Approximation by dilated averages and K-functionals

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For a positive finite measure $d\mu(\mathbf{u})$ on $\mathbb{R}^d$ normalized to satisfy $\int_{\mathbb{R}^d} d\mu(\mathbf{u}) = 1$ the dilated average of $f(\mathbf{x})$ is given by

$$A_t f(\mathbf{x}) = \int_{\mathbb{R}^d} f(\mathbf{x} - t\mathbf{u}) d\mu(\mathbf{u}).$$

It will be shown that under some mild assumptions on $d\mu(\mathbf{u})$ one has the equivalence

$$\|A_t f - f\|_B \approx \inf\{(\|f-g\|_B + t^2 \|P(D)g\|_B) : P(D)g \in B\} \quad \text{for } t > 0$$

where $\varphi(t) \approx \psi(t)$ means $c^{-1} \le \varphi(t)/\psi(t) \le c$, B is a Banach space of functions for which translations are continuous isometries and $P(D)$ is an elliptic differential operator induced by μ. Many applications are given, notable among which is the averaging operator with $d\mu(\mathbf{u}) = \frac{1}{m(S)}\chi_S(\mathbf{u})d\mathbf{u}$ where S is a bounded convex set in $\mathbb{R}^d$ with an interior point, $m(S)$ is the Lebesgue measure of S and $\chi_S(\mathbf{u})$ is the characteristic function of S. The rate of approximation by averages on the boundary of a convex set under more restrictive conditions is also shown to be equivalent to an appropriate K-functional.