

Rational Approximation with Varying Weights III

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Approximation by weighted rationals of the form $w^n r_n$, where $r_n = p_n/q_n$, p_n and q_n are polynomials of degree at most $[\alpha n]$ and $[\beta n]$, respectively, and w is an admissible weight, is investigated on compact subsets of the real line for a general class of weights and given $\alpha \geq 0$, $\beta \geq 0$, with $\alpha + \beta > 0$. Conditions that characterize the largest sets on which such approximation is possible are given. We apply the general theorems to Laguerre and Freud weights. © 2000 Academic Press

1. MAIN RESULTS

The problem of uniform approximation on compact subsets of the real line by weighted rational functions of the form $w^n r_n$, where w is an admissible weight, and r_n is a rational function, was investigated in [1, 7]. Here we further generalize the previous results and we consider applications to Laguerre and Freud weights.

For $n \in \mathbf{N}$, let \mathcal{P}_n denote the space of algebraic polynomials of degree at most n . For a compact set E , $C(E)$ denotes the space of continuous real-valued functions on E . The symbol $[\cdot]$ denotes the greatest integer function.

Let Σ be a closed regular subset of the real line \mathbf{R} and $w: \Sigma \rightarrow [0, \infty)$ be a *strongly admissible weight*, that is, w is continuous on Σ , it is positive on a set of positive capacity, and if Σ contains a neighborhood of the point ∞ , then $|x|w(x) \rightarrow 0$ as $|x| \rightarrow \infty$, $x \in \Sigma$. Let $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta > 0$ be given numbers.

We shall first consider the problem of characterizing the compact sets $E \subseteq \Sigma$ having the approximation property that every function $f \in C(E)$ is the uniform limit on E of a sequence $\{w^n r_n\}_{n=1}^{\infty}$, with $r_n = p_n/q_n$, $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$.

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From previous results from [10, 8] regarding weighted polynomial approximation (the case $\alpha = 1$, $\beta = 0$) it is known that $E \subseteq S_w$, where S_w is the support of an extremal measure μ_w , the unique probability measure that minimizes the weighted energy

$$I_w(\mu) := \iint \log \frac{1}{|z-t|w(z)w(t)} d\mu(z) d\mu(t)$$

over the set $\mathcal{M}(\Sigma)$ of all probability Borel measures μ supported on Σ . It is also known [6, 8], that S_w is a compact set, and the following representation for $w(x)$ holds on S_w :

$$w(x) = \exp(U^{\mu_w}(x) - F_w), \quad x \in S_w, \quad (1.1)$$

where F_w is a constant, and for a compactly supported Borel measure μ the logarithmic potential U^μ is defined by

$$U^\mu(z) := \int \log \frac{1}{|z-t|} d\mu(t), \quad z \in \mathbf{C}.$$

In [7, Theorem 1.5], it was shown that representation like (1.1) on an interval I with μ_w replaced by a signed measure $\mu = \mu^+ - \mu^-$ with absolutely continuous μ^\pm having densities that behave like $|t-c|^{-1/2}$ at the endpoints c of I allows approximation on I . Thus the largest set E having the approximation property is essentially the largest set E on which w can be written as the exponential of the logarithmic potential of an absolutely continuous signed measure.

Before stating the main results of the paper we introduce some notation.

Let $K \subset \mathbf{R}$ be a compact set of positive logarithmic capacity and ω_K be its equilibrium measure, that is, the measure which minimizes the unweighted logarithmic energy $I_1(\mu)$ over all measures $\mu \in \mathcal{M}(K)$. If ω_K is absolutely continuous with respect to Lebesgue measure, then by f_K we shall denote its density.

Let ν be a positive measure supported on K . For $y \in \mathbf{R}$ we define the signed measure

$$\sigma(y) := \nu - y\omega_K.$$

Let $\sigma(y) = \sigma^+(y) - \sigma^-(y)$ be the Jordan decomposition of $\sigma(y)$ and set

$$p(y) := \|\sigma^+(y)\| \quad \text{and} \quad n(y) := \|\sigma^-(y)\|.$$

Our first theorem combines and extends Theorem 1 of [1] and Theorem 1.5 of [7].

THEOREM 1.1. *Let w be a strongly admissible weight defined on a set E which is the union of finitely many closed intervals. Let $\alpha \geq 0, \beta \geq 0$ with $\alpha + \beta > 0$ be given numbers. Assume that*

$$w(u) = \exp(U^v(u) + F), \quad u \in E, \tag{1.2}$$

where F is a constant, $dv(t) = v(t) dt$ on E , and the density v is continuous and nonnegative on $\text{Int}(E)$, and at each endpoint c of E ,

$$v(t) |t - c|^{1/2} \rightarrow l_c < \infty, \quad t \rightarrow c, \quad t \in E. \tag{1.3}$$

Let $\sigma(y) = v - y\omega_E$.

First assume that there is a sequence of weighted rationals of the form $w^n p_n / q_n$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$ such that $w^n p_n / q_n \rightarrow 1$ as $n \rightarrow \infty$ uniformly on E . Then there exists $y \in \mathbf{R}$ with $p(y) \leq \alpha$ and $n(y) \leq \beta$.

Next assume that there exists $y \in \mathbf{R}$ such that $p(y) < \alpha$ and $n(y) < \beta$. Then every function $f \in C(E)$ is the uniform limit on E of a sequence of weighted rationals $\{w^n p_n / q_n\}_{n=1}^\infty$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$.

Remark 1.2. If E is a compact set with $p(x) = \alpha$ and $n(x) = \beta$ for some x , then weighted rational approximation on E of functions not vanishing on E is not always possible. This is the case for the exponential weight $w(u) = e^u$ on the interval $[0, 2\pi]$. In this case, for $\alpha = \beta = 1$, A. B. J. Kuijlaars has proved that every function $f \in C([0, 2\pi])$ that has at least one zero on $[\pi, 2\pi]$ is approximable. Hence neither of the conditions of Theorem 1.1 is both necessary and sufficient.

It turns out that for certain classes of admissible weights w the conditions of Theorem 1.1 are satisfied on each set $S_{w,\lambda}$, $\lambda > 0$.

COROLLARY 1.3. *Let $w(u) = \exp(-Q(u))$ be a positive continuous weight defined on a set $\Sigma \subset \mathbf{R}$ that is the union of finitely many closed intervals $\{I_j\}_{j=1}^m$. Assume that on each interval I_j , the external field $Q(u)$ is convex or $|u| Q'(u)$ is increasing, and for some $p \in (1, \infty)$, $w' \in L^p(\Sigma)$. Then w satisfies the conditions of Theorem 1.1. Furthermore, Theorem 1.1 holds on each set $E = S_{w,\lambda}$, $\lambda > 0$.*

Conversely, if w is a weight satisfying the conditions of Theorem 1.1 on a set E that is the union of finitely many intervals, then $E = S_{w,\lambda}$ for some $\lambda > 0$.

COROLLARY 1.4. *Let $w(u) = \exp(-Q(u))$ be an admissible weight defined on a set $\Sigma \subset \mathbf{R}$ that is the union of finitely many intervals, and assume that Q is a real-analytic function on Σ . Then w satisfies the conditions of Theorem 1.1. Furthermore, Theorem 1.1 holds on each set $E = S_{w,\lambda}$, $\lambda > 0$.*

The above results and the representation

$$w(x) = \exp(U^{(1/\lambda)} \mu_{w^\lambda}(x) - (1/\lambda) F_{w^\lambda}), \quad x \in S_{w^\lambda} \quad (1.4)$$

which follows from (1.1) suggest that it is important to study weighted rational approximation on the sets S_{w^λ} . Before we state the corresponding approximation problem we mention that by [8, Theorem IV.4.1], ([10, Lemma 5.4])

$$S_{w^{\lambda_1}} \subseteq S_{w^{\lambda_2}}, \quad \text{for } \lambda_1 > \lambda_2 > 0. \quad (1.5)$$

Now we state the first approximation problem:

(A1) For given $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha + \beta > 0$ find the largest set S_{w^λ} (or, equivalently, the smallest $\lambda > 0$) with the property that on every compact set $E \subset \text{Int}(S_{w^\lambda})$ every function $f \in C(E)$ is the uniform limit on E of a sequence $\{w^n p_n/q_n\}_{n=1}^\infty$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$.

Before stating the next theorem we introduce some notation. For $\lambda > 0$ and $y \in \mathbf{R}$ we define the signed measure

$$\sigma_\lambda(y) := \frac{1}{\lambda} \mu_{w^\lambda} - y \omega_\lambda, \quad (1.6)$$

where $\omega_\lambda := \omega_{S_{w^\lambda}}$, and we set

$$p_\lambda(y) := \|\sigma_\lambda^+(y)\|, \quad n_\lambda(y) := \|\sigma_\lambda^-(y)\|. \quad (1.7)$$

THEOREM 1.5. Let w be a strongly admissible weight defined on a closed and regular set $\Sigma \subseteq \mathbf{R}$. Assume that for every $\lambda > 0$, S_{w^λ} is the union of finitely many closed intervals, the extremal measure μ_{w^λ} is absolutely continuous on S_{w^λ} , its density v_λ is continuous and nonnegative on $\text{Int}(S_{w^\lambda})$, and at each endpoint c of S_{w^λ} ,

$$v_\lambda(t) |t - c|^{1/2} \rightarrow l_\lambda(c) < \infty, \quad t \rightarrow c, \quad t \in S_{w^\lambda}. \quad (1.8)$$

Assume further that $S_{w^{\lambda_1}} \subset S_{w^{\lambda_2}}$ for all $\lambda_1 > \lambda_2 > 0$. In particular this is true if $Q = \log(1/w)$ is real-analytic on Σ , and $v_\lambda(c) = 0$ at each endpoint c of S_{w^λ} for all $\lambda > 0$.

Then the infimum of all numbers $\lambda > 0$ such that on every compact set $E \subset S_{w^\lambda}$ every function $f \in C(E)$ is the uniform limit on E of a sequence $\{w^n p_n/q_n\}_{n=1}^\infty$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$ is the number $\lambda^* = \lambda^*(\alpha, \beta)$ defined by

$$\lambda^*(\alpha, \beta) = \inf \{ \lambda > 0 : \exists y \in \mathbf{R} : p_\lambda(y) < \alpha, n_\lambda(y) < \beta \}, \quad (1.9)$$

if $\alpha > 0$ and $\beta > 0$, and

$$\lambda^*(\alpha, 0) = \inf \{ \lambda > 0 : \exists y \in \mathbf{R} : p_\lambda(y) < \alpha, n_\lambda(y) = 0 \}, \tag{1.10}$$

$$\lambda^*(0, \beta) = \inf \{ \lambda > 0 : \exists y \in \mathbf{R} : p_\lambda(y) = 0, n_\lambda(y) < \beta \}. \tag{1.11}$$

Finally we consider approximation by weighted rationals $w^n p_n / q_n$ with $p_n \in \mathcal{P}_{[\alpha n]}$, $q_n \in \mathcal{P}_{[\beta n]}$ for $\alpha \geq 0$ and $\beta \geq 0$ with a positive sum $\alpha + \beta$ that does not exceed a given number $\gamma > 0$.

Let w be a strongly admissible weight defined on a closed and regular set $\Sigma \subset \bar{\mathbf{R}}$ and assume that w satisfies the conditions of Theorem 1.5. The second approximation problem is stated below:

(A2) For given $\gamma > 0$ find the largest set S_{w^λ} (equivalently find the smallest $\lambda > 0$) such that there exist $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha + \beta \in (0, \gamma]$ having the property that on every compact set $E \subset \text{Int}(S_{w^\lambda})$ every function $f \in C(E)$ is the uniform limit of a sequence of weighted rationals $\{w^n p_n / q_n\}_{n=1}^\infty$ with $p_n \in \mathcal{P}_{[\alpha n]}$, $q_n \in \mathcal{P}_{[\beta n]}$.

For $\lambda > 0$ and $y \in \mathbf{R}$ we set $m_\lambda(y) := p_\lambda(y) + n_\lambda(y)$, where $p_\lambda(y)$ and $n_\lambda(y)$ are defined in (1.7) and (1.6). Then $m_\lambda(y) \leq 1/\lambda + |y|$ and

$$\begin{aligned} m_\lambda(y) &= \int |d\sigma_\lambda(y)| = \int |(1/\lambda) d\mu_{w^\lambda} - y d\omega_{S_{w^\lambda}}| \\ &\geq \left| \int (1/\lambda) d\mu_{w^\lambda} - \int |y d\omega_{S_{w^\lambda}}| \right| = |1/\lambda - |y||. \end{aligned} \tag{1.12}$$

From these inequalities we get

$$m_\lambda := \inf \{ m_\lambda(y) : y \in \mathbf{R} \} = \min \{ m_\lambda(y) : y \in [0, 2/\lambda] \}. \tag{1.13}$$

Let f_λ be the equilibrium density for the set S_{w^λ} , and

$$s_\lambda(t, y) := \frac{1}{\lambda} v_\lambda(t) - y f_\lambda(t)$$

be the density of the signed measure $\sigma_\lambda(y)$.

THEOREM 1.6. *Let $\gamma > 0$ be given and w satisfy the conditions of Theorem 1.5. Assume that for every $\lambda > 0$ and $y \in \mathbf{R}$, $s_\lambda(t, y)$ has at most countably many zeros in S_{w^λ} . Then the largest set S_{w^λ} having the property that for every compact $E \subset \text{Int}(S_{w^\lambda})$ every function $f \in C(E)$ is the uniform limit on E of a sequence of weighted rationals $\{w^n p_n / q_n\}_{n=1}^\infty$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and*

$q_n \in \mathcal{P}_{[\beta n]}$ for some $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha + \beta \in (0, \gamma]$, is the set $S_{w^{\lambda(\gamma)}}$, where $\lambda(\gamma) \in (0, 1]$ is the solution of the equation

$$m_\lambda = \gamma. \quad (1.14)$$

2. PROOFS

Proof of Theorem 1.1. The proof of the necessity part of the theorem is the same as the proof of Lemma 5 of [1].

The sufficiency part follows from Theorem 1.5 in [7] and Lemma 4.4 in [9]. It is known that for a set $E = \bigcup_{j=1}^m [a_j, b_j]$ with $a_1 < b_1 < a_2 < \dots < a_m < b_m$ the equilibrium distribution ω_E has the form

$$d\omega_E(t) = f_E(t) dt = \frac{|S(t)|}{\pi \sqrt{|R(t)|}} dt, \quad t \in E, \quad (2.1)$$

where

$$R(t) = \prod_{j=1}^m ((t - a_j)(t - b_j)) \quad \text{and} \quad S(t) = \prod_{j=1}^{m-1} (t - y_j)$$

for some $y_j \in (b_j, a_{j+1})$, $j = 1, \dots, m-1$ (see, for example, [9, Lemma 4.4]). From (2.1) we see that at the endpoints of E , the density $f_E(t)$ has the same behavior as the density $v(t)$ and the result follows from Theorem 1.5 of [7]. ■

Proof of Corollary 1.3. Since $w^\lambda = \exp(-\lambda Q)$, for $\lambda > 0$ the external field for w^λ has the same properties as Q . Hence it is enough to consider w only. By [8, Theorem IV.1.10(d)], the support S_w is the union of intervals $\{J_k\}$ at most one lying in any of the intervals I_j (the components of Σ). Furthermore, if J is one of the intervals J_k , by Theorem IV.1.6(e) of [8] we have

$$\mu_{w|_J} = \mu_w|_J + \overline{\mu_w|_{(\mathbf{R} \setminus J)}},$$

where the bar denotes taking balayage onto J out of $\mathbf{C} \setminus J$. This implies that $S_{w|_J} = J$. Then by [8, Theorem IV.2.4] applied to $w|_J$, and by [8, Corollary II.4.12] according to which the measure $\overline{\mu_w|_{(\mathbf{R} \setminus J)}}$ has continuous density it follows that (1.3) holds for the density of $\mu_{w|_J}$. The representation (1.2) for w on S_w follows from (1.1).

Now suppose that E is the union of finitely many intervals, and w satisfies the conditions of Theorem 1.1 on E . In particular $w(u) = \exp(U^v(u) + F)$, $u \in E$, with density $v = v^+ - v^-$ satisfying (1.3). From (2.1) and (1.3) we

get that for $\gamma > 0$ large enough ($\gamma > \sup\{v^-(t)/f_E(t), t \in E\}$), $v_1 := v + \gamma f_E > 0$ on E . Setting $\lambda := (\|v^+\| + \gamma - \|v^-\|)^{-1} > 0$ and $F_1 := \lambda(F - \gamma \log(1/\text{cap}(E)))$ we obtain

$$w^\lambda(u) = \exp(U^{\lambda v_1}(u) + F_1), \quad u \in E,$$

and then by [8, Theorem I.3.3] we get $S_{w^\lambda} = E$ and $\mu_{w^\lambda} = \lambda v_1 dt$. ■

Proof of Corollary 1.4. For a real-analytic external field Q it was shown in [3, Theorem 38] that S_w is the union of finitely many closed intervals, the measure μ_w is absolutely continuous on S_w , and its density satisfies the conditions of Theorem 1.1. The same is true for w^λ for any $\lambda > 0$. Thus the corollary follows from Theorem 1.1. ■

For the proof of Theorem 1.5 we need a lemma.

LEMMA 2.1. *Let $E_1 \subset E_2$ be compact sets on the real line. Assume that each $E_j, j = 1, 2$ is the union of finitely many intervals. Let $\omega_j = \omega_{E_j}$ and f_j denote the equilibrium measure and density for E_j , respectively. Then $f_1 \geq f_2$ on E_1 and for every interval $I \subseteq E_1$,*

$$\omega_1(I) > \omega_2(I).$$

Proof. By Lemma 5.5 of [10] (or [8, Theorem IV.1.6(e)]) we have

$$\omega_1 = \overline{\omega_2} = \omega_2|_{E_1} + \overline{\omega_2|_{E_2 \setminus E_1}} \geq \omega_2|_{E_1},$$

where the bar denotes taking balayage onto E_1 out of $\mathbb{C} \setminus E_1$. Thus $f_1 \geq f_2$ on E_1 . We set $\nu := \omega_2|_{E_2 \setminus E_1}$.

Now assume that there is an interval $I \subset E_1$ such that $\omega_1(I) = \omega_2(I)$. Then $\bar{\nu}(I) = 0$. Let h be a continuous function on E_1 that vanishes on $E_1 \setminus I$ and is positive on $\text{Int}(I)$, and let H denote the solution of the Dirichlet problem on the domain $D = \overline{\mathbb{C}} \setminus E_1$ with boundary function h (see [8, Section I.2]). This function H is harmonic on D and continuous on $\overline{\mathbb{C}}$, and by the minimum principle, [8, Theorem I.2.4], it is also positive on D . By a property of balayage measures, [8, Theorem II.4.1(c)], we have

$$\int H d\bar{\nu} = \int H d\nu$$

which is a contradiction. Indeed the left integral is $\int_{E_1} h d\bar{\nu} = 0$ by the choice of h , and the right integral is positive since it is over $E_2 \setminus E_1 \subset D$ where $H > 0$ and by (2.1) $\nu' = f_2 > 0$. ■

Proof of Theorem 1.5. We assume that $\alpha > 0$ and $\beta > 0$, the proof in the other two cases is similar.

First let $\lambda > \lambda^*$ and $E \subset S_{w^\lambda}$ be a compact set. Let $f \in C(E)$ and $f_1 \in C(S_{w^\lambda})$ be an extension of f to S_{w^λ} . Then there is $y \in \mathbf{R}$ such that $p_\lambda(y) < \alpha$ and $n_\lambda(y) < \beta$, and by (1.4) and Theorem 1.1, f_1 is uniformly approximable on S_{w^λ} by weighted rationals $w^n p_n/q_n$ with $p_n \in \mathcal{P}_{[\alpha n]}$, $q_n \in \mathcal{P}_{[\beta n]}$ and so is f on E .

When $\lambda = \lambda^*$ we can verify the approximation property only on compact sets $E \subset \text{Int}(S_{w^{\lambda^*}})$. Indeed let E be such set. By Lemma 5.8 of [10] for every $x \in \text{Int}(S_{w^{\lambda^*}})$ there is a $\lambda(x) > \lambda^*$ such that $x \in \text{Int}(S_{w^{\lambda(x)}})$. Then

$$E \subset \bigcup_{x \in E} \text{Int}(S_{w^{\lambda(x)}})$$

and since E is compact there is a finite subcover of E , $\{\text{Int}(S_{w^{\lambda(x_i)}})\}_{i=1}^{k(E)}$. Let $\lambda := \min\{\lambda(x_i) : 1 \leq i \leq k(E)\}$. Then $\lambda > \lambda^*$, $E \subset S_{w^\lambda}$, and as we have already shown every $f \in C(E)$ is uniformly approximable on E by weighted rationals $w^n p_n/q_n$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$.

In verifying the converse it is enough to assume that $S_{w^{\lambda_1}} \subset S_{w^{\lambda_2}}$ for all $\lambda_1 > \lambda_2 > 0$. Indeed in the case when w is real-analytic on Σ we have by Lemma 2.3 of [2] for every $\lambda_0 > 0$,

$$\bigcup_{\lambda > \lambda_0} S_{w^\lambda} = \{t \in S_{w^{\lambda_0}} : v_{\lambda_0}(t) > 0\},$$

and since for every $\lambda > 0$, v_λ vanishes at the endpoints of S_{w^λ} we get $S_{w^{\lambda_1}} \subset S_{w^{\lambda_2}}$ for $\lambda_1 > \lambda_2 > 0$. By Theorem IV.4.9 of [8] (or Lemma 5.7 of [10]) with $w := w^{\lambda_2}$, $\lambda := \lambda_1/\lambda_2 > 1$, and $d\mu_{w^\lambda} = v_\lambda dt$ we have

$$\frac{\lambda_2}{\lambda_1} v_{\lambda_1} \leq v_{\lambda_2}|_{S_{w^{\lambda_1}}} - \left(1 - \frac{\lambda_2}{\lambda_1}\right) f_{S_{w^{\lambda_2}}}|_{S_{w^{\lambda_1}}}. \quad (2.2)$$

If $\lambda^* = 0$ there is nothing to prove. So assume that $\lambda^* > 0$ and let $\lambda \in (0, \lambda^*)$. In view of Theorem 1.1 it is enough to show that for all $y \in \mathbf{R}$

$$h_\lambda(y) := \max\{p_\lambda(y), (\alpha/\beta) n_\lambda(y)\} > \alpha.$$

Indeed assume that there is y_0 with $h_\lambda(y_0) = \alpha$. By definition $p_\lambda(y) \geq 0$ is a decreasing function of y and $p_\lambda(0) = 1/\lambda > 0$. Similarly $n_\lambda(y) \geq 0$ is an increasing function of y and $n_\lambda(0) = 0$. Hence $h_\lambda := \inf\{h_\lambda(y) : y \in \mathbf{R}\}$ is attained at unique $y_\lambda > 0$ and

$$h_\lambda = p_\lambda(y_\lambda) = (\alpha/\beta) n_\lambda(y_\lambda). \quad (2.3)$$

Since $\lambda < \lambda^*$ from the definition of λ^* we get

$$\alpha = h_\lambda(y_0) \geq h_\lambda = \min\{h_\lambda(y) : y \in \mathbf{R}\} \geq \alpha,$$

that is, $h_\lambda(y_0) = h_\lambda(y_\lambda) = \alpha$. Then $y_0 = y_\lambda > 0$ and $p_\lambda(y_0) = (\alpha/\beta) n_\lambda(y_0) = \alpha$.

Let $\lambda_1 \in (\lambda, \lambda^*)$. By (2.2) with $\lambda_2 = \lambda$ and Lemma 2.1, for $y > 0$ such that $p_{\lambda_1}(y) > 0$ we have

$$\begin{aligned} \frac{1}{\lambda_1} v_{\lambda_1} - y f_{S_{w^{\lambda_1}}} &\leq \frac{1}{\lambda} v_\lambda |_{S_{w^{\lambda_1}}} - \left(\frac{1}{\lambda} - \frac{1}{\lambda_1}\right) f_{S_{w^\lambda}} |_{S_{w^{\lambda_1}}} - y f_{S_{w^{\lambda_1}}} \\ &\leq \frac{1}{\lambda} v_\lambda |_{S_{w^{\lambda_1}}} - \left(\frac{1}{\lambda} - \frac{1}{\lambda_1} + y\right) f_{S_{w^\lambda}} |_{S_{w^{\lambda_1}}}. \end{aligned} \tag{2.4}$$

We integrate (2.4) over $\text{supp}(\sigma_{\lambda_1}^+(y))$. Since $S_{w^{\lambda_1}} \subset S_{w^\lambda}$, applying Lemma 2.1 we obtain

$$p_{\lambda_1}(y) < p_\lambda(y + 1/\lambda - 1/\lambda_1). \tag{2.5}$$

We set $y = y_0 - 1/\lambda + 1/\lambda_1 > 0$ for $\lambda_1 \in (\lambda, \lambda^*)$ close enough to λ (so that $1/\lambda - 1/\lambda_1 < y_0/2$), and we obtain

$$p_{\lambda_1}(y_0 - 1/\lambda + 1/\lambda_1) < p_\lambda(y_0) = \alpha.$$

Then using the identity $p_\lambda(y) - n_\lambda(y) = 1/\lambda - y$ we obtain

$$\begin{aligned} n_{\lambda_1}(y_0 - 1/\lambda + 1/\lambda_1) &= p_{\lambda_1}(y_0 - 1/\lambda + 1/\lambda_1) + y_0 - 1/\lambda \\ &< p_\lambda(y_0) + y_0 - 1/\lambda = n_\lambda(y_0) = \beta. \end{aligned}$$

We get $h_{\lambda_1}(y_0 - 1/\lambda + 1/\lambda_1) < \alpha$ which contradicts the choice of $\lambda_1 < \lambda^*$. Moreover, we have shown that h_λ is a decreasing function of $\lambda > 0$.

We proved that if $\lambda \in (0, \lambda^*)$ then $h_\lambda(y) > \alpha$ for all $y \in \mathbf{R}$.

Let $\lambda \in (0, \lambda^*)$ and let $E = S_{w^{\lambda_1}}$ for some $\lambda_1 \in (\lambda, \lambda^*)$. Then the function 1_E (the characteristic function of the set E) is not uniformly approximable on E by weighted rationals $w^n p_n/q_n$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$, because, otherwise we would have by Theorem 1.1 an $y \in \mathbf{R}$ with $h_\lambda(y) \leq \alpha$ and as we have shown this is impossible.

Theorem 1.5 is proved. ■

For the proof of Theorem 1.6 we will need the following lemma.

LEMMA 2.2. *Assume that for every $\lambda > 0$ and $y \in \mathbf{R}$ the density $s_\lambda(t, y)$ of the signed measure $\sigma_\lambda(y)$ has at most countably many zeros in S_{w^λ} . Then the function $m_\lambda(y) \in C^1(\mathbf{R})$ and there is a unique $y^* = y^*(\lambda)$ such that $m_\lambda = m_\lambda(y^*)$.*

Proof. Let $s_\lambda^\pm(t, y)$ be the densities of $\sigma_\lambda^\pm(y)$ respectively. It follows from the representation

$$m_\lambda(y) = \int_{S_{w^\lambda}} |(1/\lambda) d\mu_{w^\lambda}(t) - y d\omega_\lambda(t)|$$

that $m_\lambda(y) \in C(\mathbf{R})$. Let $y_0 \in \mathbf{R}$ be fixed. By the definition of $m_\lambda(y)$,

$$\begin{aligned} & m_\lambda(y) - m_\lambda(y_0) \\ &= \int_{S_{w^\lambda}} (s_\lambda^+(t, y) - s_\lambda^+(t, y_0)) dt + \int_{S_{w^\lambda}} (s_\lambda^-(t, y) - s_\lambda^-(t, y_0)) dt \\ &= 2 \int_{S_{w^\lambda}} (s_\lambda^+(t, y) - s_\lambda^+(t, y_0)) dt - \int_{S_{w^\lambda}} (s_\lambda(t, y) - s_\lambda(t, y_0)) dt \\ &= (y - y_0) - 2 \int_{\Delta_\lambda^+(y) \cap \Delta_\lambda^+(y_0)} (y - y_0) f_\lambda(t) dt \\ &\quad + 2 \int_{((\Delta_\lambda^+(y) \cup \Delta_\lambda^+(y_0)) \setminus (\Delta_\lambda^+(y) \cap \Delta_\lambda^+(y_0)))} (s_\lambda^+(t, y) - s_\lambda^+(t, y_0)) dt, \quad (2.6) \end{aligned}$$

where $\Delta_\lambda^\pm(y)$ is the support of $s_\lambda^\pm(t, y)$, respectively. Let \tilde{y} be the infimum of all $y \geq 0$ such that $s_\lambda(t, y)$ has at least one zero in $\text{Int}(S_{w^\lambda})$. Since $yf_\lambda(t)$ increases with y , then $\Delta_\lambda^+(y_1) \subseteq \Delta_\lambda^+(y_2)$ for $y_1 > y_2 \geq 0$ and if we assume that for some $y_1 > y_2 \geq \tilde{y}$, $\Delta_\lambda^+(y_1) \equiv \Delta_\lambda^+(y_2)$, then at $t \in \Delta_\lambda^+(y_1) \cap \Delta_\lambda^-(y_1)$ we would have

$$v_\lambda(t) = \lambda y_1 f_\lambda(t) > \lambda y_2 f_\lambda(t) = v_\lambda(t)$$

which is impossible. Furthermore, $\Delta_\lambda^+(y) \rightarrow \Delta_\lambda^+(y_0)$ in the sense that the Lebesgue measure of the set $(\Delta_\lambda^+(y) \cup \Delta_\lambda^+(y_0)) \setminus (\Delta_\lambda^+(y) \cap \Delta_\lambda^+(y_0))$ tends to zero as $y \rightarrow y_0$. Otherwise there will be a set E with positive Lebesgue measure and a number $y_0 > 0$ such that $E \subseteq \Delta_\lambda^+(y)$ for all $y \in [0, y_0)$, but $E \cap \Delta_\lambda^+(y_0) = \emptyset$. Then for $t \in E$ we will have $0 \geq s_\lambda(t, y_0) = \lim_{y \rightarrow y_0} s_\lambda(t, y) \geq 0$ hence $s_\lambda(t, y_0) = 0$ which contradicts the assumption that $s_\lambda(t, y_0)$ has countably many zeros in S_{w^λ} .

For $t \notin \Delta_\lambda^+(y) \cap \Delta_\lambda^+(y_0)$ we have

$$|s_\lambda^+(t, y) - s_\lambda^+(t, y_0)| \leq |s_\lambda(t, y) - s_\lambda(t, y_0)| = |y - y_0| f_\lambda(t),$$

and therefore the absolute value of the last integral in (2.6) is at most

$$|y - y_0| \int_{((\Delta_\lambda^+(y) \cup \Delta_\lambda^+(y_0)) \setminus (\Delta_\lambda^+(y) \cap \Delta_\lambda^+(y_0)))} f_\lambda(t) dt = o(|y - y_0|).$$

Hence from (2.6) we obtain

$$m'_\lambda(y_0) = 1 - 2 \int_{\Delta_\lambda^+(y_0)} f_\lambda(t) dt. \tag{2.7}$$

Then $m'_\lambda(y) \in C(\mathbf{R})$ follows from (2.7) and the fact that $\Delta_\lambda^+(y)$ continuously changes with y .

For $y \leq \tilde{y}$, $\Delta_\lambda^-(y) \equiv \emptyset$ and by (2.7), $m'_\lambda(y) = -1$, and $m_\lambda(y) = p_\lambda(y) = 1/\lambda - y$. For $y > \tilde{y}$, $\Delta_\lambda^+(y)$ decreases with y and by (2.7) we get that $m'_\lambda(y)$ increases on (\tilde{y}, ∞) , and $m'_\lambda(y) \rightarrow 1$ as $y \rightarrow \infty$. Then there is a unique $y^* = y^*(\lambda) > \tilde{y}$ such that $m'_\lambda(y^*) = 0$ and by (1.13), $m_\lambda = m_\lambda(y^*)$. Lemma 2.2 is proved. ■

Proof of Theorem 1.6. We first show that m_λ is a decreasing function of $\lambda > 0$. Let $\lambda_1 > \lambda > 0$. By (2.5) we have

$$p_{\lambda_1}(y) < p_\lambda(y + 1/\lambda - 1/\lambda_1), \quad y \geq 0.$$

Since $m_\lambda(y) = 2p_\lambda(y) + y - 1/\lambda$, for $y \geq 0$ we have

$$m_{\lambda_1}(y) < 2p_\lambda(y + 1/\lambda - 1/\lambda_1) + y - 1/\lambda_1 = m_\lambda(y + 1/\lambda - 1/\lambda_1). \tag{2.8}$$

Then from (1.13) and (2.8) and Lemma 2.2 ($m_\lambda(y) \in C(\mathbf{R})$) we get

$$\begin{aligned} m_{\lambda_1} &= \min\{m_{\lambda_1}(y) : y \in [0, 2/\lambda_1]\} \\ &< \min\{m_\lambda(y + 1/\lambda - 1/\lambda_1) : y \in [0, 2/\lambda_1]\} \\ &= \min\{m_\lambda(y) : y \in [1/\lambda - 1/\lambda_1, 1/\lambda + 1/\lambda_1]\}. \end{aligned} \tag{2.9}$$

By the continuity of $m_\lambda(y)$ (Lemma 2.2) the right-hand side of (2.9) tends to $\min\{m_\lambda(y) : y \in [0, 2/\lambda]\} = m_\lambda$ as $\lambda_1 \rightarrow \lambda$, $\lambda_1 > \lambda$. Hence m_λ is right-continuous and nondecreasing function of $\lambda > 0$. Now assume that for some $\lambda_2 > \lambda > 0$, $m_{\lambda_2} = m_\lambda$. Then for every $\lambda_1 \in (\lambda, \lambda_2]$, $m_{\lambda_1} = m_\lambda$. Then (2.9) implies that for every $\lambda_1 \in (\lambda, \lambda_2]$, $m_{\lambda_1} = m_\lambda = m_\lambda(y_\lambda)$ for some $y_\lambda \in [0, 1/\lambda - 1/\lambda_1) \cup (1/\lambda + 1/\lambda_1, 2/\lambda]$. By Lemma 2.2 this $y_\lambda = y^*(\lambda)$ is unique, hence $y_\lambda = 0$ or $y_\lambda = 2/\lambda$, that is $m'_\lambda(0) = 0$ or $m'_\lambda(2/\lambda) = 0$. But this is impossible since by (2.7) of Lemma 2.2 and $\Delta_\lambda^+(0) = \text{supp}(\sigma_\lambda^+(0)) = S_{w^\lambda}$ we have $m'_\lambda(0) = -1$, and by (1.12) and $m_\lambda(0) = 1/\lambda$, $y^*(\lambda) = 2/\lambda$ implies $m_\lambda = 1/\lambda$ and $s_\lambda(t, 2/\lambda) \leq 0$ on S_{w^λ} , which in view of (2.7) gives $m'_\lambda(2/\lambda) = 1$. Hence m_λ is a decreasing function of $\lambda > 0$.

Now let $\gamma > 0$ be given. First let $E \subset \text{Int}(S_{w^{\lambda(\gamma)}})$ be a compact set. As in the proof of Theorem 1.5 it follows that there is a $\lambda > \lambda(\gamma)$ such that $E \subseteq S_{w^\lambda}$. Moreover,

$$\delta := \gamma - m_\lambda = m_{\lambda(\gamma)} - m_\lambda > 0,$$

and $m_\lambda > 0$ for otherwise $s_\lambda(t, y^*(\lambda)) = 0$ on S_{w^λ} which contradicts the assumption concerning the zeros of the functions $s_\lambda(t, y)$.

Let $a_\lambda := \inf \{y > 0 : n_\lambda(y) > 0\}$, and $b_\lambda := \sup \{y > 0 : p_\lambda(y) > 0\}$. Then $0 \leq a_\lambda < b_\lambda \leq \infty$, because $p_\lambda(y)$ and $-n_\lambda(y)$ are nonincreasing functions of $y \in \mathbf{R}$, and $m_\lambda > 0$. Moreover, $y^*(\lambda) \in [a_\lambda, b_\lambda]$. Indeed if say $y^*(\lambda) < a_\lambda$, then for every $y \in (y^*(\lambda), a_\lambda)$ we would have $m_\lambda = m_\lambda(y^*(\lambda)) = p_\lambda(y^*(\lambda)) > p_\lambda(y) = m_\lambda(y)$ which contradicts the definition of m_λ . By the continuity of $m_\lambda(y)$ and hence that of $p_\lambda(y) = (m_\lambda(y) - y + 1/\lambda)/2$ and $n_\lambda(y) = m_\lambda(y) - p_\lambda(y)$, we can select $y_0 \in (a_\lambda, b_\lambda)$ with $|p_\lambda(y_0) - p_\lambda(y^*(\lambda))| \leq \delta/4$, and $|n_\lambda(y_0) - n_\lambda(y^*(\lambda))| \leq \delta/4$. Then we set $\alpha := p_\lambda(y_0) + \delta/8$ and $\beta := n_\lambda(y_0) + \delta/8$. We have $\alpha + \beta \leq m_\lambda + 3\delta/4 < \gamma$, $p_\lambda(y_0) < \alpha$, and $n_\lambda(y_0) < \beta$. Hence by Theorem 1.1, every function $f \in C(E)$ is uniformly approximable on E by a sequence of weighted rationals $\{w^n p_n/q_n\}$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$.

Conversely, let $\lambda \in (0, \lambda(\gamma))$. Then $S_{w^{\lambda(\gamma)}} \subset S_{w^\lambda}$, and $m_\lambda > m_{\lambda(\gamma)} = \gamma$. Consider the compact set $E := S_{w^{\lambda(\gamma)}}$. We recall that under the conditions of the theorem E is the union of finitely many closed intervals. Then the constant function 1 on E is not w -approximable in the sense of (A2). Indeed, assume that there are $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha + \beta \in (0, \gamma]$, and a sequence $\{w^n p_n/q_n\}$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$ that tends to 1 uniformly on E as $n \rightarrow \infty$. By Theorem 1.1 there exists $y \in \mathbf{R}$ with $p_\lambda(y) \leq \alpha$ and $n_\lambda(y) \leq \beta$. Then $m_\lambda \leq m_\lambda(y) \leq \alpha + \beta \leq \gamma$ gives a contradiction. Theorem 1.6 is proved. \blacksquare

3. WEIGHTED RATIONAL APPROXIMATION WITH LAGUERRE AND FREUD WEIGHTS

Laguerre weights. The function $w(u) = u^\theta e^{-cu}$ with $\theta \geq 0$ and $c > 0$ defined on $\Sigma = [0, \infty)$ is called *Laguerre weight*. It is known that ([8], Examples IV.1.18 and IV.5.4)

$$S_w = [a(\theta, c), b(\theta, c)] =: \Delta_{\theta, c} \quad (3.1)$$

is an interval with endpoints $a(\theta, c) = 1/c(\theta + 1 - \sqrt{2\theta + 1})$ and $b(\theta, c) = 1/c(\theta + 1 + \sqrt{2\theta + 1})$, and the extremal measure μ_w has density

$$v_w(t) = \frac{c}{\pi t} \sqrt{(t - a(\theta, c))(b(\theta, c) - t)}, \quad t \in \Delta_{\theta, c}. \quad (3.2)$$

For $\lambda > 0$ we have $w(u)^\lambda = u^{\lambda\theta} e^{-\lambda cu}$, the support $S_{w^\lambda} = \Delta_{\lambda\theta, \lambda c}$,

$$v_\lambda(t) = v_{w^\lambda}(t) = \frac{\lambda c}{\pi t} \sqrt{(t - a)(b - t)}, \quad t \in \Delta_\lambda := \Delta_{\lambda\theta, \lambda c},$$

where $a = a(\lambda\theta, \lambda c)$ and $b = b(\lambda\theta, \lambda c)$, and

$$f_\lambda(t) = \frac{1}{\pi \sqrt{(t-a)(b-t)}}, \quad t \in \Delta_\lambda$$

is the equilibrium density for the interval Δ_λ .

The approximation problem (A2) for Laguerre weights. Let $\gamma > 0$ be given. To determine m_λ for $\lambda > 0$ we consider the equation

$$v_\lambda(t) = \lambda y f_\lambda(t),$$

which is equivalent to

$$c(t-a)(b-t) = yt \quad \text{or} \quad ct^2 + t(y - c(a+b)) + cab = 0. \quad (3.3)$$

The formulas for v_λ and f_λ show that (3.3) has two real solutions $t_{1,2}(y) \in [a, b]$,

$$t_{1,2}(y) = \frac{c(a+b) - y \pm \sqrt{(c(a+b) - y)^2 - 4c^2ab}}{2c} \quad (3.4)$$

if and only if $y \in [0, c(\sqrt{b} - \sqrt{a})^2] = [0, 2/\lambda]$. For other y we have $m_\lambda(y) > m_\lambda$. By Lemma 2.2 $m_\lambda = m_\lambda(y^*)$, where y^* is the unique solution of the equation

$$\int_{t_2(y^*)}^{t_1(y^*)} f_\lambda(t) dt = \frac{1}{2}. \quad (3.5)$$

Changing variables $t = (a+b)/2 + s(b-a)/2$ in (3.5) we obtain

$$\sin^{-1}(a_1 + a_2) - \sin^{-1}(a_1 - a_2) = \pi/2, \quad (3.6)$$

where

$$a_1 = \frac{-y^*}{c(b-a)} \quad \text{and} \quad a_2 = \frac{\sqrt{(c(a+b) - y^*)^2 - 4c^2ab}}{c(b-a)}.$$

We apply the cosine function to both sides of the last equation and simplify to obtain

$$|\sqrt{(1 - (a_1 + a_2)^2)(1 - (a_1 - a_2)^2)}| = |a_1^2 - a_2^2|.$$

Simplifying further we obtain $2(a_1^2 + a_2^2) = 1$, or equivalently

$$\frac{y^{*2} + ((c(a+b) - y^*)^2 - 4c^2ab)}{c^2(b-a)^2} = \frac{1}{2},$$

which reduces to

$$4y^{*2} - 4c(a+b)y^* + c^2(b-a)^2 = 0.$$

The solutions of the last equation are $y_{1,2}^* = c(a+b \pm 2\sqrt{ab})/2$, and since $a+b = 2(\lambda\theta + 1)/(\lambda c)$ and $\sqrt{ab} = \theta/c$ (see (3.1)), we have

$$y_2^* = 1/\lambda \quad \text{and} \quad y_1^* = (2\lambda\theta + 1)/\lambda.$$

Since the range of \sin^{-1} is $[-\pi/2, \pi/2]$, Eq. (3.6) implies that $a_1 + a_2 \geq 0$ which is equivalent to $y^* \leq (2\lambda\theta + 1)/(\lambda(\lambda\theta + 1))$ and y_2^* only satisfies this condition, unless $\theta = 0$ in which case $y_1^* = y_2^*$. Hence,

$$y^* = y_2^* = c(a+b - 2\sqrt{ab})/2 = 1/\lambda. \quad (3.7)$$

Next we derive a formula for $m_\lambda(y)$ for $y \in [0, 2/\lambda]$. We have $p_\lambda(0) = 1/\lambda$ and since $m_\lambda(y) = 2p_\lambda(y) + y - 1/\lambda$,

$$p'_\lambda(y) = (m'_\lambda(y) - 1)/2 = -\int_{t_2(y)}^{t_1(y)} f_\lambda(t) dt, \quad (3.8)$$

where we used (2.7). Then with

$$s_{1,2}(y) := (2t_{1,2}(y) - a - b)/(b - a)$$

we obtain

$$p'_\lambda(y) = (\sin^{-1}(s_2(y)) - \sin^{-1}(s_1(y)))/\pi. \quad (3.9)$$

Then

$$p_\lambda(y) = 1/\lambda + \int_0^y p'_\lambda(u) du = 1/\lambda + (J_2(y) - J_1(y))/\pi$$

and

$$m_\lambda(y) = 1/\lambda + y + 2(J_2(y) - J_1(y))/\pi, \quad (3.10)$$

where

$$\begin{aligned} J_{1,2}(y) &:= \int_0^y \sin^{-1}(s_{1,2}(u)) du \\ &= y \sin^{-1}(s_{1,2}(y)) - \int_0^y \frac{us'_{1,2}(u)}{\sqrt{1 - s_{1,2}(u)^2}} du \\ &= y \sin^{-1}(s_{1,2}(y)) - \int_0^y \frac{ut'_{1,2}(u)}{\sqrt{(t_{1,2}(u) - a)(b - t_{1,2}(u))}} du. \end{aligned} \quad (3.11)$$

For $t_{1,2}(u)$ from (3.4) we get

$$t'_{1,2}(u) = \frac{\mp t_{1,2}(u)}{c(t_1(u) - t_2(u))} \tag{3.12}$$

and by (3.3)

$$\sqrt{(t_{1,2}(u) - a)(b - t_{1,2}(u))} = \sqrt{ut_{1,2}(u)/c}. \tag{3.13}$$

Then by (3.11), (3.12) and (3.13) we obtain

$$\begin{aligned} J_2(y) - J_1(y) &= y(\sin^{-1}(s_2(y)) - \sin^{-1}(s_1(y))) \\ &\quad - \frac{1}{\sqrt{c}} \int_0^y \frac{\sqrt{u}(\sqrt{t_1(u)} + \sqrt{t_2(u)})}{t_1(u) - t_2(u)} du. \end{aligned}$$

For the last integral we have by (3.3)

$$\begin{aligned} \int_0^y \frac{\sqrt{u}}{\sqrt{t_1(u)} - \sqrt{t_2(u)}} du &= \int_0^y \frac{\sqrt{u}}{\sqrt{t_1(u) + t_2(u) - 2\sqrt{t_1(u)t_2(u)}}} du \\ &= \int_0^y \frac{\sqrt{u}}{\sqrt{(a+b) - u/c - 2\sqrt{ab}}} du \\ &= \sqrt{c} \int_0^y \frac{\sqrt{u}}{\sqrt{2/\lambda - u}} du =: \frac{\sqrt{c}}{\lambda} A(\lambda y). \end{aligned}$$

To compute $A(y)$ we use change of variables $u \rightarrow 2v^2$ and integration by parts

$$\begin{aligned} A(y) &= 4 \int_0^{\sqrt{y/2}} \frac{v^2}{\sqrt{1-v^2}} dv = -4 \int_0^{\sqrt{y/2}} \sqrt{1-v^2} dv \\ &\quad + 4 \int_0^{\sqrt{y/2}} \frac{1}{\sqrt{1-v^2}} dv = -4 \sqrt{\frac{y}{2} \left(1 - \frac{y}{2}\right)} - A(y) + 4 \sin^{-1}(\sqrt{y/2}), \end{aligned}$$

and so we obtain

$$A(y) = 2 \sin^{-1}(\sqrt{y/2}) - \sqrt{y(2-y)}. \tag{3.14}$$

Then

$$J_2(y) - J_1(y) = y(\sin^{-1}(s_2(y)) - \sin^{-1}(s_1(y))) - A(\lambda y)/\lambda$$

and for $m_\lambda(y)$ from (3.8), (3.9), and (3.10) we obtain

$$m_\lambda(y) = 1/\lambda + y - 2y \int_{t_2(y)}^{t_1(y)} f_\lambda(t) dt - 2A(\lambda y)/(\lambda\pi). \quad (3.15)$$

For the minimal mass m_λ we get (using (3.5) and (3.14))

$$\begin{aligned} m_\lambda = m_\lambda(y^*) = m_\lambda(1/\lambda) &= 2/\lambda - (2/\lambda) \int_{t_2(1/\lambda)}^{t_1(1/\lambda)} f_\lambda(t) dt \\ &- 2A(1)/(\lambda\pi) = 1/\lambda - 2(\pi/2 - 1)/(\lambda\pi) = 2/(\lambda\pi). \end{aligned} \quad (3.16)$$

The quantity m_λ decreases from ∞ to m_1 as λ increases from 0 to 1. Then by Theorem 1.6 for a given $\gamma \geq m_1$ the largest interval $\Delta_\lambda := \Delta_{\lambda\theta, \lambda c}$ on which approximation by weighted rationals is possible in the sense of (A2) is the interval $\Delta_{\lambda(\gamma)}$, where $\lambda(\gamma) = 2/(\pi\gamma)$.

Freud weights. The function $w(u) = \exp(-\gamma_\tau |u|^\tau)$, with $\tau > 0$ and

$$\gamma_\tau = \frac{\Gamma(\tau/2) \Gamma(1/2)}{2\Gamma((\tau+1)/2)},$$

defined on $\Sigma = \mathbf{R}$ is called *Freud weight*. By [8, Theorem IV.5.1], $S_w = [-1, 1]$ and $\mu_{w_\tau}(t) = s_\tau(t) dt$, where

$$s_\tau(t) = \frac{\tau}{\pi} \int_{|t|}^1 \frac{u^{\tau-1}}{\sqrt{u^2 - t^2}} du, \quad t \in [-1, 1] \quad (3.17)$$

is the so called *Ullman distribution*.

The approximation problem (A2) for Freud weights. Let $\lambda > 0$. For $w(u)^\lambda = \exp(-\lambda\gamma_\tau |u|^\tau)$ it follows from the definition of the extremal measure that $S_{w^\lambda} = [-\lambda^{-1/\tau}, \lambda^{-1/\tau}] =: \Delta_\lambda$, and

$$v_\lambda(t) = v_{w^\lambda}(t) = s_\tau(\lambda^{1/\tau}t) \lambda^{1/\tau}, \quad t \in \Delta_\lambda.$$

The function s_τ is even and as we are going to show later with Lemma 3.3, for $\tau \in [1, 2]$, $s_\tau(t)$ is monotone decreasing on $[0, 1]$ and so is $v_\lambda(t)$ on $[0, \lambda^{-1/\tau}]$.

We shall restrict ourselves to Freud weights with $\tau \in [1, 2]$ since in this case the monotonicity of s_τ allows us to solve the problem completely. For $y \geq 0$ we consider the function

$$s_\lambda(t, y) = (1/\lambda) v_\lambda(t) - y f_\lambda(t), \quad t \in \Delta_\lambda,$$

where

$$f_\lambda(t) = \frac{1}{\pi \sqrt{\lambda^{-2/\tau} - t^2}}, \quad t \in \Delta_\lambda$$

is the equilibrium density for Δ_λ . The equation $s_\lambda(t, y) = 0$ has exactly two solutions $t_1(y) > 0$ and $t_2(y) = -t_1(y)$ in Δ_λ for $y \in [0, a_{\tau, \lambda})$, where

$$a_{\tau, \lambda} := \frac{v_\lambda(0)}{\lambda f_\lambda(0)} = \frac{\tau}{\lambda(\tau - 1)}.$$

By the proof of Theorem 1.6 and Lemma 2.2 we have

$$m_\lambda = \min\{m_\lambda(y) : y \in [0, a_{\tau, \lambda}]\} = m_\lambda(y^*),$$

where $y^* \in [0, a_{\tau, \lambda})$ is the unique solution of the equation

$$\frac{1}{2} = \int_{-t_1(y)}^{t_1(y)} f_\lambda(t) dt = \frac{2}{\pi} \sin^{-1}(\lambda^{1/\tau} t_1(y)).$$

Then $\lambda^{1/\tau} t_1(y^*) = \sqrt{2}/2$, and for m_λ we obtain

$$\begin{aligned} m_\lambda &= m_\lambda(y^*) = 2p_\lambda(y^*) + y^* - \frac{1}{\lambda} \\ &= \frac{2}{\lambda} \int_{-t_1(y^*)}^{t_1(y^*)} v_\lambda(t) dt - 2y^* \int_{-t_1(y^*)}^{t_1(y^*)} f_\lambda(t) dt + y^* - \frac{1}{\lambda} \\ &= \frac{2}{\lambda} \int_{-t_1(y^*)}^{t_1(y^*)} v_\lambda(t) dt - \frac{1}{\lambda} = \frac{4}{\lambda} \int_0^{\sqrt{2}/2} s_\tau(u) du - \frac{1}{\lambda}. \end{aligned} \tag{3.18}$$

To compute the last integral we need a differential equation for $s_\tau(t)$. Let $t \in (0, 1)$. With the change of variables $u \rightarrow tu_1$ and $u_1 \rightarrow 1/u$ we obtain

$$\begin{aligned} s_\tau(t) &= \frac{\tau}{\pi} \int_t^1 \frac{u^{\tau-1}}{\sqrt{u^2 - t^2}} du \\ &= \frac{\tau}{\pi} t^{\tau-1} \int_1^{1/t} \frac{u_1^{\tau-1}}{\sqrt{u_1^2 - 1}} du_1 = \frac{\tau}{\pi} t^{\tau-1} \int_t^1 \frac{u^{-\tau}}{\sqrt{1 - u^2}} du. \end{aligned} \tag{3.19}$$

Then

$$s'_\tau(t) = \frac{\tau}{\pi} \left((\tau - 1) t^{\tau-2} \int_t^1 \frac{u^{-\tau}}{\sqrt{1 - u^2}} du - t^{\tau-1} \frac{t^{-\tau}}{\sqrt{1 - t^2}} \right)$$

or equivalently

$$ts'_\tau(t) = (\tau - 1) s_\tau(t) - \frac{\tau}{\pi \sqrt{1-t^2}}. \quad (3.20)$$

For $a \in [0, 1]$ using integration by parts and (3.20) we obtain

$$\begin{aligned} I_\tau(a) &:= \int_0^a s_\tau(t) dt = as_\tau(a) - \int_0^a ts'_\tau(t) dt \\ &= as_\tau(a) + \frac{\tau}{\pi} \sin^{-1}(a) - (\tau - 1) I_\tau(a), \end{aligned}$$

hence

$$I_\tau(a) = \frac{a}{\tau} s_\tau(a) + \frac{1}{\pi} \sin^{-1}(a). \quad (3.21)$$

From (3.18) and (3.21) we obtain

$$m_\lambda = \frac{4I_\tau(\sqrt{2}/2) - 1}{\lambda} = \frac{2\sqrt{2}}{\tau\lambda} s_\tau(\sqrt{2}/2). \quad (3.22)$$

Then m_λ decreases from ∞ to m_1 as λ increases from 0 to 1. By Theorem 1.6, for given $\gamma \geq m_1$ the largest interval Δ_λ on which weighted rational approximation is possible in the sense of (A2) is the interval $\Delta_{\lambda(\gamma)}$, where (see (1.14))

$$\lambda(\gamma) = \frac{2\sqrt{2}}{\tau\gamma} s_\tau(\sqrt{2}/2). \quad (3.23)$$

The approximation problem (A1) for Freud weights. Let $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha + \beta > 0$ be given. The Freud weights satisfy the conditions of Theorem 1.5, hence by Theorem 1.5 we have

$$\lambda^*(\alpha, \beta) = \inf \{ \lambda > 0 : \exists y \in \mathbf{R} : h_\lambda(y) \leq \alpha \}.$$

As shown in the proof of Theorem 1.5, for every $\lambda > 0$ the equation $p_\lambda(y) = (\alpha/\beta) n_\lambda(y)$ has unique solution $\bar{y}(\alpha, \beta; \lambda) > 0$. Moreover, by the proof of Theorem 1.5 it follows that $\lambda^*(\alpha, \beta)$ is the unique solution of the equation

$$p_\lambda(\bar{y}(\alpha, \beta; \lambda)) = \alpha.$$

For $t \in \Delta_\lambda$ we have $u := \lambda^{1/\tau}t \in [-1, 1]$ and

$$s_\lambda(t, y) = \lambda^{1/\tau} \left(\frac{1}{\lambda} s_\tau(u) - \frac{y}{\pi \sqrt{1-u^2}} \right) =: \lambda^{1/\tau-1} \tilde{s}(u, \lambda y).$$

Then with $\tilde{p}(y) = \|\tilde{s}^+(u, y)\|$ and $\tilde{n}(y) = \|\tilde{s}^-(u, y)\|$ we have $p_\lambda(y) = \lambda^{-1} \tilde{p}(\lambda y)$ and $n_\lambda(y) = \lambda^{-1} \tilde{n}(\lambda y)$. Moreover, $\tilde{y}(\alpha, \beta; \lambda) = \lambda^{-1} \tilde{y}(\alpha, \beta)$, where $\tilde{y}(\alpha, \beta)$ is the unique solution of the equation $\tilde{p}(y) = (\alpha/\beta) \tilde{n}(y)$. Hence $\lambda^*(\alpha, \beta)$ is the unique solution of the equation $p_\lambda(\lambda^{-1} \tilde{y}(\alpha, \beta)) = \alpha$, that is, $\tilde{p}(\tilde{y}(\alpha, \beta)) = \lambda \alpha$. Therefore

$$\lambda^*(\alpha, \beta) = \tilde{p}(\tilde{y}(\alpha, \beta))/\alpha. \tag{3.24}$$

Here $\tilde{y}(\alpha, \beta) = \tilde{y}(\tau; \alpha, \beta)$ and $\lambda^*(\alpha, \beta) = \lambda^*(\tau; \alpha, \beta)$ depend on τ as well.

Now let $\tau \in [1, 2]$. In this case by Lemma 3.3 for $y \in (0, a_\tau)$ the equation $\tilde{s}(u, y) = 0$ has exactly two solutions $u_1(y) > 0$ and $u_2(y) = -u_1(y)$ in $(-1, 1)$, where $a_\tau := \sup\{y > 0 : \tilde{p}(y) > 0\}$. Then

$$\tilde{p}(y) = 2 \int_0^{u_1(y)} s_\tau(t) dt - (2y/\pi) \sin^{-1}(u_1(y)).$$

From (3.21) for the last integral we obtain

$$I_\tau(u_1(y)) = (1/\tau) u_1(y) s_\tau(u_1(y)) + (1/\pi) \sin^{-1}(u_1(y)),$$

hence

$$\tilde{p}(y) = (2/\tau) u_1(y) s_\tau(u_1(y)) + (2/\pi)(1-y) \sin^{-1}(u_1(y)). \tag{3.25}$$

On the other hand using that $\tilde{p}(y) - \tilde{n}(y) = 1 - y$ we can write the equation $\tilde{p}(y) = (\alpha/\beta) \tilde{n}(y)$ in the form $(\beta - \alpha) \tilde{p}(y) = \alpha(y - 1)$. If $\alpha \neq \beta$ by (3.24) we get

$$\lambda^*(\tau; \alpha, \beta) = \frac{\tilde{y}(\tau; \alpha, \beta) - 1}{\beta - \alpha}. \tag{3.26}$$

If $\alpha = \beta$ then $\tilde{y}(\tau; \alpha, \alpha) = 1$ and by (3.24) and (3.25),

$$\lambda^*(\tau; \alpha, \alpha) = 2u_1(1) s_\tau(u_1(1))/(\alpha\tau). \tag{3.27}$$

We now consider the special case $\tau = 2$. We have $s_2(t) = (2/\pi) \sqrt{1-t^2}$ (see (3.17)) and solving $\tilde{s}(u, y) = 0$ we get $u_{1,2}(y) = \pm \sqrt{1-y/2}$ for $y \in [0, 2)$. Hence by (3.25) we get that $\tilde{y}(2; \alpha, \beta)$ is the solution of the equation

$$(1/\pi)(\beta - \alpha)(\sqrt{y(2-y)} + 2(1-y) \sin^{-1}(\sqrt{1-y/2})) = \alpha(y - 1). \tag{3.28}$$

Then $\lambda^*(2; \alpha, \beta) = (\tilde{y}(2; \alpha, \beta) - 1)/(\beta - \alpha)$ if $\alpha \neq \beta$, and

$$\lambda^*(2; \alpha, \alpha) = \frac{\tilde{p}(\tilde{y}(2; \alpha, \alpha))}{\alpha} = \frac{\tilde{p}(1)}{\alpha} = \frac{1}{\alpha\sqrt{2}} s_2\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\pi\alpha}.$$

Next we show that the Ullman distribution s_τ for $\tau \in [1, 2]$ is monotone on $[0, 1]$.

LEMMA 3.3. *For every $\tau \in [1, 2]$ the Ullman distribution s_τ is a monotone decreasing function on the interval $[0, 1]$.*

Proof. First let $\tau \in (1, 2]$. We will show that $s'_\tau(t) < 0$ on $(0, 1)$ which in view of (3.20) is equivalent to

$$s_\tau(t) < \frac{\tau}{\pi(\tau-1)\sqrt{1-t^2}}, \quad t \in (0, 1),$$

or using (3.19) it is the same as

$$t^{\tau-1} \int_t^1 \frac{u^{-\tau}}{\sqrt{1-u^2}} du < \frac{1}{(\tau-1)\sqrt{1-t^2}}, \quad t \in (0, 1). \quad (3.29)$$

For $u \in [0, 1)$ we have the power series expansion

$$(1-u)^{-1/2} = \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} u^k =: \sum_{k=0}^{\infty} c_k u^k,$$

where $c_0 = 1$ and

$$c_k = \frac{(2k-1)!!}{k! 2^k} = O(k^{-1/2})$$

for large $k \in \mathbf{N}$. Then (3.29) is equivalent to each of the following

$$t^{\tau-1} \sum_{k=0}^{\infty} c_k \int_t^1 u^{2k-\tau} du < \frac{1}{(\tau-1)} \sum_{k=0}^{\infty} c_k t^{2k},$$

$$t^{\tau-1} \sum_{k=0}^{\infty} c_k \left(\frac{1-t^{2k+1-\tau}}{2k+1-\tau} \right) < \frac{1}{(\tau-1)} \sum_{k=0}^{\infty} c_k t^{2k},$$

and

$$\frac{t^{\tau-1}}{(\tau-1)} > \sum_{k=1}^{\infty} c_k \left(\frac{t^{\tau-1}}{(2k+1-\tau)} - \frac{2kt^{2k}}{(\tau-1)(2k+1-\tau)} \right), \quad t \in (0, 1).$$

The last inequality follows from

$$\frac{1}{(\tau - 1)} \geq \sum_{k=1}^{\infty} c_k \frac{1}{(2k + 1 - \tau)}, \quad \tau \in (1, 2]. \tag{3.30}$$

To verify (3.30) we consider the function

$$F(\tau) = \frac{1}{(\tau - 1)} - \sum_{k=1}^{\infty} c_k \frac{1}{(2k + 1 - \tau)}, \quad \tau \in (1, 2].$$

We have $F(\tau) \rightarrow \infty$ as $\tau \rightarrow 1^+$ and

$$F'(\tau) = -(\tau - 1)^{-2} - \sum_{k=1}^{\infty} c_k (2k + 1 - \tau)^{-2} < 0, \quad \tau \in (1, 2].$$

So it is enough to show that $F(2) \geq 0$. Using the same expansion as before we obtain

$$\int_t^1 \frac{u^{-2}}{\sqrt{1 - u^2}} du = \sum_{k=0}^{\infty} c_k \left(\frac{1 - t^{2k-1}}{2k - 1} \right), \quad t \in (0, 1)$$

which implies

$$\sum_{k=1}^{\infty} c_k \frac{1}{(2k - 1)} = 1 + \sum_{k=1}^{\infty} c_k \frac{t^{2k-1}}{(2k - 1)} + \int_t^1 \frac{u^{-2}}{\sqrt{1 - u^2}} du - \frac{1}{t}$$

for $t \in (0, 1)$. Next for $t \in (0, 1)$ from (3.19) we get

$$\int_t^1 \frac{u^{-2}}{\sqrt{1 - u^2}} du - \frac{1}{t} = \frac{1}{t} \left(\frac{\pi}{2} s_2(t) - 1 \right) = \frac{-t}{\sqrt{1 - t^2} + 1}.$$

Taking a limit as $t \rightarrow 0^+$ in the last two equations we obtain

$$F(2) = 1 - \sum_{k=1}^{\infty} c_k \frac{1}{(2k - 1)} = 0.$$

For $\tau = 1$ by (3.17) we get

$$s_1(t) = (1/\pi)(\ln(1/t) + \ln(1 + \sqrt{1 - t^2})), \quad t \in (0, 1],$$

a decreasing function on $(0, 1]$. This completes the proof of Lemma 3.3. ■

REFERENCES

1. P. Borwein, E. A. Rakhmanov, and E. B. Saff, Weighted rational approximation with varying weights, *Constr. Approx.* **2** (1996), 223–240.
2. S. B. Damelin and A. B. J. Kuijlaars, The support of the equilibrium measure in the presence of a monomial external field on $[-1, 1]$, *Trans. Amer. Math. Soc.* **351** (1999), 4561–4584.
3. P. Deift, T. Kriecherbauer, and K. T.-R. McLaughlin, New results on the equilibrium measure for logarithmic potentials in the presence of an external field, *J. Approx. Theory* **95** (1998), 388–475.
4. A. B. J. Kuijlaars, The role of the endpoint in weighted polynomial approximation with varying weights, *Constr. Approx.* **2** (1996), 287–301.
5. A. B. J. Kuijlaars, Weighted approximation with varying weights: the case of a power-type singularity, *J. Math. Anal. Appl.* **204** (1996), 409–418.
6. H. N. Mhaskar and E. B. Saff, Weighted analogues of capacity, transfinite diameter, and Chebyshev constant, *Constr. Approx.* **8** (1992), 105–124.
7. E. A. Rakhmanov, E. B. Saff, and P. C. Simeonov, Rational approximation with varying weights II, *J. Approx. Theory* **92** (1998), 331–338.
8. E. B. Saff and V. Totik, “Logarithmic Potentials with External Fields,” *Grundlehren der mathematischen Wissenschaften*, Vol. 316, Springer-Verlag, Heidelberg, 1997.
9. H. Stahl and V. Totik, “General Orthogonal Polynomials,” *Encyclopedia of Math.*, Vol. 43, Cambridge University Press, New York, 1992.
10. V. Totik, “Weighted Polynomial Approximation with Varying Weights,” *Lecture Notes in Math.*, Vol. 1569, Springer-Verlag, Heidelberg, 1994.