

**MATH 226 - CS 255**  
**Homework assignment 2,**  
 due September 23, 2005.

*Abiding by the Vanderbilt Honor Code, I pledge  
 that I have neither given nor received unauthorized aid on this assignment.*

**Signature:** .....

1. Use Theorem 2.2 to show that  $2^{-x}$  has a unique fixed point on  $[\frac{1}{3}, 1]$ . Use fixed-point iteration to find an approximation to the fixed point accurate to within  $10^{-4}$ . Use Corollary 2.4 to estimate the number of iterations required to achieve  $10^{-4}$  accuracy, and compare this theoretical estimate to the number actually needed. [Hint: Look at graph of derivative to estimate its range of values].

2. In this question, you will compare the different root-finding methods that we have presented. To do so:

(a) Find the root of

$$f(x) = \cos(x) - \sqrt{x},$$

where  $f$  is defined in  $[0, \frac{\pi}{2}]$ : Do by hand the first three iteration steps using each of the methods: (i) bisection, (ii) fixed point, (iii) Newton-Raphson, (iv) secant. Justify your work (for (ii), (iii), (iv), show how you apply the steps of each method). Make a table of your results, as in the example below:

Iteration	Bisection	Fixed-Point	Newton-Raphson	Secant
1	<i>value</i>	<i>value</i>	<i>value</i>	<i>value</i>
2	<i>value</i>	<i>value</i>	<i>value</i>	<i>value</i>
3	...	...	...	...
⋮	⋮	⋮	⋮	⋮

(b) How many iterations are required in order to get a solution accurate to at least  $10^{-15}$  using (i) the bisection method and (ii) the fixed point method with  $x_0 = \frac{\pi}{4}$ ?

3. Implement in matlab the above root-finding methods and use the programs to find the root of

$$f(x) = \cos(x) - \sqrt{x}.$$

In particular:

(a) The matlab code available at

<http://www.math.vanderbilt.edu/~daphne/courses/math226/September13/bisectionfun.m>

asks for the user to specify the number of bisection method iterations. Modify the code, so that the program will repeat the iteration steps until the possible error is below a given tolerance. The program should ask the user for the maximum allowable approximation error and perform the required number of iterations. Explain how the number of iteration steps is calculated.

(b) Write three matlab programs, each of which will implement the fixed point iteration, the Newton-Raphson method, and the secant method respectively. Use your programs to find the root of  $f(x) = \cos(x) - \sqrt{x}$ . All three programs should compare successive approximations of the root or fixed point, and stop if the difference between successive approximations is below a user-supplied tolerance. How many iterations are required before subsequent estimates differ by less than  $10^{-15}$ , if  $x_0 = \frac{\pi}{4}$ ? Choose a suitable  $x_1$  for the secant method.

Compare the rate of convergence of the methods.

4. In class, we saw that the function

$$g(x) = rx \left(1 - \frac{x}{2}\right)$$

defined in  $[0, 2]$  has a fixed point, but the fixed point iteration  $p_n = g(p_{n-1})$  may or may not converge, depending on the value of  $r$ . By calculating the value of  $g'$  at the fixed point, derive conditions on  $r$ , so that the iteration will converge to the fixed point.

5. The following table lists three sequences produced by different iterative methods. Which of these sequences appear to converge? Justify your answer. For the sequences that converge, determine the rate of convergence.

$x_1 = -2.888888889$	$y_1 = 0.$	$z_1 = 0.8359550561$
$x_2 = -2.880177515$	$y_2 = 1.$	$z_2 = 0.7026921574$
$x_3 = -2.879451589$	$y_3 = -.41614683$	$z_3 = 0.6782967522$
$x_4 = -2.879390800$	$y_4 = 0.67318141$	$z_4 = 0.6774204883$
$x_5 = -2.879385707$	$y_5 = 0.22255410$	$z_5 = 0.67741935484$
...	$y_6 = 0.90256407$	$z_6 = 0.67741935483$
	$y_7 = -.232193130$	...
	$y_8 = 0.894096612$	
	$y_9 = -.215688525$	
	...	