segment Earth-Sun during one year is simply the area of the region inside the ellipse. Note that the Cartesian equation of the ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{with } b^2 = a^2 - c^2 = a^2\left(1-\left(\frac{c}{a}\right)^2\right) = a^2(1-e^2).$$

For the upper half of the ellipse, we have: $y = b\sqrt{1-\left(\frac{x}{a}\right)^2}$.

The required area is given by:

$$A = 2\times\int_{-a}^{a} b\sqrt{1-\left(\frac{x}{a}\right)^2}\,dx = \frac{b}{a}\times\Big[\underbrace{2\times\int_{-a}^{a} a\sqrt{1-\left(\frac{x}{a}\right)^2}\,dx}_{\text{area of a circle of radius } a \text{ (that is: an ellipse for which } b=a)}\Big]$$

$$= \frac{b}{a}\times\pi a^2$$

$$\boxed{A = \pi a b}$$

Remark: the ellipse is deduced from the circle of radius a by a contraction by a factor $\frac{b}{a}$ in the y-direction. It is therefore intuitive that its area will be the area of the circle

(d) Based on the polar representation of the ellipse, for example, we can write its length as:

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta .$$

We calculate : $\frac{dr}{d\theta} = (1+e)\, r_- \times \frac{-e\sin\theta}{(1+e\cos\theta)^2}$.

It follows : $L = (1+e) r_- \int_0^{2\pi} \sqrt{\frac{(1+e\cos\theta)^2 + e^2\sin^2\theta}{(1+e\cos\theta)^4}}\, d\theta$

or $\boxed{L = (1+e) r_- \int_0^{2\pi} \frac{\sqrt{(1+e\cos\theta)^2 + e^2\sin^2\theta}}{(1+e\cos\theta)^2}\, d\theta}$

3/ (a) We have : $x = r\cos\theta = \sin^2\theta$

$y = r\sin\theta = \frac{\sin^3\theta}{\cos\theta}$

Thus: $\frac{dx}{d\theta} = 2\sin\theta\cos\theta$, $\frac{dy}{d\theta} = \frac{3\sin^2\theta\cos^2\theta + \sin^4\theta}{\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta}(3\cos^2\theta + \sin^2\theta)$

$\frac{dy}{d\theta} = \frac{\sin^2\theta}{\cos^2\theta}\left(2\cos^2\theta + 1\right)$

We get: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin^2\theta\,(2\cos^2\theta+1)}{2\sin\theta\cos^3\theta}$, $\boxed{\frac{dy}{dx} = \frac{\sin\theta(2\cos^2\theta+1)}{2\cos^3\theta}}$

(b) To find horizontal tangent lines, we must solve $\frac{dy}{dx} = 0$ for $\theta \in (-\frac{\pi}{2}, 0]$, which gives $\sin\theta = 0$, or $\theta = 0$

Horizontal tangent line at the pole. (since $\theta = 0$ implies $r = 0$)

To find vertical tangent lines, we must solve $2\cos^3\theta = 0$ for $\theta \in (-\frac{\pi}{2}, 0]$. There are no solutions, so

(c) First, let us calculate $\frac{d}{d\theta}\left(\frac{dy}{dx}\right)$. According to (a), we have

$$\frac{d}{d\theta}\left(2\frac{dy}{dx}\right) = \frac{1}{\cos^6\theta}\left[\left\{\cos\theta(2\cos^2\theta+1)+\sin\theta(-4\cos\theta\sin\theta)\right\}\cos^3\theta + 3\cos^2\theta\sin^2\theta(2\cos^2\theta+1)\right]$$

$$= \frac{1}{\cos^6\theta}\left[\cos^4\theta\left(2\cos^2\theta+1-4\sin^2\theta+6\sin^2\theta\right)+3\cos^2\theta\sin^2\theta\right]$$

$$= \frac{1}{\cos^4\theta}\left[\cos^2\theta\left(1+\underbrace{2\cos^2\theta+2\sin^2\theta}_{=2}\right)+3\sin^2\theta\right] = \frac{1}{\cos^4\theta}\left[\underbrace{3\cos^2\theta+3\sin^2\theta}_{=3}\right]$$

$$\frac{d}{d\theta}\left(\frac{dy}{dx}\right) = \frac{3}{2\cos^4\theta}.$$

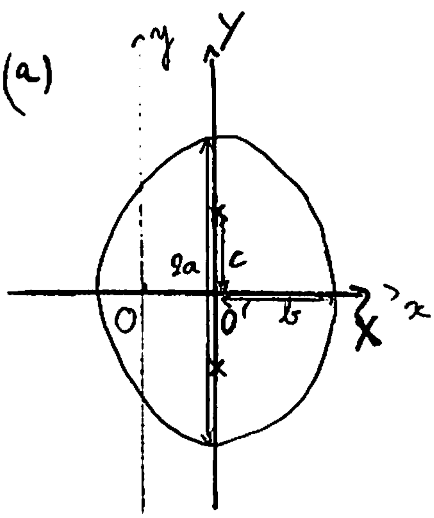
In view of $\frac{dx}{d\theta} = 2\sin\theta\cos\theta$, we then obtain:

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)/d\theta}{dx/d\theta} = \boxed{\frac{3}{4\sin\theta\cos^5\theta}}$$

Since $\cos\theta > 0$ and $\sin\theta < 0$ for $\theta \in \left(-\frac{\pi}{2}, 0\right)$, we derive that

$\frac{d^2y}{dx^2} < 0$, so that the curve is <u>concave down.</u>

4/ (a)



For the required ellipse, we are given:

$2a = 4$, that is $a = 2$

and $c = 1$, so that $b^2 = a^2 - c^2 = 3$.

In the Cartesian system of coordinates of origin O', the equation of the ellipse is:

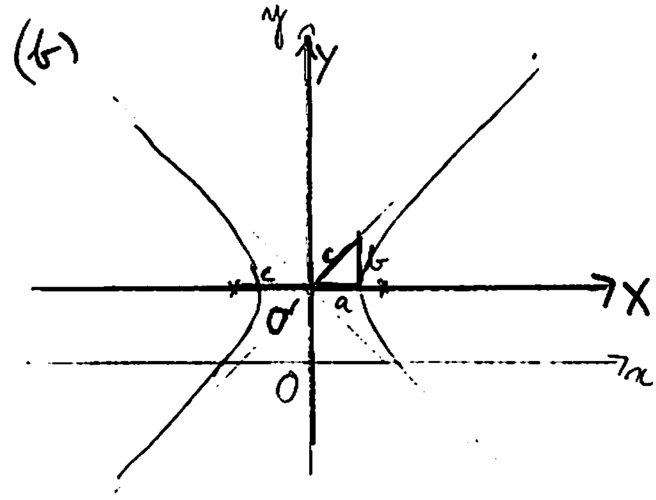
$$\frac{X^2}{b^2} + \frac{Y^2}{a^2} = 1, \text{ that is } \frac{X^2}{3} + \frac{Y^2}{4} = 1$$

Observe that $X = x - 1$ and that $Y = y$.

The equation of the ellipse is therefore:

$$\frac{(x-1)^2}{3} + \frac{y^2}{4} = 1.$$

(b)



For the required hyperbola, we are given:

$c = 1$ and $\frac{b}{a} = 1$.

The relation $c^2 = a^2 + b^2$ gives

$1 = 2a^2 = 2b^2$, so that $a^2 = b^2 = \frac{1}{2}$.

In the Cartesian system of coordinates of origin O', the equation of the ellipse is:

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1 \quad , \text{ that is: } 2X^2 - 2Y^2 = 1 .$$

Observe that $X = x$ and that $Y = y - 1$. The equation of the hyperbola becomes

$$\boxed{2x^2 - 2(y-1)^2 = 1}$$

5/ (a) With $r = \frac{\sin^2\theta}{\cos\theta}$, we have $x = r\cos\theta = \sin^2\theta$

$y = r\sin\theta = \frac{\sin^3\theta}{\cos\theta}$

Observe that $y^2 = \frac{\sin^6\theta}{\cos^2\theta}$, $x^3 = \sin^6\theta$, and $x - 1 = -\cos^2\theta$,

so that $x^3 + (x-1)y^2 = 0$.

We get $y^2 = \frac{x^3}{1-x}$ (note that $0 \leq x < 1$), and since y is positive in the region $0 \leq \theta \leq \frac{\pi}{4}$, we deduce: $\boxed{y = \sqrt{\frac{x^3}{1-x}}}$

(b) Let $x(t) = \frac{t^2}{1+t^2}$ and $y(t) = \frac{t^3}{1+t^2}$. We calculate

$1 - x = \frac{1+t^2-t^2}{1+t^2} = \frac{1}{1+t^2}$, thus $\sqrt{\frac{x^3}{1-x}} = \sqrt{\frac{t^6}{(1+t^2)^3}(1+t^2)} = \frac{t^3}{(1+t^2)} = y(t)$, Note that, with $0 \le t \le t_{max}$, the range of $x(t)$ is $\left[0, \frac{t_{max}^2}{1+t_{max}^2}\right]$, because $x(t)$ increases with t. (Note: do not differentiate to see this, simply write $x(t) = 1 - \frac{1}{1+t^2}$, and observe that $1+t^2$ is increasing, so $\frac{1}{1+t^2}$ is decreasing, so $-\frac{1}{1+t^2}$ is increasing).

But, for the curve under consideration, the range of x is $\left[0, \frac{1}{2}\right]$.

Thus, we must solve $\frac{t_{max}^2}{1+t_{max}^2} = \frac{1}{2}$, which gives $\underline{t_{max} = 1}$

(c) The surface area of the region obtained by rotating the curve about the y-axis is:

$$S = \int_0^1 2\pi\, x(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Note that
$$x'(t) = \frac{2t(1+t^2) - 2t \cdot t^2}{(1+t^2)^2} = \frac{2t}{(1+t^2)^2}$$
$$y'(t) = \frac{3t^2(1+t^2) - 2t \cdot t^3}{(1+t^2)^2} = \frac{3t^2 + t^4}{(1+t^2)^2}$$

We get: $$S = \int_0^1 2\pi \frac{t^2}{1+t^2} \sqrt{\frac{4t^2}{(1+t^2)^4} + \frac{t^4(3+t^2)^2}{(1+t^2)^4}}\, dt$$

$$\boxed{S = 2\pi \int_0^1 \frac{t^3}{(1+t^2)^3} \sqrt{4 + t^2(3+t^2)^2}\, dt}$$