

**INTRODUCTION TO PARTIAL
DIFFERENTIAL EQUATIONS**
MATH 334-I, SPRING 2010; MWF 10:10–11:00,
Stevenson 1312

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**Office Hours: 1416 Stevenson, MW 11-12 or by
Appointment**

Course Description

The course builds on the basics about the Laplace, Wave and Heat equations. It covers the theory of single and double layer potentials, integral equations, their relation to the Dirichlet and Neuman problem and the eigenvalue problems. First order equations are introduced along with their physical origin and basic solvability issues are addressed. Generation of singularities, shocks, and the notions of viscosity solution are presented. The Kruzhkov uniqueness theorem will be given a central role.

The course will move to more advanced topics such as elliptic equations with measurable coefficients. Their weak setting in some Sobolev space and various solvability methods (variational, Galerkin etc.) Part of this will be a spectrum of techniques of a priori estimates in the sup and Hölder norms.

TEXTBOOK

No textbook is required. The class notes should suffice. While I will follow my book [3] (2nd Ed) as a general guiding line, a variety of topics and problems will be taken from other texts, mainly from Courant–Hilbert [2], Garabedian [6], John [8], Weinberger [21]. The books in the bibliography list below have been put on reserve for this course in the library.

SYLLABUS

1 The Double Layer Potential and Boundary Value Problems

- The double layer potential
- On the Integral defining the Double Layer Potential
- The Jump Condition Across ∂E
- More on the Jump Condition Across ∂E
- The Normal Derivative Across ∂E
- The Dirichlet Problem by Integral Equations
- The Neumann Problem by Integral Equations
- Green's Function for the Neumann Problem
- Eigenvalue Problems
- Compact Kernels in $L^2(E)$
- Compact kernels in $L^2(E)$ and Green's function

2 Integral Equations and Eigenvalue Problems

- Integral Equations
- The Homogeneous and Adjoint Equations
- Existence of Solutions for Small λ
- Separable Kernels
- Solving the Homogeneous Equation
- Solving the Inhomogeneous Equation
- Small Perturbations of Separable Kernels
- General Kernels
- Potential Kernels
- Applications to the Neumann Problem
- Integral Equations and Operators in $L^2(E)$
- Kernels in $L^2(E)$

- Integral Equations in $L^2(E)$
- Solving Integral Equations in $L^2(E)$
- Existence of Solutions for Small λ
- The Eigenvalue Problem
- The first eigenvalue and eigenfunction
- The Sequence of Eigenvalues
- Questions of Completeness and the Hilbert-Schmidt Theorem
- The Eigenvalue Problem for the Laplacean
- An Expansion of the Green Function

3 Equations of First Order and Conservation Laws

- Quasilinear Equations
- The Cauchy Problem
- Solving the Cauchy Problem
- Equations in Divergence Form and Weak Solutions
- Surface of Discontinuity
- The Shock Line
- The Initial Value Problem
- Conservation Laws in 1 Space Dimension
- Weak Solutions and Shocks
- Lack of Uniqueness
- Lax Notion of Weak Solution
- Constructing Weak Solution
- The Theorems of Existence and Stability
- Existence of Solutions
- Stability
- A Representation Formula

- Initial Data in the Sense of $L^1_{\text{loc}}(\mathbb{R})$
- Sup-Estimates and Invariants
- More on Sup-Bounds and Invariants
- Compactly Supported Initial Data
- The Entropy Condition
- Entropy Solutions
- Remarks on the Shocks and Entropy Conditions
- The Kruzhkov Uniqueness Theorem
- The Maximum Principle for Entropy Solutions
- Stability in $L^1(\mathbb{R}^N)$
- Asymptotic Behaviour of Solutions of Burgers Equation

4 Linear Elliptic Equations with Measurable Coefficients

- Weak Formulations and Weak Derivatives
- Introduction to Sobolev Spaces
- Embedding Theorems
- Traces and their characterizations
- The Dirichlet and Neumann Problems
- Solvability by Functional Analytical Methods
- Solvability by variational methods
- Solvability by Galerkin methods
- Eigenvalue problems
- A-priori bounds for local and global solutions
- Quasi-Linear Equations and their solvability

References

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