Partial Differential Equations MATH 334, Spring 2012; T-Th 9:35–10:50, Stevenson SC 1431

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E.DiBenedetto

Office Hours: 1416 Stevenson, T,Th, Fri 12:00-13:00 or by Appointment

Course Description

The course is an interplay of Partial Differential Equations with an array of analytical techniques needed to their weak solvability. Solvability of linear and quasi-linear elliptic equations with measurable coefficients, motivates the theory of Sobolev spaces, traces and their embeddings. The theory leads to DeGiorgi classes and their properties, including the Harnack inequality. The course includes some topics of BMO and limiting Sobolev embeddings. If time permits we will introduce the basic theory of weak solvability of quasi-linear parabolic equations and the basic properties of their solutions. Depending on time we will introduce the basic Hopf theory of solvability of Hamilton-Jacobi equations, based on their connection to Classical Mechanics.

The items marked with ** will be omitted in a first pass and covered if there is interest.

The course assumes a working knowledge of Real Analysis, Measure Theory, and L^p-spaces.

The main part of the course will consists of (see syllabus below)

- Linear Elliptic Equations with Measurable Coefficients
- DeGiorgi Classes
- Introduction to Sobolev Spaces Part 1
- Introduction to Sobolev Spaces Part 2

One of more of the following topics will be chosen according to the interest of the students (see syllabus below)

- Topics in Harmonic Analysis, BMO and Limiting Sobolev Embeddings
- Quasi-Linear Equations of the First Order
- Non-Linear Equations of the First Order

TEXTBOOK AND SUGGESTED BOOKS

No textbook will be required. I will pass along class notes which should be sufficient. The list of books below is meant to complement the lectures and for consultation.

SYLLABUS SPRING 2012

Linear Elliptic Equations With Measurable Coefficients

1. Weak Formulations and Weak Derivatives
2. Embeddings of $W^{1,p}(E)$. See Below Syllabus on Sobolev Spaces
3. Some Consequences of the Multiplicative Embedding Inequalities
4. The Homogeneous Dirichlet Problem
5. Solving the Homogeneous Dirichlet Problem by the Riesz Representation Theorem
6. Solving the Homogeneous Dirichlet Problem by Variational Methods
7. The Case $N = 2$
8. Gâteaux Derivatives and The Euler Equation of a Variational Functional
9. Solving the Homogeneous Dirichlet Problem by Galerkin Approximations
10. On the Selection of an Orthonormal System in $W_o^{1,2}(E)$
11. Conditions on $\mathbf{f}$ and f for the Solvability of the Dirichlet Problem
12. Traces on ∂E of Functions in $W^{1,p}(E)$. See Below Syllabus on Sobolev Spaces
13. The Inhomogeneous Dirichlet Problem
14. The Neumann Problem
15. The Eigenvalue Problem
16. Constructing Eigenvalues
17. The Sequence of Eigenvalues and Eigenfunctions
18. A Priori $L^\infty(E)$ Estimates for Solutions of the Dirichlet Problem
19. An Auxiliary Lemma on Fast Geometric Convergence
20. A Priori $L^\infty(E)$ Estimates for Solutions of the Neumann Problem
21. Miscellaneous Remarks on Further Regularity
22. **Problems and Complements**

DeGiorgi Classes

1. Quasi-Linear Equations and DeGiorgi Classes
2. DeGiorgi Classes
3. Local Boundedness of Functions in the DeGiorgi Classes
4. Hölder Continuity of Functions in the DeGiorgi Classes
5. Estimating the Values of u by the Measure of the Set Where u Is Either Large or Small
6. Reducing the Measure of the Set Where u Is Either Large or Small
7. The Discrete Isoperimetric Inequality

8. Proof of the Hölder Continuity Theorem
9. Boundary DeGiorgi Classes: Dirichlet Data
10. Continuity up to ∂E of Functions in the Boundary DG Classes (Dirichlet Data)
11. Boundary DeGiorgi Classes: Neumann Data
12. Continuity up to ∂E of Functions in the Boundary DG Classes (Neumann Data)
13. The Harnack Inequality
14. Expansion of Positivity
15. Proof of the Harnack Inequality
16. Harnack Inequality and Hölder Continuity
17. Local Clustering of the Positivity Set of Functions in $W^{1,1}(E)$
18. A Proof of the Harnack Inequality Independent of Hölder Continuity

Introduction to Sobolev Spaces. Part 1

Spaces of Continuous Functions, Distributions, and Weak Derivatives

1. ** Bounded Linear Functionals on $C_o(\mathbb{R}^N)$
2. ** Positive Linear Functionals on $C_o(\mathbb{R}^N)$
3. ** Relating Radon Measures and Positive Functionals
4. ** Partition of Unity
5. ** Proof of the Representation Theorem: constructing μ
6. ** Proof of the Representation Theorem: representing a functional
7. ** Characterizing Bounded Linear Functionals on $C_o(\mathbb{R}^N)$
8. ** A Topology for $C_o^\infty(E)$ for an Open set $E \subset \mathbb{R}^N$
9. ** A Metric Topology for $C_o^\infty(E)$ and $D(E)$
10. ** $D(E)$ is not Complete
11. ** A Topology for $C_o^\infty(K)$ for a Compact Set $K \subset E$ and $\mathcal{D}(K)$
12. ** $\mathcal{D}(K)$ is Complete
13. ** Relating the Topology of $D(E)$ to the Topology of $\mathcal{D}(K)$
14. ** The Schwartz Topology of $\mathcal{D}(E)$
15. ** $\mathcal{D}(E)$ is Complete
16. ** Cauchy Sequences in $\mathcal{D}(E)$ and Completeness
17. ** The Topology of $\mathcal{D}(E)$ is not Metrizable
18. ** Continuous Maps and Functionals
19. ** Distributions in E

20. ** Continuous Linear Maps $T:\mathcal{D}(E)\to\mathcal{D}(E)$

21. Distributional Derivatives

22. Fundamental Solutions

23. The Fundamental Solution of the Wave Operator in $\mathbb{R}^2$

24. The Fundamental Solution of the Laplace Operator

25. Weak Derivatives and Main Properties

26. Domains and their Boundaries

27. ∂E of Class C^1

28. Positive Geometric Density and ∂E Piecewise Smooth

29. The Segment Property

30. The Cone Property

31. On the Various Properties of ∂E

32. More on Smooth Approximations

33. Extensions into $\mathbb{R}^N$

34. The Chain Rule

35. Steklov Averagings

36. Characterizing $W^{1,p}(E)$ for $1<p<\infty$

37. Remarks on $W^{1,\infty}(E)$

38. The Rademacher's Theorem

Introduction to Sobolev Spaces. Part 2

Embeddings of $W^{1,p}(E)$ into $L^q(E)$

1. Multiplicative Embeddings of $W_o^{1,p}(E)$

2. Proof of Theorem for $N=1$

3. Proof of Theorem for $1\le p<N$

4. Proof of Theorem for $p\ge N>1$

5. On the Limiting Case $p=N$

6. Embeddings of $W^{1,p}(E)$

7. Poincaré Inequalities

8. Multiplicative Poincaré Inequalities

9. The Discrete Isoperimetric Inequality

10. Morrey Spaces

11. Embeddings for Functions in the Morrey Spaces

12. Limiting Embedding of $W^{1,N}(E)$
13. Compact Embeddings
14. Fractional Sobolev Spaces in $\mathbb{R}^N$
15. Traces
16. Traces and Fractional Sobolev Spaces
17. Traces on ∂E of Functions in $W^{1,p}(E)$
18. Traces and Fractional Sobolev Spaces
19. Multiplicative Embeddings of $W^{1,p}(E)$
20. Proof of the Multiplicative Theorem. A Special Case
21. Constructing a Map Between E and Q
22. Multiplicative Embeddings of $W_o^{1,p}(E)$
23. Compact Embeddings
24. Traces and Fractional Sobolev Spaces
25. Characterizing Functions in $W^{1-\frac{1}{p},p}(\mathbb{R}^N)$ as Traces
26. Traces on ∂E of Functions in $W^{1,p}(E)$
27. Traces on a Sphere

Topics on Harmonic Analysis, BMO and Limiting Sobolev Embedding

1. A Vitali-Type Covering
2. The Maximal Function
3. Strong L^p Estimates for the Maximal Function
4. Estimates of Weak and Strong Type
5. The Calderón-Zygmund Decomposition Theorem
6. Functions of Bounded Mean Oscillation
7. Proof of John-Nirenberg Inequality
8. The Sharp Maximal Function
9. Proof of the Fefferman-Stein Theorem
10. The Marcinkiewicz Interpolation Theorem
11. Quasi–Linear Maps and Interpolation
12. Proof of the Marcinkiewicz Theorem
13. Rearranging the Values of a Function
14. Basic Properties of Rearrangements

15. Symmetric Rearrangements

16. A Convolution Inequality for Rearrangements

17. Approximations by Simple Functions

18. Reduction to Finite Union of Intervals

19. Proof of Theorem 14.1. The Case $T + S \leq R$

20. Proof of Theorem 14.1. The Case $S + T > R$

21. Hardy's Inequality

22. A Convolution-Type Inequality

23. Proof of Hardy's Inequality

24. Equivalent Forms

25. N-Dimensional Versions

26. L^p Estimates of Riesz Potentials

27. The Limiting Case $p = N$

28. Functions of Bounded Mean Oscillation

29. Generalized Riesz Potentials

30. Motivating Riesz Potentials as Embeddings

31. The Limiting Case $p = \frac{N}{\alpha}$

Quasi-Linear Equations of First Order

1. Quasi-Linear Equations

2. The Cauchy Problem

3. The Case of Two Independent Variables

4. The Case of N Independent Variables

5. Solving the Cauchy Problem

6. Constant Coefficients

7. Solutions in Implicit Form

8. Equations in Divergence Form and Weak Solutions

9. Surfaces of Discontinuity

10. The Shock Line

11. The Initial Value Problem

12. Conservation Laws

13. Conservation Laws in One Space Dimension

14. Weak Solutions and Shocks

15. Lack of Uniqueness

16. Hopf Solution of The Burgers Equation

17. Weak Solutions When $a(\cdot)$ is Strictly Increasing

18. Lax Variational Solution

19. Constructing Variational Solutions

20. The Theorems of Existence and Stability

21. The Representation Formula

22. Initial Datum in the Sense of $L^1_{\text{loc}}(\mathbb{R})$

23. The Entropy Condition

24. Entropy Solutions

25. Variational Solutions Are Entropy Solutions

26. Remarks on the Shock and the Entropy Conditions

27. The Kruzhkov Uniqueness Theorem

28. Proof of the Uniqueness Theorem

29. Stability in $L^1(\mathbb{R}^N)$

30. The Maximum Principle for Entropy Solutions

31. **Problems and Complements**

Non-Linear Equations of First Order

1. Integral Surfaces and Monge's Cones

2. Constructing Monge's Cones

3. The Symmetric Equation of Monge's Cones

4. Characteristic Curves and Characteristic Strips

5. Characteristic Strips

6. The Cauchy Problem

7. Identifying the Initial Data

8. Constructing the Characteristic Strips

9. Solving the Cauchy Problem

10. A Quasi-Linear Example in $\mathbb{R}^2$

11. The Cauchy Problem for the Equation of Geometrical Optics

12. Wave Fronts, Light Rays, Local Solutions and Caustics

13. The Initial Value Problem for Hamilton–Jacobi Equations

14. The Cauchy Problem in Terms of the Lagrangian

15. The Hopf Variational Solution
16. The First Hopf Variational Formula
17. The Second Hopf Variational Formula
18. Semigroup Property of Hopf Variational Solutions
19. Regularity of Hopf Variational Solutions
20. Hopf Variational Solutions Are Weak Solutions of the Cauchy Problem
21. Some Examples
22. Uniqueness
23. More on Uniqueness and Stability
24. Stability in $L^p(\mathbb{R}^N)$ for All $p \geq 1$
25. Comparison Principle
26. Semi-Concave Solutions of the Cauchy Problem
27. Uniqueness of Semi-Concave Solutions
28. A Weak Notion of Semi-Concavity
29. Semi-Concavity of Hopf Variational Solutions
30. Weak Semi-Concavity of Hopf Variational Solutions Induced by the Initial Datum
31. Strictly Convex Hamiltonian
32. Uniqueness of Weakly Semi-Concave Variational Hopf Solutions

References

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