Mathematical delights

Simon Foucart

Some time in ’06

Abstract

I have collected, from various sources, a sample of recreational mathematical problems. Some are dead easy, some more intricate, some are *exercices de style*,... Most of them [and I may further expand the list] are familiar to a trained mathematician. I hope nonetheless that they can provide some sort of entertainment, even though they have been selected on the basis of my biased taste. Be aware that some solutions are my own and may not be the best. Note also that I did not edit any picture but that the reader is strongly advised to draw his/her own whenever necessary.

1. A small ship, not supporting more than 100kg, is the only available option for three fugitives, weighting 40kg, 50kg and 80kg, to cross a river. How can they do it?

2. We encircle a basket-ball with a piece of string which is one centimeter longer than its perimeter and we do the same with a golf-ball. In which situation is the distance from the ball to the string the largest?

3. Equipped with two pairs of surgical gloves, how many operations can a surgeon perform? The only requirement is sterility: no part that has been in contact with someone can be in contact with someone else.

4. This game is played on a round table, with a stack of (identical) coins. The first player puts a coin on the table, then the second player puts another coin on the table, not overlapping the first one, then it is the turn of the first player again, who applies the same rule, and so on. The loser is the one who is unable to add an extra coin on the table. Is it better for me to play first or second?

5. Me and my spouse are invited to a party, along with four other couples. Some of us shake hands at the beginning at the evening, but of course a couple don't shake hands between themselves. Later in the evening, I enquire about how many hands each guest has shaken and I am surprised not to get two similar answers. How many hands have my spouse shaken?

6. We are piling up some playing cards above the edge of a table, trying to get as far away as possible from this edge. How long can the stack of cards be?

7. Using the numbers 1, 5, 6 and 7 once and only once, the four standard operations $+$, $-$, $\times$ and $/$ and any parenthesizing you like, obtain the number 21.

8. Suppose that the real numbers a, b and c satisfy

$$\cos a + \cos b + \cos c = 0, \qquad \sin a + \sin b + \sin c = 0.$$

Prove that

$$\cos 2a + \cos 2b + \cos 2c = 0, \qquad \sin 2a + \sin 2b + \sin 2c = 0.$$

9. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $|f'(t)| = 1$ for all $t \in [a, b]$. Prove that either $f'(t) = 1$ for all $t \in [a, b]$ or $f'(t) = 1$ for all $t \in [a, b]$.

10. Prove that $\displaystyle\sum_{k=1}^{n} k^3 = \left[\sum_{k=1}^{n} k\right]^2$, without calling upon mathematical induction. By the way, how does one justify mathematical induction?

11. Consider a $2^n \times 2^n$ checkerboard from which an arbitrary square has been removed. Can it be paved with polyominos, that is L-shaped tiles covering three squares each?

12. Give the nicest (surely the shortest) proof of Pythagoras theorem.

13. Consider a pyramid P made up with four equilateral triangles on a square basis. Consider also a tetrahedron T whose faces are equal to the equilateral triangles of P. One puts P and T together by sticking a face of T to a triangular face of P. How many faces does the resulting polyhedron have?

14. Consider a rectangle which is paved by some smaller rectangles. For each of these small rectangles, at least one side is an integer. Is it true that at least one side of the large rectangle is an integer?

15. Two pencil marks are made at random on a stick, which is then broken into k pieces at random. What is the probability that the two marks are on the same piece of stick?

16. Find the bijections f of $[0,1]$ satisfying $f(2x - f(x)) = x$ for all $x \in [0,1]$.

17. Prove that there is an uncountable family $(S_a)_{a\in A}$ of infinite subsets of $\mathbb{N}$ such that $S_a \cap S_b$ is finite for $a \neq b$.

18. For $a_1, a_2, \ldots, a_n > 0$ and for $t_1, t_2, \ldots, t_n > 0$ satisfying $t_1 + t_2 + \cdots + t_n = 1$, establish that

$$a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n} \leq t_1 a_1 + t_2 a_2 + \cdots + t_n a_n,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

19. Let M be a $n \times n$ nilpotent matrix. Prove that $M^n = 0$.

20. Let X, Y be two sets for which there exist injections $f : X \to Y$ and $g : Y \to X$. Prove that there exists a bijection from X onto Y.

21. Let f be a map from an inner product space X into itself which satisfies

(1) $$f(ix) = i\,f(x), \qquad x \in X,$$

(2) $$\|f(x) - f(y)\| = \|x - y\|, \qquad x, y \in X.$$

Prove that f is linear.

22. Let K be a compact metric space and let $f : K \longrightarrow K$ be a map satisfying

$$d(f(x), f(y)) \geq d(x,y), \qquad x, y \in K.$$

Prove that f is an isometry from K onto K.

23. Let X be a complete metric space with a countable base for the topology. Consider a continuous map $f : X \to X$ which is topologically transitive, i.e. $\bigcup_{n\geq 0} f^{-n}(U)$ is dense in X for any non-empty open subset U of X. Prove that the points $x \in X$ such that the orbit $\{f^n(x),\ n \geq 0\}$ is dense in X constitute a dense subset of X.

24. Consider a continuous function $f : [0,1] \to [0,1]$. For $x \in [0,1]$, prove that

$$[f^n(x) \text{ converges}] \iff \left[f^{n+1}(x) - f^n(x) \longrightarrow 0\right].$$

25. Prove that the distance between two consecutive prime numbers is unbounded. Prove also that the distance between two consecutive square-free numbers is unbounded.

26. Let α be an algebraic number with degree $n > 1$. Prove that for all $p/q \in \mathbb{Q}$ ($q > 0$), one has
$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^n} \quad \text{for some constant } c = c(\alpha) > 0.$$

27. Let $P(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$ be a polynomial with integer coefficients. Prove that, if a_0, a_n and $P(1)$ are odd, then P has no rational zero.

28. Is it possible for two polynomials to achieve simultaneously one value and to also achieve simultaneously another value?

29. Let A and B be two square matrices. Suppose that A and B commute and that A is a normal matrix (i.e. $AA^* = A^*A$). Prove that A^* and B commute. Deduce that the product of two commuting normal matrices is also normal.

30. Let x, y be two elements of a unital associative ring. Prove that $1 - xy$ is invertible if and only if $1 - yx$ is. Prove also that x is invertible as soon as it admits a unique right inverse.

---

1. The lightest two persons cross the river and one of them comes back. He/she stays on the riverbank while the heaviest person makes his/her journey. Then the light person already at destination brings back the boat and can pick up the remaining light person.

2. These distances are the same! And it would also work with the earth, say. Indeed, if r is the radius (in centimeters) of the given ball, the length of the piece of string is $2\pi r + 1 = 2\pi(r + 1/2\pi)$, so that the distance from the ball to the string is always $1/2\pi$.

3. Three: he starts off by wearing the two pairs of glove, one on top of the other, and carries out the first operation. Then he removes the outside pair of gloves, leaving them inside-out. He carries out the second operation and then puts the first pair of gloves on again. The soiled sides of the gloves are in contact with each other, but the external surface is clean. Thus he can perform a third operation.

4. To play first. The key word here is symmetry. I put the first coin right in the center. Then the tabletop is symmetrical and wherever my opponent puts his/her coin, I put mine at

the symmetrical location, hence keeping the tabletop symmetrical at each step. Therefore I cannot lose.

5. The answer must be four, as the problem would be the same if one asked how many people my spouse ignored, and the sum of these two numbers equals eight. Here is a proper justification. There is a guest who shook eight hands, i.e. has shaken hands with everybody except his/her spouse, hence the guest who ignored everybody must be his/her spouse. Let us discard this couple, as everybody else has shaken hands with one and only one of them. The problem is just the same, except for the total number of guests (eight instead of ten) and the solution which decreases by one. Do this thrice. With only two couples, it is now rather clear that my spouse has shaken one hand. The required solution is consequently four.

6. One can get as far away from the edge as desired. Consider a stack of n cards, each of them of length $2L$, say. Measuring from the edge of the table, let x_k be the position of the center of mass of the k upper cards and let y_k be the position of the center of mass of the k-th card from the top. Obviously, one has $x_{k+1} = \dfrac{kx_k + y_{k+1}}{k+1}$. The whole system is in equilibrium when no system of k upper cards, $k \in [\![1, n]\!]$, is off balance, hence at the limit of the equilibrium, one has

$$x_k = y_{k+1} + L, \quad \text{i.e., expressing } y_{k+1} \text{ in terms of } x_k \text{ and } x_{k+1}, \quad (k+1)(x_k - x_{k+1}) = L.$$

In view of $x_n = 0$, it follows, , that

$$y_1 = x_1 = (x_1 - x_2) + (x_2 - x_3) + \cdots + (x_{n-1} - x_n) = \frac{L}{2} + \frac{L}{3} + \cdots + \frac{L}{n}.$$

The position of the top card grows without bound, although rather slowly.

7.

$$21 = \frac{6}{1 - \frac{5}{7}}.$$

8. What we need to show is that

$$e^{i2a} + e^{i2b} + e^{i2c} = 0,$$

knowing that $e^{ia} + e^{ib} + e^{ic} = 0$. In order to conclude, we just have to write

$$0 = (e^{ia} + e^{ib} + e^{ic})^2 = e^{i2a} + e^{i2b} + e^{i2c} + 2e^{i(a+b+c)}(\underbrace{e^{-ia} + e^{-ib} + e^{-ic}}_{=0}).$$

9. The result would have been obvious if we had assumed that f was continuously differentiable, but we hadn't. Still, if we take for granted the fact that a differentiable function obeys the intermediate value theorem, there is no difficulty either... Let us not use this fact. We select a point $t \in [a, b]$ with e.g. $f'(t) = 1$. Then we define a continuous function $f_t : [a, b] \to \mathbb{R}$ by

$$f_t(u) := \begin{cases} \dfrac{f(u) - f(t)}{u - t}, & u \neq t, \\ 1, & u = t. \end{cases}$$

By the mean value theorem, one has $|f_t(u)| = 1$ for each $u \in [a, b]$, and the continuity of f_t yields $f_t(u) = 1$ for each $u \in [a, b]$. In turn, we obtain $f'(u) = 1$ for each $u \in [a, b]$.

10. Consider a square of side $\displaystyle\sum_{k=1}^{n} k$, which is decomposed into a square of side 1 and $(n-1)$ L-shaped polygons of width k for $k \in [\![2, n]\!]$. One needs to know that $\displaystyle\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ to derive that the area of the L-shaped polygon of width k is $2 \times k \times \dfrac{k(k+1)}{2} - k^2 = k^3$. Expressing the area of the whole square yields the result. As for the principle of induction – saying that, if a proposition P_{n+1} holds as soon as P_n does, $n \in \mathbb{N}$, and if P_1 holds, then P_n holds for any $n \in \mathbb{N}$ – it follows from the existence of a minimal element to any non-empty subset of $\mathbb{N}$. Indeed, one needs to prove that $S := \{n \in \mathbb{N} : P_n \text{ does not hold}\}$ is empty. If not, it would have a minimal element m, say, with $m \geq 2$. Then by definition, $m - 1 \notin S$, i.e. P_{m-1} holds, so that P_m also holds, i.e. $m \notin S$. This is a contradiction.

11. Note that the number of squares to be covered is $4^n - 1 = (4-1)(4^{n-1} + \cdots + 1)$, which is a multiple of three. On this ground, the given area *might* be paved with these particular tiles, and indeed *can* be. This can be seen by induction. A 2×2 checkerboard minus one square has exactly the same shape as a polyomino, hence the result for $n = 1$. Suppose now that the result holds for n and consider a $2^{n+1} \times 2^{n+1}$ checkerboard minus a certain square. One divides this board into four $2^n \times 2^n$ checkerboards, call them A, B, C and D. The missing square belongs to D, say, and the induction hypothesis yields a paving of D minus this square. As for the remaining area, consider a polyomino which covers one square in each of A, B and C. Each of these checkerboards minus the corresponding square can be paved according to the induction hypothesis. Putting everything together, we obtain a paving of the original checkerboard minus the given square. This establishes that the result holds for $n + 1$, and completes the induction.

12. In a rectangular triangle, the height on the hypotenuse c gives rise to two subtriangles, of hypotenuses a and b, which are scaled versions the original one. One has to be convinced that the area of a rectangular triangle is $\mathrm{cst} \times$ hypotenuse^2 [cst being unchanged upon rescaling] to state that the area of the original triangle is the sum of the areas of the subtriangles and to simplify by cst to conclude that $c^2 = a^2 + b^2$.

13. It is very tempting to answer that it has $5 + 4 - 2 = 7$ faces. But by doing so, one overlooks the fact that a face of P and a face of T may combine into a single face. And this is actually the case here. Indeed, join two pyramids by a common edge so that their bases lie on a same – horizontal, say – plane. The shape 'between' the pyramids is seen to be the regular tetrahedron T [the distance between the two pyramid tops is clearly equal to the length of the other edges]. Thus, removing one pyramid, the desired polyhedron remains and we derive that it possesses only 5 faces.

14. The answer is yes, and the easiest way to see this is to notice that at least one side of a rectangle $[a, b] \times [c, d]$ is an integer if and only if

$$\int_{[a,b]\times[c,d]} \exp\left(i2\pi(x+y)\right) dx\, dy = \int_a^b \exp\left(i2\pi x\right) dx \cdot \int_c^d \exp\left(i2\pi y\right) dy = 0.$$

Then, for the large rectangle R paved by the small rectangles $R_1, \ldots, R_n$, one has

$$\int_R \exp\left(i2\pi(x+y)\right) dx\, dy = \sum_{k=1}^{n} \int_{R_k} \exp\left(i2\pi(x+y)\right) dx\, dy = 0.$$

This means that at least one side of R is an integer.

15. The $(k-1)$ break-points and the 2 mark-points constitutes altogether $(k+1)$ points. There are $\binom{k+1}{2}$ ways of choosing the 2 mark-points out of these $(k+1)$ points, and the number of ways for them to appear on the same piece of stick, i.e. to appear consecutively, is k. Thus the required probability is $k/\binom{k+1}{2} = 2/(k+1)$.
As a result, we obtain the identity

$$\int_{x_i \geq 0,\ x_1+\cdots+x_k=1} \left(x_1^2 + \cdots + x_k^2\right) dx_1 \ldots dx_k = \frac{2}{k+1}.$$

Indeed, if x_j is the length of the j-th piece, the probability is

$$\int_{x_i\geq 0,\ x_1+\cdots+x_k=1} \sum_{j=1}^{k} P(\text{marks 1 and 2 on } j\text{-th piece})\, dx_1 \dots dx_k =$$

$$\int_{x_i\geq 0,\ x_1+\cdots+x_k=1} \sum_{j=1}^{k} \underbrace{P(\text{mark 1 on } j\text{-th piece})}_{=x_j} \times \underbrace{P(\text{mark 2 on } j\text{-th piece})}_{=x_j}\, dx_1 \dots dx_k.$$

16. For $x \in [0,1]$, define a sequence $(u_n)_{n\in\mathbb{N}}$ by

$$u_1 = x, \qquad u_{n+1} = 2u_n - f(u_n),\ n \in \mathbb{N}.$$

Note that the sequence $(u_n)_{n\in\mathbb{N}}$ must lie in $[0,1]$, the domain of f, and that

$$u_{n+2} = 2u_{n+1} - f(u_{n+1}) = 2u_{n+1} - u_n.$$

Since the sequences $(1)_{n\in\mathbb{N}}$ and $(n)_{n\in\mathbb{N}}$ obviously belong to the two-dimensional vector space of solutions of this difference equation, we can write $u_n = \lambda + \mu n$ for some constants λ, μ. Since u_n must remain bounded, the constant μ must equal zero and the sequence $(u_n)_{n\in\mathbb{N}}$ must consequently be stationary. The equality $u_2 = u_1$ means that $f(x) = x$. Thus, if such a map exists, it must be the identity. Conversely, the map $f = \mathrm{id}_{[0,1]}$ is clearly a solution of the problem.

17. Let $\mathbb{Q} =: \{r_n, n \in \mathbb{N}\}$. For $a \in \mathbb{R} \setminus \mathbb{Q}$, we consider a sequence $(r_{n_k})_{k\in\mathbb{N}}$ converging to a, and we define the set S_a to be $S_a := \{n_k, k \in \mathbb{N}\}$. This does the trick.

18. Set $G := a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}$ and $A := t_1 a_1 + t_2 a_2 + \cdots + t_n a_n$. Without loss of generality, one may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. Since $a_1 \leq G \leq a_n$, we consider the integer $k \in [\![1, n-1]\!]$ such that $a_k \leq G \leq a_{k+1}$. Then one can write

$$\sum_{i=1}^{k} t_i \int_{a_i}^{G} \left(\frac{1}{x} - \frac{1}{G}\right) dx + \sum_{i=k+1}^{n} t_i \int_{G}^{a_i} \left(\frac{1}{G} - \frac{1}{x}\right) dx \geq 0. \tag{3}$$

It follows that

$$\sum_{i=1}^{n} t_i \int_{G}^{a_i} \frac{dx}{G} \geq \sum_{i=1}^{n} t_i \int_{G}^{a_i} \frac{dx}{x},$$

that is to say

$$\frac{A}{G} - 1 \geq \sum_{i=1}^{n} t_i \ln \frac{a_i}{G} = 0.$$

This is clearly what was required. Note that equality here implies equality in (3), hence $a_1 = \cdots = a_k = G$ and $a_{k+1} = \cdots = a_n = G$.

19. Notice first the chain of inclusions

$$\ker M \subseteq \ker M^2 \subseteq \cdots \subseteq \ker M^k \subseteq \ker M^{k+1} \subseteq \cdots .$$

Now suppose that $M^n \neq 0$, so that the sequence $(\dim \ker M, \dots, \dim \ker M^n)$ is non-decreasing and has values in $[\![1, n-1]\!]$. Thus there exists $p \in [\![1, n-1]\!]$ such that $\ker M^{p+1} = \ker M^p$. Hence, for all $q \geq p$, one has $\ker M^q = \ker M^p$, as

$$M^q x = 0 \Rightarrow M^{q-p-1}x \in \ker M^{p+1} = \ker M^p \Rightarrow M^{q-1}x = 0 \Rightarrow \cdots \Rightarrow M^p x = 0.$$

But $M^q = 0$ for some integer $q \geq n$, i.e. $\ker M^q = \mathbb{K}^n$. This yields $\ker M^p = \mathbb{K}^n$, i.e. $M^p = 0$, a contradiction.
There is another elegant solution using Hamilton–Cayley theorem. Suppose that

$$p := \min \left\{ k \in \mathbb{N} : M^k = 0 \right\} \geq n+1.$$

Let us write the characteristic polynomial of M in the form

$$P_M(\lambda) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_m \lambda^m, \quad a_m \neq 0.$$

Multiplying $P_M(M) = 0$ by M^{p-1-m} (justified because $p - 1 - m \geq 0$), we obtain $a_m M^{p-1} = 0$, hence $M^{p-1} = 0$, which contradicts the definition of p.

20. Define a map $\phi : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\phi(A) := X \setminus g[Y \setminus f(A)]$, $A \in \mathcal{P}(X)$. Assume for the moment that there exists $B \in \mathcal{P}(X)$ satisfying $\phi(B) = B$, i.e. $g[Y \setminus f(B)] = X \setminus B$. Since f induces a bijection from B onto $f(B)$ and g induces a bijection from $Y \setminus f(B)$ onto $X \setminus B$, one easily constructs a bijection from $X = B \cup [X \setminus B]$ onto $Y = f(B) \cup [Y \setminus f(B)]$. It remains to establish the existence of the set B. Since ϕ preserves inclusion and since $\phi(X) \subseteq X$, one has

$$B_0 := X \supseteq B_1 := \phi(X) \supseteq B_2 := \phi^2(X) \supseteq \cdots \supseteq B_n := \phi^n(X) \supseteq \cdots ,$$

and we set $B := \bigcap_{n\geq 0} B_n$, which is seen to be suitable as follow

$$\begin{aligned}\phi(B) = X \setminus g[Y \setminus f(\bigcap_{n\geq 0} B_n)] &\underset{(f \text{ inj.})}{=} X \setminus g[Y \setminus \bigcap_{n\geq 0} f(B_n)] = X \setminus g[\bigcup_{n\geq 0} (Y \setminus f(B_n))] \\ &= X \setminus \bigcup_{n\geq 0} g[Y \setminus f(B_n)] = \bigcap_{n\geq 0} (X \setminus g[Y \setminus f(B_n)]) = \bigcap_{n\geq 0} \phi(B_n) = \bigcap_{n\geq 1} B_n = B.\end{aligned}$$

21. We use (2) and the identities $f(ix) = if(x)$ [(1)], $f(-x) = -f(x)$ [(1) used twice] and $f(-ix) = -if(x)$ [(1) used thrice] in the polarization formula to obtain

$$\begin{aligned}
&4\langle f(x), f(y)\rangle \\
&\quad = \|f(x)+f(y)\|^2 - \|f(x)-f(y)\|^2 + i\|f(x)+if(y)\|^2 - i\|f(x)-if(y)\|^2 \\
&\quad = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 4\langle x, y\rangle,
\end{aligned}$$

so that f preserves the inner product. We now need to show that $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$ for any scalars λ, μ and any vectors x, y, so we consider the norm of the difference and prove that it vanishes as follow:

$$\begin{aligned}
&\|f(\lambda x+\mu y) - \lambda f(x) - \mu f(y)\|^2 \\
&\quad = \langle f(\lambda x+\mu y) - \lambda f(x) - \mu f(y), f(\lambda x+\mu y) - \lambda f(x) - \mu f(y)\rangle \\
&\quad = \langle f(\lambda x+\mu y), f(\lambda x+\mu y)\rangle - \overline{\lambda}\langle f(\lambda x+\mu y), f(x)\rangle - \overline{\mu}\langle f(\lambda x+\mu y), f(y)\rangle \cdots \\
&\quad = \langle \lambda x+\mu y, \lambda x+\mu y\rangle - \overline{\lambda}\langle \lambda x+\mu y, x\rangle - \overline{\mu}\langle \lambda x+\mu y, y\rangle \cdots \\
&\quad = \|(\lambda x+\mu y) - \lambda x - \mu y\|^2 = 0.
\end{aligned}$$

22. There are two things to do here: firstly, to show that f preserves the distances, secondly, to show that f is surjective.
For the first part, let us consider $x, y \in K$. As a preliminary, we establish that

$$\forall \varepsilon > 0, \quad \exists n \in \mathbb{N} \text{ such that } d(f^n(x), x) \le \varepsilon \text{ and } d(f^n(y), y) \le \varepsilon. \tag{4}$$

Indeed, due to the compactness of $K \times K$, the sequence $(f^n(x), f^n(y))$ admits a convergent subsequence. Given $\varepsilon > 0$, this accounts for the existence of $m, n \in \mathbb{N}$ such that

$$d(f^{m+n}(x), f^m(x)) \le \varepsilon \quad \text{and} \quad d(f^{m+n}(y), f^m(y)) \le \varepsilon.$$

The claim then follows from $d(f^n(x), x) \le d(f^{m+n}(x), f^m(x))$ and $d(f^n(y), y) \le d(f^{m+n}(y), f^m(y))$.
We then derive

$$\begin{aligned}
d(f(x), f(y)) &\le d(f^n(x), f^n(y)) \\
&\le d(f^n(x), x) + d(x, y) + d(y, f^n(y)) \le 2\varepsilon + d(x, y).
\end{aligned}$$

This being true for any $\varepsilon > 0$, we obtain $d(f(x), f(y)) \le d(x, y)$ and consequently $d(f(x), f(y)) = d(x, y)$.
Now for the second part, we notice that (4) establishes the density of $f(K)$ in K. Besides, as

the continuous image of a compact set, $f(K)$ is also compact, hence closed. We conclude that $f(K) = K$.

23. Let $\{U_i,\ i \geq 0\}$ be a countable base for the topology. Since the $\bigcup_{n\geq 0} f^{-n}(U_i)$'s are dense open subsets of X, Baire theorem yields that $Y := \bigcap_{i\geq 0} \bigcup_{n\geq 0} f^{-n}(U_i)$ is dense in X. One claims that the orbit $\{f^n(y),\ n \geq 0\}$ is dense in X for any $y \in Y$. Indeed, given a non-empty open set V, there exists $j \geq 0$ such that $U_j \subseteq V$, thus $y \in \bigcup_{n\geq 0} f^{-n}(U_j) \subseteq \bigcup_{n\geq 0} f^{-n}(V)$, which means that $f^n(y) \in V$ for some $n \geq 0$, hence the claim.

24. The direct implication is obvious. We now assume that $f^{n+1}(x) - f^n(x) \underset{n\infty}{\longrightarrow} 0$. Let $\mathcal{A}$ denote the set of limit points of the sequence $(f^n(x))_{n\in\mathbb{N}}$, which is non-empty by compactness. We are going to establish that $\mathcal{A}$ is a singleton. But first we remark that $f(a) = a$ for any $a \in \mathcal{A}$. Indeed, there is a subsequence $(f^{n_k}(x))_{k\in\mathbb{N}}$ such that

$$f^{n_k}(x) \underset{k\infty}{\longrightarrow} a, \quad \text{hence } f^{n_k+1}(x) \underset{k\infty}{\longrightarrow} f(a) \ \text{ by continuity of } f,$$

and the assertion $f(a) = a$ follows from

$$|f(a) - a| \leq |f(a) - f^{n_k+1}(x)| + |f^{n_k+1}(x) - f^{n_k}(x)| + |f^{n_k}(x) - a| \underset{k\infty}{\longrightarrow} 0.$$

Let us now consider $a, b \in \mathcal{A}$ with $a < b$. We claim that any $c \in (a, b)$ is also in $\mathcal{A}$. For this purpose, let us fix $\varepsilon > 0$ (for which $(c - 2\varepsilon, c + 2\varepsilon) \subseteq (a, b)$ without loss of generality). We pick $n_2 \geq n_1 \geq N$ satisfying

$$|f^{n+1}(x) - f^n(x)| < 2\varepsilon \quad \text{for } n \geq N, \qquad |f^{n_1}(x) - a| < \varepsilon, \qquad |f^{n_2}(x) - b| < \varepsilon.$$

We set $k := \max\{n \in [\![n_1, n_2]\!] : f^n(x) \leq c - \varepsilon\} < n_2$, so that

$$f^{k+1}(x) > c - \varepsilon \quad \text{and} \quad f^{k+1}(x) = [f^{k+1}(x) - f^k(x)] + f^k(x) < 2\varepsilon + c - \varepsilon = c + \varepsilon.$$

We have thus justified our claim. We now consider a given $c \in (a, b)$. Since it is a limit point of $(f^n(x))_{n\in\mathbb{N}}$, there exists $m \in \mathbb{N}$ such that $f^m(x) \in (a, b)$. But then $f^m(x) \in \mathcal{A}$, so that $f^n(x) = f^m(x)$ for any $n \geq m$, which contradicts our hypothesis. Consequently, the set $\mathcal{A}$ is a singleton, say $\mathcal{A} = \{a\}$. We are now ready to complete the proof. We assume that $f^n(x)$ does not converge to a. In this case, for some $\varepsilon > 0$, we can construct a subsequence $(f^{n_k}(x))$ satisfying $|f^{n_k}(x) - a| \geq \varepsilon$ for all $k \in \mathbb{N}$. By compactness, this subsequence admits a limit point b, which is also a limit point of $(f^n(x))_{n\in\mathbb{N}}$. But the two conditions $|b - a| \geq \varepsilon$ and $\mathcal{A} = \{a\}$ are contradictory, hence the conclusion.

25. Given an integer n, remark first that $2, 3, \ldots, n$ divide into $n! + 2, n! + 3, \ldots, n! + n$ respectively, thus separating two prime numbers by a distance at least n. This is of course unbounded as n grows to infinity.
Let now $p_1, \ldots, p_n$ denote the first n prime numbers. By the Chinese remainder theorem, the system of congruence

$$x \equiv a_i \pmod{p_i^2}, \qquad i \in [\![1, n]\!],$$

is solvable for any $a_1, \ldots, a_n$, since the p_i^2's are pairwise relatively prime. In particular, there exists a number x for which $x \equiv -1 \pmod{p_1^2}, \ldots, x \equiv -n \pmod{p_n^2}$, i.e. $p_1^2, \ldots, p_n^2$ divide into $x+1, \ldots, x+n$, thus separating two square-free numbers (and a fortiori two prime numbers) by a distance at least $n+1$.

26. Let P be the minimal polynomial for α: P has integer coefficients, which are relatively prime, a positive leading coefficient and of course $P(\alpha) = 0$. Let δ denote the distance from α to the nearest other zero of P. We may assume that $|\alpha - p/q| < \delta$, for otherwise the conclusion straightforward. By the mean value theorem, we get

$$|P(p/q)| \leq \|P'\|_{[\alpha-\delta,\alpha+\delta]} \left|\alpha - \frac{p}{q}\right| =: \frac{1}{c}\left|\alpha - \frac{p}{q}\right|.$$

It remains to observe that $q^n P(p/q)$ is a non-zero integer, so that $|P(p/q)| \geq \dfrac{1}{q^n}$.

27. Assume that p/q (with $\gcd(p, q) = 1$) is a zero of P, so that

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0.$$

Note that there is an odd number of odd coefficients a_i, because $a_n + a_{n-1} + \cdots + a_1 + a_0$ is odd, therefore it is enough to prove that p and q (and hence all the $p^{n-i}q^i$) are odd to derive that the left-hand side of the previous equation is odd, a contradiction. Since p divides into $-(a_n p^n + a_1 p q^{n-1}) = a_0 q^n$ and since p and q are coprime, we deduce that p divides a_0. The latter being odd, this implies that p is odd. We proceed likewise to establish that q is odd. And we're done.

28. The answer depends on the interpretation of polynomials. If one considers real polynomials, the answer is yes: the polynomials X^2 and X^4 achieve the value 0 on the set $\{0\}$ and the value 1 on the set $\{-1, 1\}$. But if one considers complex polynomials, the answer is no. Let P, Q be two polynomials (non-constant, with $\deg[P] \geq \deg[Q]$) such that, for some $a \neq b$,

$$\mathcal{Z}[P - a] = \mathcal{Z}[Q - a] \qquad \text{and} \qquad \mathcal{Z}[P - b] = \mathcal{Z}[Q - b].$$

The notation $\mathcal{Z}$ is self-evident, it stands for the set of zeros. Let now z be a zero of order k of $(P-a)(P-b)$, e.g. z is a zero of order k of $P-a$. Since z is a zero of $Q-a$ as well, one gets $P(z)=Q(z)=a$, i.e. z is a zero of $P-Q$. Since z is also a zero of order $k-1$ of P', one derives that z is a zero of order (at least) k of $P'(P-Q)$. This implies that $(P-a)(P-b)$ divides into $P'(P-Q)$. But one has

$$\deg[(P-a)(P-b)] = 2\deg[P] \qquad \text{and} \qquad \deg[P'(P-Q)] \leq 2\deg[P]-1,$$

a contradiction.

29. One needs to prove that $A^*B - BA^* = 0$, knowing that $AB = BA$ and $AA^* = A^*A$. Remember that a square matrix C is zero if and only if $\|C\| := \mathrm{tr}[CC^*] = 0$. Here one has

$$\begin{aligned}\mathrm{tr}[(A^*B-BA^*)(A^*B-BA^*)^*] &= \mathrm{tr}[(A^*B-BA^*)(B^*A-AB^*)] \\ &= \mathrm{tr}[A^*BB^*A] - \mathrm{tr}[A^*BAB^*] - \mathrm{tr}[BA^*B^*A] + \mathrm{tr}[BA^*AB^*].\end{aligned}$$

To conclude, it is enough to remark that

$$\begin{aligned}\mathrm{tr}[A^*BAB^*] &= \mathrm{tr}[A^*ABB^*] = \mathrm{tr}[AA^*BB^*] = \mathrm{tr}[A^*BB^*A], \\ \mathrm{tr}[BA^*AB^*] &= \mathrm{tr}[BAA^*B^*] = \mathrm{tr}[ABA^*B^*] = \mathrm{tr}[BA^*B^*A].\end{aligned}$$

Now assume furthermore that B is normal (i.e. $BB^* = B^*B$). Using what has just been done, it is immediate to derive that AB is normal, as

$$(AB)(AB)^* = ABB^*A^* = AB^*BA^* = B^*AA^*B = B^*A^*AB = (AB)^*(AB).$$

30. Suppose that $1-xy$ admits an inverse z, say. One needs to find a suitable candidate for the inverse of $1-yx$, and this can be guessed by considering a more familiar situation. Let A, B be two square matrices satisfying $\|A\|, \|B\| < 1$, then $I-AB$ and $I-BA$ are invertible, with

$$(I-BA)^{-1} = I+BA+\cdots+(BA)^n+\cdots = I+B[I+AB+\cdots+(AB)^{n-1}+\cdots]A = I+B(I-AB)^{-1}A.$$

This suggests that the inverse of $1-yx$ is $1+yzx$, which is checked as follow:

$$\begin{aligned}(1-yx)(1+yzx) &= 1-yx+y\,\overbrace{(1-xy)z}^{=1}\,x = 1, \\ (1+yzx)(1-yx) &= 1-yx+y\,\underbrace{z(1-xy)}_{=1}\,x = 1.\end{aligned}$$

Note that everything remains valid if one merely deals with right inverses instead of inverses.

Now suppose that x has a unique right inverse z, i.e. that $(1 - x')z = 1$ where $x' := 1 - x$. According to the previous argument (with $y = 1$), one infers that $(1 - x')(1 + zx') = 1$. By uniqueness, one derives $1 + zx' = z$, that is $1 = z(1 - x') = zx$. This shows that x is invertible.
