Rank and Nielsen equivalence in hyperbolic extensions

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December 20, 2018

Abstract

In this note, we generalize a theorem of Juan Souto on rank and Nielsen equivalence in the fundamental group of a hyperbolic fibered 3–manifold to a large class of hyperbolic group extensions. This includes all hyperbolic extensions of surfaces groups as well as hyperbolic extensions of free groups by convex cocompact subgroups of $\mathrm{Out}(F_n)$.

1 Introduction

Perhaps the most basic invariant of a finitely generated group is its **rank**, that is, the minimal cardinality of a generating set. Despite its simple definition, rank is notoriously difficult to calculate even for well-behaved groups. For example, work of Baumslag, Miller, and Short [BMS] shows that the rank problem is unsolvable for hyperbolic groups. In this note we calculate the rank for a large class of hyperbolic group extensions and furthermore show that, up to Nielsen equivalence, all minimal (i.e., minimal-cardinality) generating sets are of a standard form.

Let $1 \to H \to G \to \Gamma \to 1$ be an exact sequence of infinite hyperbolic groups. We say that the extension has the **Scott–Swarup property** if each finitely generated, infinite index subgroup of H is quasiconvex as a subgroup of G. Every subgroup $\Delta \le \Gamma$ induces a new short exact sequence $1 \to H \to G_\Delta \to \Delta \to 1$, where G_Δ is the full preimage of Δ under the surjection $G \to \Gamma$. Our main theorem is the following; for the statement $\ell_\Gamma(\cdot)$ denotes conjugacy length with respect to any finite generating set for Γ.

Theorem 1.1. *Let $1 \to H \to G \to \Gamma \to 1$ be an exact sequence of infinite hyperbolic groups that has the Scott–Swarup property and torsion-free kernel H. For every $r \ge 0$ there is an $N \ge 0$ such that if $\Delta \le \Gamma$ is a finitely generated subgroup with $\mathrm{rank}(\Delta) \le r$ and $\ell_\Gamma(\delta) \ge N$ for each $\delta \in \Delta \setminus \{1\}$, then*

$$\mathrm{rank}(G_\Delta) = \mathrm{rank}(H) + \mathrm{rank}(\Delta).$$

Moreover, every minimal generating set for G_Δ is Nielsen equivalent to a generating set which contains a minimal generating set for H and projects to a minimal generating set for Δ.

Examples of subgroups $\Delta \le \Gamma$ satisfying these conditions can easily be constructed. Indeed, for any set $\delta_1, \ldots, \delta_r$ of pairwise independent infinite order elements of Γ, Theorem 1.1 applies to

Key words and phrases: Hyperbolic group extensions, rank, Nielsen equivalence, convex cocompact subgroups

2010 Mathematics Subject Classification: Primary 20F67, 20E22; Secondary 20F65, 20F05, 20F10

*The first named author was supported by NSF grant DMS-1711089; the second named author was supported by NSF grants DMS-1400498 and DMS-1744551.

$\Delta = \langle \delta_1^m, \ldots, \delta_r^m \rangle$ for all sufficiently large m. Alternately, one can build finite-index subgroups $K \leq \Gamma$ such that Theorem 1.1 applies to every rank r subgroup of K.

Theorem 1.1 generalizes a theorem of Juan Souto [Sou], who established this result when $\Gamma \cong \mathbb{Z}$ and H is the fundamental group of a closed orientable surface S_g of genus $g \geq 2$. Here the extension is induced by a hyperbolic S_g–bundle over S^1 with pseudo-Anosov monodromy $f\colon S_g \to S_g$, so that G is the fundamental group of the mapping torus M_f of f. In this language, Souto proves that the rank of $\pi_1(M_{f^N}) \cong G_{\langle f^N \rangle}$ is equal to $2g+1$ for N sufficiently large. Moreover, any two minimal generating sets in this situation are Nielsen equivalent. See also the work of Biringer–Souto [BS] for more on this special case. In this paper, we use techniques previously established by Kapovich and Weidmann [KW2, KW1] to generalize Souto's result to Theorem 1.1.

Theorem 1.1 applies to all hyperbolic extensions of surface groups [FM, Ham, KL] as well as all hyperbolic extensions of free groups by convex cocompact subgroups of $\mathrm{Out}(F_n)$ [DT2, HH, DT1]. We thus obtain the following corollary:

Corollary 1.2. *The conclusions of Theorem 1.1 hold for all extensions of the following forms:*

i. Extensions $1 \to \pi_1(S_g) \to G \to \Gamma \to 1$ with G and Γ both infinite and hyperbolic.

ii. Extensions $1 \to F_g \to G \to \Gamma \to 1$ such that G is hyperbolic and the induced outer action $\Gamma \to \mathrm{Out}(F_g)$ has convex cocompact image.

Proof. Since the kernels of the above extensions are torsion-free, it suffices to verify the Scott–Swarup property. For the surface group extensions in (i), this was established by Scott and Swarup in the case that $\Gamma \cong \mathbb{Z}$ [SS] and by Dowdall–Kent–Leininger in the general case [DKL] (see also [MR]). For the free group extensions in (ii), Mitra [Mit] established the Scott–Swarup property when $\Gamma \cong \mathbb{Z}$ and the general case was proven by the authors in [DT1] and by Mj–Rafi in [MR]. □

We note that Souto's theorem is exactly case (i) above with Γ a cyclic group; the other cases of Corollary 1.2 are all new. In particular, the result is new even for free-by-cyclic groups $G = F \rtimes_\phi \mathbb{Z}$ with fully irreducible and atoroidal monodromy $\phi \in \mathrm{Out}(F_g)$, where the conclusion is that $F_g \rtimes_{\phi^N} \mathbb{Z}$ has rank $g+1$ for all sufficiently large N.

The following counter examples show that neither the torsion-free hypothesis on H nor the Scott–Swarup hypothesis on the extension can be dropped from Theorem 1.1.

Counter Example 1.3 (Lack of Scott–Swarup property)**.** In [Bri, Section 1.1.1], Brinkmann builds a hyperbolic automorphism ϕ of the free group $F = F_m * \langle a_1, \ldots a_n \rangle$, where $m \geq 3$, of the form

$$\phi(F_m) = F_m,$$
$$\phi(a_i) = \begin{cases} a_{i+1} & \text{if } 1 \leq i < n \\ w a_1 v & \text{if } i = n, \end{cases}$$

where $w, v \in F_m$. Notice that the induced extension $G_\phi = F \rtimes_\phi \mathbb{Z}$ does not have the Scott–Swarup property: F_m is not quasiconvex in $F_m \rtimes_\phi \mathbb{Z}$ (which is hyperbolic) and hence not quasiconvex in G_ϕ. Focusing on the case where $n = 2$, one sees that for each k odd, ϕ^k has the property that $\phi^k(a_1) = w_k a_2 v_k$ and $\phi^k(a_2) = w'_k a_1 v'_k$ for some $w_k, v_k, w'_k, v'_k \in F_m$. Hence, when k is odd, G_{ϕ^k} is generated by F_m, a_1, and a generator of $\mathbb{Z}$, making its rank at most $m+2 < \mathrm{rank}(F)+1$.

Counter Example 1.4 (Torsion in H)**.** Here we exploit the failure of Lemma 3.1.iii in the presence of torsion. Fix $m \geq 3$ and a prime q, and let $F, Z \leq H$ be the groups

$$H = \langle a_1, \ldots, a_m, s \mid s^q = 1, [a_i, s] = 1\, \forall i \rangle, \quad F = \langle a_1, \ldots, a_m \rangle \leq H, \quad \text{and} \quad Z = \langle s \rangle \leq H.$$

Thus F is free with $\operatorname{rank}(F) = m$ and H decomposes as a direct product $H = F \times Z$ with $\operatorname{rank}(H) = m+1$ and $[H : F] = q$. Let $\rho \colon H \to F$ denote the projection onto the F factor and $\iota \colon F \to H$ the inclusion of F into H. Let $\beta \colon H \to H$ be the homomorphism defined by the assignments

$$\beta(s) = s \qquad \text{and} \qquad \beta(a_i) = a_i s \quad \text{for each } i = 1, \dots, m.$$

Observe that β^q is the identity, thus β is in fact an automorphism of H. Since $\beta(a_1) \notin F$, we have $\beta(F) \neq F$. Let $\tau \in \operatorname{Aut}(F)$ be any fully irreducible and atoroidal automorphism. Using the product structure of H, set $\alpha = \tau \times \mathrm{id}_Z$. We note that α is an automorphism of H and that

$$\rho\iota = \mathrm{id}_F, \qquad \rho\beta = \rho, \qquad \rho\alpha = \tau\rho, \qquad \text{and} \qquad \rho\alpha\beta = \tau\rho.$$

Now let $\phi = \alpha\beta \in \operatorname{Aut}(H)$ and consider the extension $G = H \rtimes_\phi \mathbb{Z}$. Notice that $\phi(F) \neq F$. However, since H contains only finitely many index q subgroups, we may choose $n > 1$ so that $\phi^n(F) = F$. Let $G_n \leq G$ be the preimage of $n\mathbb{Z}$ under the projection $G \to \mathbb{Z}$; this is an index n subgroup with $G_n \cong H \rtimes_{\phi^n} \mathbb{Z}$. The further subgroup $G_n' \cong F \rtimes_{\phi^n|_F} \mathbb{Z}$ has $[G_n : G_n'] = q$. Since $\phi^n|_F = \rho\phi^n\iota = \tau^n$ is fully irreducible and atoroidal, G_n' is hyperbolic [BF2] and each finitely generated infinite index subgroup of F is quasiconvex in G_n' [Mit]. Since $[G : G_n']$ is finite, our extension $G = H \rtimes_\phi \mathbb{Z}$ is also hyperbolic and has the Scott–Swarup property. However, the extension G does not satisfy the conclusion of Theorem 1.1: For all $k \geq 1$, the observation $\langle F, \phi^{kn+1}(F)\rangle = H$ implies that the subextension $H \rtimes_{\phi^{kn+1}} \mathbb{Z}$ is generated by F and the stable letter and thus rank at most $m + 1 < \operatorname{rank}(H) + 1$.

Acknowledgments: This work drew inspiration from Souto's paper [Sou] and owe's an intellectual debt to the powerful machinery provided by Kapovich and Weidmann [KW1, KW2]. We thank the referee for helpful suggestions.

2 Setup

Fix a group G with a finite, symmetric generating set S and let $X = \operatorname{Cay}(G,S)$ be its Cayley graph. Equip X with the path metric d in which each edge has length 1, making (X,d) into a proper, geodesic metric space. For subsets $A, B \subset X$, define $d(A,B) = \inf\{d(a,b) \mid a \in A, b \in B\}$ and declare the **ε–neighborhood** of A to be $\mathcal{N}_\varepsilon(A) = \{x \in X \mid d(\{x\}, A) < \varepsilon\}$. The **Hausdorff distance** between sets is defined as

$$d_{\mathrm{Haus}}(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{N}_\varepsilon(B) \text{ and } B \subset \mathcal{N}_\varepsilon(A)\}.$$

We identify G with the vertices of X and define the **wordlength** of $g \in G$ by $|g|_S = d(e,g)$, where e is the identity element of G. A **tuple** in G is a (possibly empty) ordered list $L = (g_1, \dots, g_n)$ elements of g. The **length** of a tuple $L = (g_1, \dots, g_n)$ is the number $\ell(L) = n$ of entries of the list, and its **magnitude** is defined to be $\|L\| = \max_i |g_i|_S$; for $h \in G$ we denote the tuple $(hg_1h^{-1}, \dots, hg_nh^{-1})$ by hLh^{-1}. We define the **conjugacy magnitude** of a tuple L to be $\mathcal{C}(L) = \min_{h \in G} \left\|hLh^{-1}\right\|$. The following three operations are called **elementary Nielsen moves** on a tuple $L = (g_1, \dots, g_n)$:

- For some $i \in \{1, \dots, n\}$, replace g_i by g_i^{-1} in L.
- For some $i, j \in \{1, \dots, n\}$ with $i \neq j$, interchange g_i and g_j in L.
- For some $i, j \in \{1, \dots, n\}$ with $i \neq j$, replace g_i by $g_i g_j$ in L.

Two tuples are **Nielsen equivalent** if one may be transformed into the other via a finite chain of elementary Nielsen moves. Nielsen proved that any two minimal generating sets of a finitely generated free group are Nielsen equivalent [Nie]. Hence, two tuples L_1 and L_2 of length n are Nielsen equivalent if and only if there is an automorphism $\psi\colon F_n \to F_n$ such that $\phi_1 = \phi_2 \circ \psi$, where $\phi_i\colon F_n \to G$ is the homomorphism taking the jth element of a (fixed) basis for F_n to the jth element of L_i. Note that Nielsen equivalent tuples generate the same subgroup of G.

Following Kapovich–Weidmann [KW2, Definition 6.2], we consider the following variation:

Definition 2.1. A **partitioned tuple** in G is a list $M = (Y_1,\dots,Y_s;T)$ of tuples $Y_1,\dots,Y_s,T$ of G with $s \geq 0$ such that (1) either $s > 0$ or $\ell(T) > 0$, and (2) $\langle Y_i\rangle \neq \{e\}$ for each $i > 0$. Thus $(;T)$ (where $\ell(T) > 0$) and $(Y_1;)$ (where $\langle Y_1\rangle \neq \{e\}$) are examples of partitioned tuples. The length of M is defined to be $\ell(M) = \ell(Y_1)+\dots+\ell(Y_s)+\ell(T)$. The **underlying tuple** of M is the $\ell(M)$–tuple $\mathcal{U}(M) = (Y_1,\dots,Y_s,T)$ obtained by concatenating $Y_1,\dots,Y_s,T$. The **elementary moves** on a partitioned tuple $M = (Y_1,\dots,Y_s;(t_1,\dots,t_n))$ consist of:

- For some $i \in \{1,\dots,s\}$ and $g \in \langle(\cup_{j\neq i}Y_j)\cup\{t_1,\dots,t_n\}\rangle$, replace Y_i by gY_ig^{-1}.
- For some $k \in \{1,\dots,n\}$ and elements $u,u' \in \langle(\cup_j Y_j)\cup\{t_1,\dots,t_{k-1},t_{k+1},\dots,t_n\}\rangle$, replace t_k by ut_ku'.

Two partitioned tuples M and M' are **equivalent** if M can be transformed into M' via a finite chain of elementary moves. In this case, it is easy to see that the underlying tuples $\mathcal{U}(M)$ and $\mathcal{U}(M')$ are Nielsen equivalent.

We henceforth assume that G is a **hyperbolic group**, which is equivalent to requiring that X be **δ–hyperbolic** for some fixed $\delta \geq 0$. This means that every geodesic triangle $\triangle(a,b,c)$ in X is **δ–thin** in the sense that each side is contained in the δ–neighborhood of the union of the other two. A **geodesic** in X is a map $\gamma\colon \mathbf{J} \to X$ of an interval $\mathbf{J} \subset \mathbb{R}$ such that $|s-t| = d(\gamma(s),\gamma(t))$ for all $s,t \in \mathbf{J}$. Two geodesic rays $\gamma_1,\gamma_2\colon \mathbb{R}_+ \to X$ are **asymptotic** if $d_{\mathrm{Haus}}(\gamma_1(\mathbb{R}_+),\gamma_2(\mathbb{R}_+)) < \infty$. The **Gromov boundary** of X is defined to be the set ∂X of equivalence classes of geodesic rays in X. Note that every isometry of X induces a self-bijection of ∂X. The equivalence class or **endpoint** of a ray $\gamma\colon \mathbb{R}_+ \to X$ is denoted $\gamma(\infty) \in \partial X$, and γ is said to **join** $\gamma(0)$ to $\gamma(\infty)$. A biinfinite geodesic $\gamma\colon \mathbb{R} \to X$ determines two rays and is said to **join** their respective endpoints $\gamma(-\infty)$ and $\gamma(\infty)$. The fact that X is proper and δ–hyperbolic ensures that any two points of $X \cup \partial X$ can be joined by a geodesic segment, ray, or line; see [KB, KW1]. The **convex hull** of a set $Y \subset X \cup \partial X$ is the union $\mathrm{Conv}(Y)$ of all geodesics joining points of Y (including degenerate geodesics of the form $\{0\} \to Y$). The set Y is **ε–quasiconvex** if $\mathrm{Conv}(Y) \subset \mathcal{N}_\varepsilon(Y)$. A subgroup $U \leq G$ is ε–quasiconvex if it is so when viewed as a subset of X. We refer the reader to [Gro, GdlH, BH] for further background on hyperbolic groups.

A sequence $\{x_n\}$ in X is said to **converge** to $\zeta \in \partial X$ if for some (equivalently every) geodesic $\gamma\colon \mathbb{R}_+ \to X$ in the class ζ and sequence $\{t_m\}$ in $\mathbb{R}_+$ with $t_m \to \infty$, one has

$$\lim_{n,m}\left(d(x_n,x_0)+d(\gamma(t_m),x_0)-d(x_n,\gamma(t_m))\right) = \infty.$$

The **limit set** of a subgroup $U \leq G$ is the set $\Lambda(U)$ accumulation points $\zeta \in \partial X$ of an orbit $U\cdot x_0 \subset X$; the fact that any two orbits of U have finite Hausdorff distance implies that this is independent of the point x_0. Following Kapovich–Weidmann [KW1, Definition 4.2] we define the **hull** of a subgroup U to be

$$\mathcal{H}(U) = \overline{\mathrm{Conv}\Big(\mathrm{Conv}\left(\Lambda(U)\cup\{x\in X \mid d(x,u\cdot x)\leq 100\delta \text{ for some } u\in U\setminus\{e\}\}\right)\Big)}.$$

We leave the following fact as an exercise for the reader. Alternatively, it follows from a slight modification of [KW1, Lemma 4.10 and Lemma 10.3].

Lemma 2.2. *There is a constant $A = A(\varepsilon)$ for each $\varepsilon \geq 0$ such that $d_{\text{Haus}}(U, \mathcal{H}(U)) \leq A$ for every torsion-free ε–quasiconvex subgroup U of G.*

By noting that there are only finitely many subgroups of G that may be generated by elements from the finite set $\mathcal{N}_r(\{e\})$, we have the following lemma:

Lemma 2.3. *There is a constant $c = c(r)$ for each $r > 0$ such that every quasiconvex subgroup $U \leq G$ generated by elements from the r–ball $\mathcal{N}_r(\{e\})$ is c–quasiconvex.*

The following technical result of Kapovich and Weidmann is a key ingredient in our argument:

Theorem 2.4 (Kapovich–Weidmann [KW2, Theorem 6.7], c.f. [KW1, Theorem 2.4])**.** *For every $m \geq 1$ there exists a constant $K = K(m) \geq 0$ with the following property. Suppose that $M = (Y_1, \dots, Y_s; T)$ is a partitioned tuple in G with $\ell(M) = m$ and let $H = \langle \mathcal{U}(M) \rangle$ be the subgroup generated by the underlying tuple of M. Then either*

$$H = \langle Y_1 \rangle * \cdots * \langle Y_s \rangle * \langle T \rangle,$$

with $\langle T \rangle$ free on the basis T, or else M is equivalent to a partitioned tuple $M' = (Y'_1, \dots, Y'_s; T')$ for which one of the following occurs:

1. *There are $i, j \in \{1, \dots, s\}$ with $i \neq j$ and $d(\mathcal{H}(\langle Y'_i \rangle), \mathcal{H}(\langle Y'_j \rangle)) \leq K$.*
2. *There is some $i \in \{1, \dots, s\}$ and $t \in T'$ such that $d(\mathcal{H}(\langle Y'_i \rangle), t \cdot \mathcal{H}(\langle Y'_i \rangle)) \leq K$.*
3. *There exists an element $t \in T'$ with a conjugate in G of wordlength at most K.*

We conclude this section with the following lemma, which ties into the conclusions of Theorem 2.4 and is an adaptation of [KW2, Propositions 7.3–7.4] to our context. Since the hypotheses of [KW2] are not satisfied here, we include a short proof.

Lemma 2.5. *For every $K, r > 0$ there is a constant $B = B(K, r)$ with the following property: Let Y_1, Y_2, Y_3 be tuples in G generating torsion-free quasiconvex subgroups $U_i = \langle Y_i \rangle$ and satisfying $\mathcal{C}(Y_i) \leq r$ for each $i = 1, 2, 3$.*

- *If $d(\mathcal{H}(U_1), \mathcal{H}(U_2)) \leq K$, then (Y_1, Y_2) is Nielsen equivalent to a tuple Y satisfying $\mathcal{C}(Y) \leq B$.*
- *If $d(\mathcal{H}(U_3), g \cdot \mathcal{H}(U_3)) \leq K$ for $g \in G$, then $(Y_3, (g))$ is Nielsen equivalent to a tuple Z with $\mathcal{C}(Z) \leq B$.*

Proof. For brevity, we prove the claims simultaneously. By assumption, we may choose points $x_1 \in \mathcal{H}(U_1)$, $x_2 \in \mathcal{H}(U_2)$ and $z_3, z_4 \in \mathcal{H}(U_3)$ with $d(x_1, x_2) \leq K$ and $d(z_3, gz_4) \leq K$. For $i = 1, 2, 3$, we also choose $h_i \in G$ such that $\left\| h_i Y_i h_i^{-1} \right\| \leq r$. The subgroups $U'_i = h_i U_i h_i^{-1}$ are then $c(r)$–quasiconvex by Lemma 2.3 and hence satisfy $d_{\text{Haus}}(U'_i, \mathcal{H}(U'_i)) \leq A(c(r))$ by Lemma 2.2. Noting that $\mathcal{H}(U'_i) = h_i \mathcal{H}(U_i)$, we may choose $u_i \in U_i$ for $i = 1, 2$ such that $d(h_i u_i h_i^{-1}, h_i x_i) \leq A(c(r))$. Similarly choose $w_j \in U_3$ so that $d(h_3 w_j h_3^{-1}, h_3 z_j) \leq A(c(r))$ for $j = 3, 4$. Set $B = 4A(c(r)) + 2K + r$.

To conclude the second claim, observe that

$$\begin{aligned} \left| h_3 (w_3^{-1} g w_4) h_3^{-1} \right|_S &= d(w_3 h_3^{-1}, g w_4 h_3^{-1}) \\ &\leq d(w_3 h_3^{-1}, z_3) + d(z_3, g z_4) + d(g z_4, g w_4 h_3^{-1}) \\ &\leq 2A(c(r)) + K. \end{aligned}$$

Since $\|h_3 Y_3 h_3^{-1}\| \le r$ as well, the concatenated tuple $Z = (Y_3, (w_3^{-1} g w_4))$ clearly satisfies $\mathcal{C}(Y') \le B$. Further, since $w_3, w_4 \in \langle Y_3 \rangle$, it is immediate that Z is Nielsen equivalent to $(Y_3, (g))$.

For the first claim, set $f = h_1 u_1^{-1} u_2 h_2^{-1}$ and use the triangle inequality to observe

$$\begin{aligned}|f|_S &= d(u_1 h_1^{-1}, u_2 h_2^{-1}) \\ &\le d(u_1 h_1^{-1}, x_1) + d(x_1, x_2) + d(x_2, u_2 h_2^{-1}) \\ &\le 2A(c(r)) + K.\end{aligned}$$

Since $\|h_2 Y_2 h_2^{-1}\| \le r$, another use of the triangle inequality gives

$$\|h_1(u_1^{-1} u_2 Y_2 u_2^{-1} u_1) h_1^{-1}\| = \|f(h_2 Y_2 h_2^{-1}) f^{-1}\| \le 4A(c(r)) + 2K + r = B.$$

The concatenated tuple $Y = (Y_1, u_1^{-1} u_2 Y_2 u_2 u_1^{-1})$ thus evidently satisfies $\mathcal{C}(Y) \le B$. To complete the proof, it only remains to show that (Y_1, Y_2) is Nielsen equivalent to Y. But this is clear: since $u_2 \in \langle Y_2 \rangle$ the tuple (Y_1, Y_2) is equivalent to $(Y_1, u_2 Y_2 u_2^{-1})$ which, since $u_1^{-1} \in \langle Y_1 \rangle$, is in turn equivalent to Y. □

3 Proof of the main result

Suppose now that our fixed group G fits into a short exact sequence

$$1 \longrightarrow H \longrightarrow G \xrightarrow{p} \Gamma \longrightarrow 1 \qquad (1)$$

of infinite hyperbolic groups that enjoys the Scott–Swarup property with torsion-free kernel H. Recall that the conjugation action of G on H induces a homomorphism $\Phi\colon \Gamma \to \mathrm{Out}(H)$ and that, since G is hyperbolic, Φ has finite kernel. For any subgroup $\Delta \le \Gamma$, we set $G_\Delta = p^{-1}(\Delta) \le G$, and note that this subgroup of G fits into the sequence $1 \to H \to G_\Delta \to \Delta \to 1$.

The follow lemma summarizes some of the basic properties we will require.

Lemma 3.1. *For the sequence (1), we have the following:*

i. For every infinite order $g \in \Gamma$, $\Phi(g) \in \mathrm{Out}(H)$ does not fix the conjugacy class of any infinite index, finitely generated subgroup of H.

ii. The kernel H is either free of rank at least 3 or else isomorphic to the fundamental group of a closed surface of genus at least 2.

iii. Every proper subgroup $U \lneq H$ is either quasiconvex in G or else has $\mathrm{rank}(U) > \mathrm{rank}(H)$.

Proof. To prove item (i), suppose towards a contradiction that $g \in \Gamma$ of infinite order fixes the conjugacy class of an infinite index, finitely generated subgroup A of H. Then, after applying an inner automorphism of H, we see that the semidirect product $A \rtimes_\phi \mathbb{Z}$ is contained in G, where ϕ is an automorphism in the class $\Phi(g)$. However, it is well-known that the subgroup A is distorted (i.e. not quasi-isometrically embedded) in $A \rtimes_\phi \mathbb{Z}$ and hence distorted in G. This, however, contradicts the Scott–Swarup property and proves item (i).

Next, the theory of JSJ decompositions for hyperbolic groups [RS] (see also [Lev]) shows that a sequence of hyperbolic groups as in (1) with torsion-free kernel H must have H isomorphic to the free product $(*_{i=1}^k \Sigma_i) * F_n$, where F_n is free of rank n and each Σ_i is the fundamental group of a closed surface. We must show that this factorization is trivial, i.e. either $k = 0$ or $n = 0$. This follows from the fact that such a nontrivial free product decomposition is canonical (e.g. [SW, Theorem 3.5]) and so is preserved under *any* automorphism of H (up to permuting the factors). Hence, for each infinite order $g \in \Gamma$, some power of $\Phi(g)$ fixes the conjugacy class of a surface group factor of H, contradicting item (i) above unless $k = 0$ or $n = 0$. This proves (ii).

For (iii), let $J = [U : H] > 1$. If $J = \infty$, then U is quasiconvex in G by the Scott–Swarup property. Otherwise basic covering space theory implies $\text{rank}(U) = m(1-J) + J\,\text{rank}(H)$ for $m \in \{1,2\}$ depending, respectively, on whether H is free or the fundamental group of a closed surface. $\square$

The following lemma is essential proven in [KK, Corollary 11] in the case where H is free and Γ is cyclic. We sketch the argument for the reader.

Lemma 3.2. *If $1 \to H \to G \to \Gamma \to 1$ is a sequence of infinite hyperbolic groups such that H is torsion-free and G has the Scott–Swarup property, then G does not split over a cyclic (or trivial) group. Moreover, the same holds for $G_\Delta \le G$ whenever the subgroup $\Delta \le \Gamma$ is infinite.*

Proof. We prove the moreover statement since it is clearly stronger. Let $\Delta \le \Gamma$ be an infinite subgroup. Suppose towards a contradiction that G_Δ has a minimal, nontrivial action on a simplicial tree T with cyclic (or trivial) edge stabilizers. Since H is normal in G_Δ, the action $H \curvearrowright T$ is also minimal. Hence the main theorem of [BF1], implies that T/H is a finite graph. Notice that Δ acts on the corresponding graph of groups decomposition of H (via $\Phi\colon \Gamma \to \text{Out}(H)$). First, this decomposition must have trivial edge groups: an infinite cyclic edge stabilizer would be fixed under some infinite order $g \in \Delta \le \Gamma$, contradicting that G is hyperbolic. Hence, the nontrivial graph of groups T/H has trivial edge stabilizers, but this implies that Δ virtually fixes this splitting of H. From this we obtain an infinite order element $g \in \Delta \le \Gamma$ which fixes a vertex group A of the splitting. Since A is finitely generated and has infinite index in H, we have a contradiction to Lemma 3.1.i. This completes the proof. $\square$

Let us establish notation and specify the constants for the proof Theorem 1.1. Let $\bar{S} \subset \Gamma$ be the image of our fixed generating set $S \subset G$. We assume that $\ell_\Gamma(\cdot)$ is conjugacy length in Γ with respect to $\bar{S}$. For the given r, let K be the maximum of the constants $K(1), \ldots, K(\text{rank}(H)+r)$ provided by Theorem 2.4. Set $D_0 = K$ and use Lemma 2.5 recursively to define $D_{n+1} = \max\{D_n, B(K, D_n)\}$ for each $n \in \mathbb{N}$. Set $N = 1 + D_{2\,\text{rank}(H)}$ and suppose that $\Delta \le \Gamma$ is any subgroup with $\text{rank}(\Delta) \le r$ and $\ell_\Gamma(\delta) \ge N$ for all $\delta \in \Delta \setminus \{1\}$. Let G_Δ be the preimage of Δ under the projection $p\colon G \to \Gamma$. We make the following observations:

Lemma 3.3. *If Y is a tuple in G with $Y \subset G_\Delta$ and $\mathcal{C}(Y) < N$, then $\langle Y \rangle \le H$.*

Proof. Choose $g \in G$ so that $\|gYg^{-1}\| < N$. Then for each $y \in Y$ we have

$$|p(g)p(y)p(g)^{-1}|_{\bar{S}} = |p(gyg^{-1})|_{\bar{S}} \le |gyg^{-1}|_S < N$$

which shows that $\ell_\Gamma(p(y)) < N$. Since we also have $p(y) \in \Delta$ by assumption, this gives $p(y) = 1$ and hence $y \in H$ by the hypothesis on Δ. Thus $\langle Y \rangle \le H$. $\square$

Lemma 3.4. *Fix $n \in \{0, \ldots, 2\,\text{rank}(H)-1\}$ and suppose that $M = (Y_1, \ldots, Y_s; T)$ is a partitioned tuple with $\langle \mathcal{U}(M) \rangle = G_\Delta$ and $\ell(M) \le (\text{rank}(H)+r)$ such that for each $i \in \{1, \ldots s\}$ we have $\mathcal{C}(Y_i) \le D_n$ with $\langle Y_i \rangle$ quasiconvex. Then there is a partitioned tuple $\tilde{M} = (\tilde{Y}_1, \ldots \tilde{Y}_{\tilde{s}}; \tilde{T})$ satisfying $\mathcal{C}(\tilde{Y}_j) \le D_{n+1}$ for each $j \in \{1, \ldots, \tilde{s}\}$ such that $\mathcal{U}(\tilde{M})$ is Nielsen equivalent to $\mathcal{U}(M)$ and either*

a. *$\ell(\tilde{T}) < \ell(T)$ with $\tilde{s} \le s+1$ or else*

b. *$\ell(\tilde{T}) = \ell(T)$ with $\tilde{s} < s$.*

Proof. Since $\ell(M) \leq \text{rank}(H) + r$ and $\langle \mathcal{U}(M) \rangle = G_\Delta$ does not split as a nontrivial free product (Lemma 3.2), we may apply Theorem 2.4 to obtain a partitioned tuple $M' = (Y_1', \ldots, Y_s'; T')$ that is equivalent to M and satisfies one of the three conclusions of that theorem. Since all elementary moves on a partitioned tuple $(W_1, \ldots, W_p; V)$ preserve the conjugacy class of each tuple W_i, we have $\mathcal{C}(Y_i') \leq D_n$ with $\langle Y_i' \rangle$ quasiconvex for each i. As $D_n < N$, Lemma 3.3 gives $\langle Y_i' \rangle \leq H$ and so ensures that $\langle Y_i' \rangle$ is torsion-free.

We now analyze the conclusions of Theorem 2.4: If M' satisfies conclusion (1), then after reordering we may assume $d(\mathcal{H}(\langle Y_1' \rangle), \mathcal{H}(\langle Y_2' \rangle)) \leq K$ and use Lemma 2.5 to find a tuple Y Nielsen equivalent to (Y_1', Y_2') with $\mathcal{C}(Y) \leq D_{n+1}$. The partitioned tuple $(Y, Y_3', \ldots, Y_s'; T')$ then satisfies the claim. If M satisfies (2), then after reordering we have $d(\mathcal{H}(\langle Y_1' \rangle), t \cdot \mathcal{H}(\langle Y_1' \rangle)) \leq K$ for some $t \in T'$ and so may use Lemma 2.5 to find a tuple Z equivalent to $(Y_1', (t))$ with $\mathcal{C}(Z) \leq D_{n+1}$. Here we take $\tilde{M} = (Z, Y_2', \ldots, Y_s'; T' \setminus \{t\})$ to complete the claim. If M' satisfies (3), then T' contains an element t with $\mathcal{C}((t)) \leq K \leq D_{n+1}$ and the partitioned tuple $(Y_1', \ldots, Y_s', (t); T' \setminus \{t\})$ satisfies the claim. $\square$

The pieces are now in place to prove our main theorem:

Proof of Theorem 1.1. Let L be any minimal-length tuple with $\langle L \rangle = G_\Delta$. Since G_Δ has a standard generating set of size $\text{rank}(H) + \text{rank}(\Delta)$, we have $\ell(L) \leq \text{rank}(H) + r$. Set $M_0 = (;L)$ and observe that M_0 satisfies Lemma 3.4 with $n = 0$. We may therefore inductively apply Lemma 3.4 (with $n = 0, 1, \ldots$) to obtain a sequence $M_0, M_1, \ldots$ of partitioned tuples each with $\mathcal{U}(M_i)$ Nielsen equivalent to L. After inducting as many times as possible, we obtain a partitioned tuple $M_k = (Y_1, \ldots, Y_s; T)$ that satisfies $\mathcal{C}(Y_i) \leq D_k$ for each i (by construction) but violates the hypotheses of Lemma 3.4, either because $k = 2\,\text{rank}(H)$ or because some $\langle Y_i \rangle$ fails to be quasiconvex. Since $\mathcal{C}(Y_i) \leq D_k < N$, Lemma 3.3 ensures that $\langle Y_i \rangle \leq H$ for each i. Since $G_\Delta = \langle \mathcal{U}(M_k) \rangle$ surjects onto Δ, it follows that $\ell(T) \geq \text{rank}(\Delta)$. Thus at most $\ell(L) - \text{rank}(\Delta)$ applications of Lemma 3.4 could have reduced the length of T (option a) and so at least $k - \ell(L) + \text{rank}(\Delta)$ applications must have combined Y_i's (option b). It now follows that $k < 2\,\text{rank}(H)$, for otherwise k applications of the claim would necessarily produce a tuple Y_i with $\ell(Y_i) > \text{rank}(H)$, contradicting $\ell(Y_i) + \ell(T) \leq \text{rank}(H) + \text{rank}(\Delta)$.

Since M_k violates Lemma 3.4 but $k < 2\,\text{rank}(H)$, it must be that some $\langle Y_i \rangle$ fails to be quasiconvex. After reordering, let us assume $\langle Y_1 \rangle \leq H$ is not quasiconvex. Note that we also cannot have $\text{rank}(\langle Y_i \rangle) > \text{rank}(H)$, for otherwise $\ell(Y_i) + \ell(T) > \text{rank}(H) + \text{rank}(\Delta)$ contradicting our choice of L. The only possibility afforded by Lemma 3.1.iii is therefore $\langle Y_1 \rangle = H$ with $\ell(Y_1) = \text{rank}(H)$. Since $\ell(M_k) \leq \text{rank}(H) + \text{rank}(\Delta)$, it follows that M_k is of the form $M_k = (Y_1; T)$ with $\ell(Y_1) = \text{rank}(H)$ and $\ell(T) = \text{rank}(\Delta)$. Therefore M_k is a standard generating set for G_Δ that is Nielsen equivalent to L. $\square$

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