Math 170, Fall 2008, Section 2
Assignment 1

Honor code: You should neither give nor receive help on this assignment.
Organize your work in a concise way and show every step of your argument. Write legibly.

1. (4pts) Solve the initial-value problem
$$e^x y\frac{dy}{dx} = e^{-y} + e^{-x-y} \quad \text{subject to} \quad y(0) = -1.$$
Give your result in the form $x = f(y)$ for some function f.

2. (4pts) A trough, whose ends are half-circles of radius $R = 50$cm, is filled with water. Find the hydrostatic force on one end of the trough. Give the literal expression first and verify the unit consistency (note: the unit of g is m.s^{-2} or N.kg^{-1}). Determine the numerical value of the force. How would it change if the radius was twice as large?

3. (4pts) Find the general solution of the differential equation
$$\frac{dy}{dx} = \frac{2x + 2xy^2}{x^4 + 2x^2 + 1}.$$

4. (8pts) We consider the function $f(x) = \frac{1}{2}x^2$.
   (a) Let L be the length of the curve $y = f(x)$ between the points $(1, 1/2)$ and $(2, 2)$. Write L as a definite integral, and find its value.
   [Hint: do not use substitution; instead, integrate by parts to obtain an expression depending on L on the right-hand side; then solve the resulting equation; at some point you will have to remark that $x^2 = (x^2 + 1) - 1$; you will also need to use $\int \frac{dx}{\sqrt{1+x^2}} = \operatorname{arcsinh}(x)$]
   (b) The previous curve is rotated around the x-axis. Write the area A of the resulting surface as a definite integral, and find its value.
   [Hint: again, start with an integration by parts to obtain an equation in A; you will need to isolate $\int_1^2 \sqrt{1+x^2}\,dx$, which you should have calculated in (a)]
   (c) Find the coordinates of the centroid of the region delimited by the curves
   $$y = 0, \quad y = f(x), \quad x = 1, \quad x = 2.$$
   (d) Find the volume of the region delimited by the surface of revolution defined in (b).
   [Hint: use the theorem of Pappus]

Math 170, Assignment 1 : Solution

1/ Write the differential equation as: $e^{x} y \frac{dy}{dx} = e^{-y}(1+e^{-x})$,

and separate the variables : $e^{y} y \, dy = e^{-x}(1+e^{-x})\, dx$.

Therefore, we have : $\int e^{y} y \, dy = \int (e^{-x} + e^{-2x})\, dx$.

The integral on the right-hand side is : $-e^{-x} - \frac{1}{2} e^{-2x} + C_1$.

As for the integral on the left-hand side, we use integration by parts:

$$\int e^{y} y \, dy = e^{y} y - \int e^{y} \, dy = e^{y} y - e^{y} + C_2 .$$

Thus, we have :
$$e^{y}(y-1) = -e^{-x} - \frac{1}{2} e^{-2x} + C_3$$
$$= -\frac{1}{2}\left[(2e^{-x} + e^{-2x} + 1) + C\right]$$
$$2e^{y}(1-y) = (e^{-x}+1)^2 + C$$

Using the initial condition $y(0) = -1$, we get:

$$2 \times e^{-1} \times 2 = 2^2 + C \quad , \quad \text{so that } C = 4(e^{-1}-1)$$

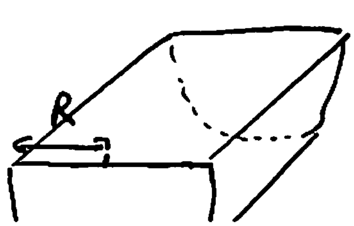
It follows that : $2e^{y}(1-y) - 4(e^{-1}-1) = (e^{-x}+1)^2$

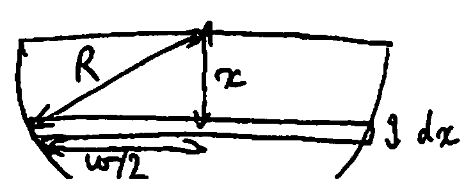
hence : $\sqrt{2e^{y}(1-y) + 4(1-e^{-1})} = e^{-x}+1$

We obtain:
$$e^{-x} = \sqrt{2e^{y}(1-y) + 4(1-e^{-1})} - 1$$
$$-x = \ln\left(\sqrt{2e^{y}(1-y) + 4(1-e^{-1})} - 1\right)$$

Finally: $\boxed{x = \ln\left(\frac{1}{\sqrt{2e^{y}(1-y) + 4(1-e^{-1})} - 1}\right)}$

2/





The pressure at any point of the rectangular surface represented on the figure is: $P = \rho g x$.

The area of this surface is, $dS = w\,dx$.

Note that $\frac{w}{2} = \sqrt{R^2 - x^2}$, so that $dS = 2\sqrt{R^2-x^2}\,dx$.

Hence, the force on the rectangular surface is:

$$dF = P \times dS = \rho g x \times 2\sqrt{R^2-x^2}\,dx = 2\rho g\ x\sqrt{R^2-x^2}\,dx.$$

To obtain the total force on the end of the trough, we integrate dF for x between 0 and R:

$$F = \int_{x=0}^{x=R} dF = 2\rho g \int_0^R x\sqrt{R^2-x^2}\,dx = 2\rho g\left[-\frac{1}{3}\left(R^2-x^2\right)^{\frac{3}{2}}\right]_0^R$$

$$= 2\rho g\left[0 - \left(-\frac{1}{3}(R^2)^{\frac{3}{2}}\right)\right], \quad \boxed{F = \frac{2}{3}\rho g R^3}$$

Units: $[\rho] = kg.m^{-3}$, $[g] = N.kg^{-1}$, $[R] = m$, so $\left[\frac{2}{3}\rho g R^3\right] = kg.m^{-3}.N.kg^{-1}.m^3 = N$ ✓

Numerical value: $\rho = 1000\,kg.m^{-3}$ (water), $g = 9.81\,N.kg^{-1}$, $R = 0.5\,m$.

$$\underline{F \approx 817.5\ N}$$

If R was doubled, then R^3 would be multiplied by 8, and so would F.

3/ The differential equation can be written as:

$$\frac{dy}{dx} = \frac{2x(1+y^2)}{(x^2+1)^2}.$$

Separate the variables: $\frac{dy}{1+y^2} = \frac{2x}{(x^2+1)^2}\,dx$,

and integrate: $\mathrm{Arctan}(y) = \int \frac{2x\,dx}{(x^2+1)} \underset{\substack{u = x^2+1 \\ du = 2x\,dx}}{=} \int \frac{du}{u^2}$

Thus: $\operatorname{Arctan}(y) = -\frac{1}{u} + C = -\frac{1}{x^2+1} + C$.

We conclude that : $\boxed{y = \tan\left(-\frac{1}{x^2+1} + C\right)}$

[Caution: the solution may not be defined for all x's]

4/ With $f(x) = \frac{1}{2}x^2$, we have $f'(x) = x$.

(a) The length of the curve $y = f(x)$ between the points $(1, 1/2)$ and $(2, 2)$ is:

$$L = \int_1^2 \sqrt{1+x^2}\, dx.$$

Following the indications provided by the hint, we write:

$$L = \int_1^2 1 \times \sqrt{1+x^2}\, dx = \left[x \times \sqrt{1+x^2}\right]_1^2 - \int_1^2 x \times \frac{1}{2} \times \frac{2x}{\sqrt{1+x^2}}\, dx$$

$$= 2\sqrt{5} - \sqrt{2} - \int_1^2 \frac{x^2}{\sqrt{1+x^2}}\, dx = 2\sqrt{5} - \sqrt{2} - \int_1^2 \frac{(x^2+1) - 1}{\sqrt{1+x^2}}\, dx$$

$$L = 2\sqrt{5} - \sqrt{2} - \int_1^2 \sqrt{1+x^2}\, dx + \int_1^2 \frac{1}{\sqrt{1+x^2}}\, dx$$

$$L = 2\sqrt{5} - \sqrt{2} - L + \left[\operatorname{Arcsinh}(x)\right]_1^2$$

Rearranging this equation, we obtain

$$2L = 2\sqrt{5} - \sqrt{2} + \operatorname{Arcsinh}(2) - \operatorname{Arcsinh}(1)$$

Finally : $L = \sqrt{5} - \frac{1}{2}\sqrt{2} + \frac{1}{2}\operatorname{Arcsinh}(2) - \frac{1}{2}\operatorname{Arcsinh}(1)$

(b) The area of the surface is given by

$A = \int_1^2 2\pi \times \frac{1}{2}x^2 \times \sqrt{1+x^2}\, dx$, that is $A = \pi \int_1^2 x^2\sqrt{1+x^2}\, dx$

Using integration by parts, we get:

$$\frac{A}{\pi} = \int_1^2 x \times x\sqrt{1+x^2}\, dx = \left[x \times \frac{1}{3}(1+x^2)^{\frac{3}{2}}\right]_1^2 - \int_1^2 1 \times \frac{1}{3}(1+x^2)^{3/2}\, dx$$

$$= \frac{2}{3} \times 5^{3/2} - \frac{1}{3} \times 2^{3/2} - \frac{1}{3}\int_1^2 (1+x^2)\sqrt{1+x^2}\, dx$$

$$\frac{A}{\pi} = \frac{2\times 5}{3}\sqrt{5} - \frac{2}{3}\sqrt{2} - \frac{1}{3}\underbrace{\int_1^2 \sqrt{1+x^2}\, dx}_{= L \text{ (see (a))}} - \frac{1}{3}\underbrace{\int_1^2 x^2\sqrt{1+x^2}\, dx}_{= \frac{A}{\pi}}$$

Rearranging this equation, we get

$$\frac{4}{3}\frac{A}{\pi} = \frac{2\times 5}{3}\sqrt{5} - \frac{2}{3}\sqrt{2} - \frac{1}{3}\left(\sqrt{5} - \frac{1}{2}\sqrt{2} + \frac{1}{2}\operatorname{Arcsinh}(2) - \frac{1}{2}\operatorname{Arcsinh}(1)\right)$$

$$= 3\sqrt{5} - \frac{1}{2}\sqrt{2} - \frac{1}{3}\times\frac{1}{2}\operatorname{Arcsinh}(2) + \frac{1}{3}\times\frac{1}{2}\times \operatorname{Arcsinh}(1)$$

Finally: $\boxed{A = \frac{9}{4}\pi\sqrt{5} - \frac{3}{8}\pi\sqrt{2} - \frac{\pi}{8}\operatorname{Arcsinh}(2) + \frac{\pi}{8}\operatorname{Arcsinh}(1).}$

(c) The coordinates of the centroid are given by

$$x_c = \frac{1}{S}\int_1^2 x \times \frac{1}{2}x^2\, dx, \quad y_c = \frac{1}{S}\int_1^2 \frac{1}{2}\left(\frac{1}{2}x^2\right)^2 dx,$$

where $S = \int_1^2 \frac{1}{2}x^2\, dx = \left[\frac{1}{6}x^3\right]_1^2 = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}.$

thus: $x_c = \frac{6}{7}\times\left[\frac{x^4}{8}\right]_1^2 = \frac{3}{7\times 4}[16-1] = \frac{3\times 15}{7\times 4}$

$y_c = \frac{6}{7}\times\left[\frac{x^5}{8\times 5}\right]_1^2 = \frac{3}{7\times 4\times 5}[32-1] = \frac{3\times 31}{7\times 4\times 5}$

$$\boxed{x_c = \frac{45}{28}} \quad \boxed{y_c = \frac{93}{140}}$$

(d) The centroid is at distance y_c from the axis of rotation, so it travels a distance $D = 2\pi \times y_c$, that is: $D = \frac{93}{70}\pi$

By the theorem of Pappus, the volume V of the region under consideration is:

$V = S \times D$, thus $V = \frac{7}{6}\times\frac{93}{70}\pi$, that is $\boxed{V = \frac{31}{20}\pi}$