Group-type Subfactors and Hadamard Matrices

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Abstract

A hyperfinite II_1 subfactor may be obtained from a symmetric commuting square via iteration of the basic construction. For certain commuting squares constructed from Hadamard matrices, we describe this subfactor as a group-type inclusion $R^H \subset R \rtimes K$, where H and K are finite groups with outer actions on the hyperfinite II_1 factor R. We find the group of outer automorphisms generated by H and K, and use the method of Bisch and Haagerup to determine the principal and dual principal graphs. In some cases a complete classification is obtained by examining the element of $H^3(H * K/\mathrm{Int}R)$ associated with the action.

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1 Introduction

In [14], Jones described the basic construction on a finite-index subfactor $M_0 \subset M_1$ of type II_1. Iterating this construction gives the tower of factors

$$M_0 \subset M_1 \subset M_2 \subset M_3 \subset \ldots$$

Taking relative commutants yields two towers of finite dimensional algebras

$$\begin{array}{ccccccc} \mathbb{C} = M_0' \cap M_0 & \subset & M_0' \cap M_1 & \subset & M_0' \cap M_2 & \subset & \dots \\ & & \cup & & \cup & & \\ & \mathbb{C} = & M_1' \cap M_1 & \subset & M_1' \cap M_2 & \subset & \dots \end{array}$$

This is the standard invariant of the subfactor. The principal graph and dual principal graph are obtained from the Bratteli diagrams of these inclusions. While usually not a complete invariant, these graphs summarize much important data about the subfactor. The standard invariant is a complete invariant for amenable subfactors [21].

The classification problem is fundamental in the study of subfactors. In this paper we address this problem for a family of hyperfinite II_1 subfactors constructed from finite data. We will provide principal graphs, and in some cases a full classification up to subfactor isomorphism.

We recall the definition of commuting squares from [20]. Let

$$\begin{array}{ccc} C & \subset & D \\ \cup & & \cup \\ A & \subset & B \end{array}$$

be a quadrilateral of von Neumann algebras, with trace. We may construct the Hilbert space $L^2(D)$ and the conditional expectations E_B, E_C onto $L^2(B)$, $L^2(C)$ respectively. This quadrilateral is a commuting square if E_B and E_C commute.

A commuting square is specified by its four constituent algebras, the various inclusions, and certain additional data indicating how the towers $A \subset B \subset D$ and $A \subset C \subset D$ are related. This data can be summarized as the biunitary connection, which is an element of the multi-matrix algebra $A' \cap D$.

Goodman, de la Harpe, and Jones [10] showed how to construct a hyperfinite II_1 subfactor from a commuting square of finite-dimensional C^*-algebras, via iteration of the basic construction.

As described in [16], the standard invariant of a commuting-square subfactor is computable to any number of levels in finite time. However, the time required grows exponentially with the level, so this method cannot be used to find the full principal graph except in the most trivial examples.

If the commuting square is flat, then the standard invariant of the corresponding subfactor may be found by inspection (see [9]). Likewise, the standard invariant may be easily computed if the subfactor is depth 2. A few more complex examples have also been studied (see [17], [22]). For a general commuting square, however, even determining finite depth or amenability of its subfactor is an intractible problem.

A Hadamard matrix H is a real n by n matrix all of whose entries are ± 1, with $HH^T = n1$. n must be 1, 2, or a multiple of 4, but it is not known if Hadamard matrices exist for all such n. These matrices have been studied for over a century, with connections to areas as diverse as signals processing, cryptography, and group cohomology. A complex Hadamard matrix may be defined similarly as a unitary matrix all of whose entries have the same complex modulus [11].

For any complex Hadamard matrix, the quadrilateral

$$\begin{array}{ccc} \mathbb{C}^n & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} & \subset & H\mathbb{C}^n H^* \end{array}$$

commutes, and induces a commuting-square subfactor. For $n \geq 6$ many families of such matrices exist, giving a wide variety of examples of these Hadamard subfactors. Subfactors of this form were first examined in [15]. Their planar algebras (or equivalently, their standard invariants) are described by spin models, which makes computing the first few levels of the standard invariant relatively straightforward. For example, the first relative commutant of a Hadamard subfactor is always abelian [15].

Since Jones' 1999 paper, only a few Hadamard subfactors have been studied. The $n \times n$ Fourier matrix is defined by $F_{ij} = \xi^{ij}$, where ξ is a primitive nth root of unity. It may be easily computed using the profile matrix of [15] that Fourier matrices and their tensor products give depth-2 subfactors. Some other examples of small index were studied by Camp and Nicoara in [8], with full computations of the principal graph for a few index-4 examples. For practically all Hadamard subfactors, nothing is known about the standard invariant beyond the first few levels.

In this paper we will partially classify a family of Hadamard subfactors obtained from tensor products of Fourier matrices (and slight generalizations thereof) with a certain additional twist. As well as all previously known examples, this family includes a wide variety of subfactors at every composite integer index, of both finite and infinite depth. The twisted tensor product construction was suggested to the author by Jones. As we will show, these subfactors may be described as inclusions of the form $R^H \subset R \rtimes K$, for appropriate actions of finite abelian groups H and K on the hyperfinite II_1 factor R.

These group type subfactors were studied by Bisch and Haagerup in [3]. In this paper, the authors give a method for finding the principal graph of any such subfactor from the image of the free product of H and K in $\mathrm{Out}R$. We will show how to explicitly compute this image for the twisted tensor product Hadamard subfactors.

To analyze our Hadamard examples, we will first (section 2) discuss automorphisms of the hyperfinite II_1 factor which are compatible with the structure of the Jones tower in a particular way. For such automorphisms, determining outerness reduces to a problem in finite-dimensional linear algebra.

In section 3 we will discuss those Hadamard matrices which produce depth-2 subfactors. Taking the twisted tensor product of two such matrices gives a new Hadamard matrix, whose subfactor is of Bisch-Haagerup type. We will show that in this case the group actions have the compatibility property mentioned above, and so the principal and dual principal graphs may be readily computed.

The principal graph is not a complete invariant even for finite-depth subfactors. To classify the Bisch-Haagerup subfactors up to isomorphism, it is also necessary to consider a certain scalar 3-cocycle $\omega \in H^3(H * K/\mathrm{Int}R)$ associated with the group action. We will discuss this cohomological data in section 4, using on the conjugacy invariants of [13].

In section 5 we will use these methods to describe several examples. As well as examining certain finite-depth cases, we will provide infinite-depth Hadamard subfactors of every composite index. We will use the results of section 4 to fully classify all index-4 Hadamard subfactors.

Throughout this paper, if A is a von Neumann algebra acting on a Hilbert space H, we will take A' to be the commutant of A in the set $\mathcal{B}(H)$ of bounded linear operators on H. $A' \cap A = Z(A)$ is the center of A.

2 Compatible Automorphisms of the Hyperfinite II_1 Factor

Let $B_0 \subset B_1$ be a connected inclusion of finite-dimensional C^* algebras, along with its Markov trace. Iterating the basic construction gives a tower of algebras

$$B_0 \subset B_1 \subset B_2 \subset ... \subset B_\infty$$

where B_∞ is the hyperfinite II_1 factor. We examine automorphisms of B_∞ which are compatible with the structure of the tower.

We recall some basic properties of the iterated basic construction on finite-dimensional von Neumann algebras. This discussion is largely taken from [16].

Let $B_0 \subset B_1$ be a connected inclusion of finite-dimensional Von Neumann algebras. An inclusion is connected if the commutant $B_0 \cap B_1'$ is equal to $\mathbb{C}$. Defining a trace tr on B_1 makes $L^2(B_1)$ into a Hilbert space with inner product $\langle x, y\rangle = \mathrm{tr}(y^*x)$, on which B_1 acts by left multiplication. We may then define the conditional expectation $e = E_{B_0}$, which is the orthogonal projection onto the closed subspace $L^2(B_0) \subset L^2(B_1)$. This allows us to perform the basic construction on $B_0 \subset B_1$, obtaining $B_2 = \{B_1, e\}'' \subset \mathcal{B}(L^2(B_1))$. If we extend tr to a trace on B_2, we may then iterate this procedure, obtaining the Jones tower

$$B_0 \subset B_1 \subset B_2 \subset B_3 \subset ...$$

There is a unique trace on the original B_1 (the Markov trace, of some modulus τ) which extends to a trace on the entire tower. With this choice of trace, we may apply the GNS construction and take the closure of $\cup_i B_i$ to obtain the hyperfinite II_1 factor B_∞. We label the Jones projections by $B_i = \{B_{i-1}, e_i\}''$ for $i \geq 2$. Then $\mathrm{tr}(e_i) = \tau$ for all i, and the e_i's obey the relations

- $e_i e_j = e_j e_i$ for $|i - j| > 1$
- $e_i e_{i\pm1} e_i = \tau e_i$

We also have $e_i x e_i = e_i E_{B_{i-2}}(x)$ for $x \in B_{i-1}$.

Now we discuss automorphisms of Jones towers. We will require all of our automorphisms to be trace-preserving and respect the adjoint operation.

Theorem 2.1. *Let the tower of B_i's be as above. Let a be an automorphism of B_1 which leaves B_0 invariant. Then there is a unique (trace-preserving, $*$-) automorphism α of B_∞ such that $\alpha(e_i) = e_i$, $\alpha(B_i) = B_i$, and $\alpha|_{B_1} = a$.*

Proof. We construct α inductively. Let $\alpha_1 = a$. Let α_i be a trace-preserving $*$-automorphism of B_i, leaving B_j invariant and fixing e_j for $i \leq j$. For

$x, y \in B_i$, define α_{i+1} by $\alpha_{i+1}(xe_{i+1}y) = \alpha_i(x)e_{i+1}\alpha_i(y)$. It is not immediately clear that this is a defined map; in principal we might have $\sum_k x_k e_{i+1} y_k = 0$ with $\sum_k \alpha_i(x_k)e_{i+1}\alpha(y_k) \neq 0$, but we will show that this possibility does not arise.

We compute $\alpha_{i+1}(ae_{i+1}b)\alpha_{i+1}(ce_{i+1}d)$. This is

$$\alpha_i(a)e_{i+1}\alpha_i(b)\alpha_i(c)e_{i+1}\alpha_i(d)$$

$$= \alpha_i(a)E_{B_{i-1}}(\alpha_i(bc))e_{i+1}\alpha_i(d)$$

α_i leaves B_{i-1} invariant, and therefore commutes with $E_{B_{i-1}}$, so this is

$$\alpha_i(a)\alpha_i(E_{B_{i-1}}(bc))e_{i+1}\alpha_i(d)$$

$$= \alpha_{i+1}(aE_{B_{i-1}}(bc)e_{i+1}d)$$
$$= \alpha_{i+1}(ae_{i+1}bce_{i+1}d)$$

Therefore α_{i+1} is a homomorphism on elements of the form $ae_{i+1}b$; $B_i e_{i+1} B_i = B_{i+1}$ [16], so α is a homomorphism on all of B_{i+1}.

α_{i+1} clearly sends adjoints to adjoints. Since α_i is a trace-preserving homomorphism, we have

$$\mathrm{tr}(xe_{i+1}y) = \mathrm{tr}(e_{i+1})\mathrm{tr}(yx)$$

$$= \mathrm{tr}(e_{i+1})\mathrm{tr}(\alpha_i(yx)) = \mathrm{tr}(\alpha_i(x)e_{i+1}\alpha_i(y))$$

by the properties of the Jones projections. So α_{i+1} is trace-preserving as well. A trace-preserving $*$-homomorphism is an isometry. This tells us that α_{i+1} is a defined map on B_{i+1}, since $||x||_2 = 0 \to ||\alpha_{i+1}(x)||_2 = 0$, and that it is injective, since $||\alpha_{i+1}(x)||_2 = 0 \to ||x||_2 = 0$.

Therefore α_{i+1} is a (trace-preserving, $*$-) automorphism of B_{i+1}, which leaves B_j invariant and fixes e_j for $j \leq i+1$.

For $x \in B_i$, $\alpha_{i+1}(x)e_{i+1} = \alpha_{i+1}(xe_{i+1}) = \alpha_i(x)e_{i+1}$. Since $\alpha_i(x)$ and $\alpha_{i+1}(x)$ are in B_i, by properties of the Jones projections this means $\alpha_{i+1}(x) = \alpha_i(x)$. In other words $\alpha_{i+1}|B_i = \alpha_i$. So α_{i+1} is an extension of α_i, and we can define α_∞ on $\cup_i B_i$ by $\alpha_\infty(x) = \alpha_j(x)$ for $x \in B_j$. All the α_i's are norm-1, so α_∞ is bounded in norm. This means it extends to the closure of $\cup_i B_i$, giving an automorphism α of B_∞.

$\alpha|_{B_i} = \alpha_i$, so $\alpha|_{B_1} = \alpha_1 = a$, and $\alpha(e_i) = \alpha_i(e_i) = e_i$.

Uniqueness of α is immediate since B_1 and the e_i's generate B_∞. $\square$

Alternatively, if α is a automorphism of B_∞ which leaves the B_i's invariant and fixes the e_i's, then it is equal to the extension of $\alpha|_{B_1}$ to B_∞ as above.

These conditions are a bit stronger than necessary. If α fixes the e_i's and leaves B_1 invariant, then it also leaves $\{e_2\}' \cap B_1 = B_0$ and $\{B_1, e_2, ..., e_i\}'' = B_i$ invariant.

Maps of this form may be said to be compatible with the tower.

Definition 2.1. *If $\alpha \in \mathrm{Aut}(B_\infty)$ fixes e_i and leaves B_i invariant for all i, then α is a* **compatible** *automorphism.*

These may be thought of as a finite-dimensional version of the automorphisms discussed in [19], which are compatible with a Jones tower of II_1 factors. Some specific automorphisms compatible with a tower of finite-dimensional algebras were described in [22]. In this paper, Svendsen constructed a factor as the closure of the tower, and used limit arguments to show that these particular automorphisms were outer on this factor. We will use similar methods to determine when an arbitrary compatible automorphism is outer, by considering the relationship between compatible automorphisms and the canonical shift.

The Bratteli diagram of an inclusion of finite-dimensional Von Neumann algebras is a graphical depiction of the inclusion matrix (see [16]). For $0 \leq i \leq j$, both $B_i' \cap B_j$ and $B_{i+2}' \cap B_{j+2}$ may be implemented as the algebra of length $(j-i)$ paths on the Bratteli diagram of $B_0 \subset B_1$ [16]. It follows that these two algebras are isomorphic. The isomorphism is the canonical shift, and we may construct it as follows.

Lemma 2.1. *Let*

$$T_{ij} = \tau^{-(j-i)/2} e_{j+2} e_{j+1} ... e_{i+3} e_{i+2}$$

For all $x \in B_i' \cap B_j$, there is a unique $y \in B_{i+2}' \cap B_{j+2}$ such that $T_{ij} y T_{ij}^* = e_{j+2} x$, and the map Θ defined by $y = \Theta(x)$ is a $*$-isomorphism from $B_i' \cap B_j$ to $B_{i+2}' \cap B_{j+2}$.

Proof. We have

$$T_{ij}^* T_{ij} = \tau^{-(j-i)} e_{i+2} e_{i+3} ... e_{j+1} e_{j+2} e_{j+1} ... e_{i+2}$$

which is equal to e_{i+2} by the properties of the Jones projections.

Then let x be an element of $B_{i+2}' \cap B_{j+2}$. $T_{ij} x T_{ij}^*$ is an element of $e_{j+2} B_{j+2} e_{j+2}$. From the properties of the basic construction, it follows that there is a unique y in B_k such that $T_{ij}xT_{ij}^* = e_{j+2}y$. Since T_k, x, and e_{j+2} all commute with B_i, y must do so as well.

Now we may define $\rho_{ij} : B'_{i+2} \cap B_{j+2} \to B'_i \cap B_j$ by $T_{ij}xT_{ij}^* = e_{j+2}\rho_{ij}(x)$. It follows immediately that $\rho(x^*) = \rho(x)^*$.

For $x, y \in B'_{i+2} \cap B_{j+2}$, we then have

$$T_{ij}xT_{ij}^*T_{ij}yT_{ij}^* = e_{j+2}\rho_{ij}(x)e_{j+2}\rho_{ij}(y)$$

Since $\rho_{ij}(x)$ is in B_j, it commutes with e_{g+2}, and this is $e_{g+2}\rho_{ij}(x)\rho_j(y)$.

We also have $T_{ij}xT_{ij}^*T_{ij}yT_{ij}^* = T_{ij}xe_{i+2}yT_{ij}^*$. Since x commutes with e_{i+2} and $T_{ij}e_i = T_{ij}$, this implies that $T_{ij}xyT_{ij}^* = e_{j+2}\rho_{ij}(x)\rho_{ij}(y)$, so $e_{j+2}\rho_{ij}(xy) = e_{j+2}\rho_k(x)\rho_k(y)$. $e_{j+2}a = 0$ for $a \in B_j$ only if $a = 0$, so this means that ρ_{ij} is a homomorphism.

Now we investigate the norm of ρ_{ij}. Let x be an element of $B'_{i+2} \cap B_{j+2}$. We have $\mathrm{tr}(e_{j+2}\rho_{ij}(x)) = \tau\mathrm{tr}(\rho_{ij}(x))$. This is the same as $\mathrm{tr}(T_{ij}xT_{ij}^*) = \mathrm{tr}(xT_{ij}^*T_{ij}) = \mathrm{tr}(xe_{i+2})$. There is a trace-preserving conditional expectation onto $B'_{i+2} \cap B_{j+2}$, so we find

$$\mathrm{tr}(xe_{i+2}) = \mathrm{tr}(E_{B'_{i+2}\cap B_{j+2}}(xe_{i+2})) = \mathrm{tr}(xE_{B'_{i+2}\cap B_{j+2}}(e_{i+2}))$$

The quadrilateral

$$\begin{array}{ccc} B'_{i+2} \cap B_{j+2} & \subset & B_{j+2} \\ \cup & & \cup \\ Z(B_{i+2}) & \subset & B_{i+2} \end{array}$$

commutes, since the conditional expectation onto B_{i+2} preserves B'_{i+2}. So $E_{B'_{i+2}\cap B_{j+2}}(e_{i+2}) = E_{Z(B_{i+2})}(e_{i+2})$.

Definition 2.2. *The* **central support** *of a projection p in a finite von Neumann algebra A is the smallest central projection $q \in Z(A)$ with $pq = p$. If $q = 1$, we say that p has full central support.*

We have $e_{i+2} < 1$, since it is a projection. Also e_{i+2} has full central support in B_{i+2} [16]. Since $Z(B_{i+2})$ is finite-dimensional, this means there is some $\lambda > 0$ with $\lambda 1 < E_{Z(B_{i+2})}(e_{i+2}) < 1$. Therefore $\lambda\mathrm{tr}(x) < \mathrm{tr}(xE_{Z(B_{i+2})}(e_{i+2})) < \mathrm{tr}(x)$, implying $\tau^{-1}\lambda\mathrm{tr}(x) < \mathrm{tr}(\rho_{ij}(x)) < \tau^{-1}\mathrm{tr}(x)$. We then find $c > 0$ with $c < \tau^{-1}\lambda, c^{-1} > \tau^{-1}$.

Since ρ_{ij} is a $*$-homomorphism, for any $x \in B'_{i+2} \cap B_{j+2}$ we have

$$||\rho_{ij}(x)||_2^2 = \mathrm{tr}(\rho_{ij}(x)^*\rho(x)) = \mathrm{tr}(\rho(x^*x))$$

The above inequality on trace then implies

$$c||x||_2^2 < ||\rho_{ij}(x)||_2^2 < c^{-1}||x||_2^2$$

ρ_{ij} is thus an injective homomorphism. $B'_{i+2} \cap B_{j+2}$ is isomorphic to $B'_i \cap B_j$, so in fact ρ_{ij} is an isomorphism. It follows that there exists $\Theta_{ij} : B'_i \cap B_j \to B'_{i+2} \cap B_{j+2}$ such that $\rho_{ij}\Theta_{ij}$ is the identity, and Θ_{ij} is also a $*$-isomorphism.

By the definition of ρ_{ij}, $\Theta_{ij}(x)$ is then the unique element of $B'_{i+2} \cap B_{j+2}$ obeying the relation $T_{ij}\Theta_{ij}(x)T_{ij}^* = e_{j+2}x$ for $x \in B'_i \cap B_j$. □

Using the same constant c as above, we must have

$$c||x||_2^2 < ||\Theta_{ij}(x)||_2^2 < c^{-1}||x||_2^2$$

since $\Theta_{ij}\rho_{ij}$ is the identity. This map Θ_{ij} is the canonical shift on $B'_i \cap B_j$. This is essentially a finite-dimensional version of the shift on the higher relative commutants of a subfactor (see e.g. [5]).

We now recall some results from [16] and [10], based on Perron-Frobenius theory.

Let $\mathbf{s^{(i)}}$ be the size vector for B_i, i.e. the xth minimal central projection $p_x \in B_i$ has $p_x B_i = M_{s_x^{(i)}}(\mathbb{C})$. Then as n goes to infinity, $\tau^{2n}\mathbf{s^{(i+2n)}}$ converges to some vector $\mathbf{v}$, which is a Perron-Frobenius eigenvector for the inclusion matrix of $B_i \subset B_{i+2}$. Every component of each $\mathbf{s^{(k)}}$ is positive, and this is true of $\mathbf{v}$ as well, so for all x labeling a central projection of B_i, the set $\{\tau^n(s^{(i+2n)})_x | n \in \mathbb{N})\}$ is bounded and bounded away from zero. Since $Z(B_i)$ is finite-dimensional, there is some constant $c > 0$ with $c < \tau^n(s^{(i+2n)})_x < c^{-1}$ for all $n \in \mathbb{N}$, $1 \leq x \leq \dim Z(B_i)$.

Likewise, let $\mathbf{t^{(j)}}$ be the trace vector for B_j, with $t_y^{(j)}$ equal to the trace of a minimal projection in $p_y B_j$. From the Markov property of the trace on $B_j \subset B_{j+1}$, we have $\mathbf{t^{(j+2n)}} = \tau^n \mathbf{t^{(j)}}$ [10]. Again, finite dimensionality of $Z(B_j)$ implies that there is $d > 0$ with $d < \tau^{-n} t_y^{(j)} < d^{-1}$ for all $n \in \mathbb{N}$, $1 \leq y \leq \dim Z(B_j)$.

This implies that the traces of certain projections in the tower of relative commutants are bounded away from zero.

Lemma 2.2. *Choose $0 \leq i \leq j$. There exists $\epsilon > 0$ such that for all $n \geq 0$ and $p > 0$ a projection in $B'_{i+2n} \cap B_{j+2n}$, $\mathrm{tr}p \geq \epsilon$.*

Proof. Let p be a minimal projection in $B'_{i+2n} \cap B_{j+2n}$.

From the path algebra model of[16], the trace of p is equal to $s_x^{(i+2n)} t_y^{(j+2n)}$ for some $1 \leq x \leq \dim Z(B_i)$, $1 \leq y \leq \dim Z(B_j)$.

The above Perron-Froebenius argument implies that there are constants $c > 0$, $d > 0$ such that $c < \tau^{-n} s_x^{(i+2n)}$, $d < \tau^n t_y^{(i+2n)}$ for all x, y, n. This means that $\epsilon = cd < \mathrm{tr} p$. □

The **iterated shift** Θ_{ij}^n is defined as

$$\Theta_{ij}^n = \Theta_{i+2(n-1),j+2(n-1)} \Theta_{i+2(n-2),j+2(n-2)} ... \Theta_{i+2,j+2} \Theta_{ij}$$

This is a $*$-isomorphism from $B'_i \cap B_j$ to $B'_{i+2n} \cap B_{j+_2n}$.

Theorem 2.2. *For all i, j there exists $c > 0$ such that for all $x \in B'_i \cap B_j$ and all $n > 0$ we have*

$$c||x||_2 \leq ||\Theta_{ij}^n(x)||_2 \leq c^{-1}||x||$$

Proof. Let p be a minimal projection in $B'_i \cap B_j$. By the previous lemma we have $\epsilon > 0$ such that $\epsilon \leq \mathrm{tr} p \leq 1$, $\epsilon \leq \mathrm{tr} \Theta_{ij}^n(p) \leq 1$ for all n. It follows that

$$\epsilon^2 \mathrm{tr} p \leq \mathrm{tr} \Theta_{ij}^n(p) \leq \epsilon^{-2} \mathrm{tr} p$$

Any positive element is a linear combination of minimal projections, so this inequality holds for all $a > 0$ in $B'_i \cap B_j$. Applying this to x^*x we get

$$\epsilon^2 \mathrm{tr} x^*x \leq \mathrm{tr} \Theta_{ij}^n(x)^* \Theta_{ij}^n(x) \leq \epsilon^{-2} \mathrm{tr} x^*x$$

since Θ_{ij}^n is a $*$-isomorphism. This gives

$$\epsilon||x||_2 \leq ||\Theta_{ij}^n(x)||_2 \leq \epsilon^{-1}||x||_2$$

□

Let ω be a free ultrafilter of the natural numbers. If R is the hyperfinite II_1 factor, we define the ultrapower R^ω as the set of bounded functions from the natural numbers to R, modulo those which approach zero strongly along the ultrafilter. Convergence along the ultrafilter is defined using the ultralimit(see [9]): for a sequence of points (x_i) in some topological space, we say that $\lim_{i \to \omega}(x_i) = L$ if for any neighborhood N of L there is a set $S \subset \mathbb{N}$ in the ultrafilter such that $x_i \in N$ for all $i \in S$.

R embeds in R^ω as constant sequences. The central sequence algebra R_ω is then defined as the subalgebra $R' \cap R^\omega$, and both R^ω and R_ω are nonseparable II_1 factors [9]. If $x = (x_i)$ is an element of R^ω, then $\mathrm{tr}(x)$ is $\lim_{i\to\omega} \mathrm{tr}(x_i)$.

Take $0 \leq i \leq j$. Theorem 2.2 gives a map from $B_i' \cap B_j$ into the central sequence algebra $(B_\infty)_\omega$.

Lemma 2.3. *Let $\tilde{\Theta}$ from $B_i' \cap B_j$ to $l^\infty(B_\infty)$ defined by $\tilde{\Theta}(x) = (\Theta_{ij}^n(x))$. Then $\tilde{\Theta}$ is an injective homomorphism from $B_i' \cap B_j$ into $(B_\infty)_\omega$.*

Proof. From theorem 2.2, the sequence $\tilde{\Theta}(x)$ is bounded in ∞-norm, so it defines an element of B_∞^ω.

From the definition of the iterated shift Θ_{ij}^n, this element asymptotically commutes with all the B_i's, i.e. $\lim_{n\to\omega} ||[\Theta_{ij}^n(x), y]||_2 = 0$ for y in any B_i. Since the union of the B_i's are dense in B_∞, $\tilde{\Theta}(x)$ asymptotically commutes with every element of B_∞ and is contained in $(B_\infty)_\omega$.

$\tilde{\Theta}$ is a homomorphism, since each Θ_{ij}^n is. From lemma 2.2, the iterated shift is bounded away from zero in 2-norm. So if $x \neq 0$, the sequence $\tilde{\Theta}(x)$ does not approach zero in 2-norm, and gives a nonzero element of the central sequence algebra. In other words, $\tilde{\Theta}$ is injective. ☐

Since x has finite spectrum, this implies that $\tilde{\Theta}$ preserves the spectrum of x. In particular $||x||_\infty = \tilde{\Theta}(||x||)_\infty$.

Let α be a compatible automorphism. Take $0 < i < j$, $x \in B_{i+2}' \cap B_{j+2}$. Then since α fixes the Jones projections, we have

$$e_{j+2}\alpha(\rho_{ij}(x)) = \alpha(e_{j+2}\rho_{ij}(x))$$

$$= \alpha(T_{ij}xT_{ij}^*) = \alpha(T_{ij})\alpha(x)\alpha(T_{ij}^*) = T_{ij}\alpha(x)T_{ij}^*$$

This is the same as $e_{j+2}\rho_{ij}(\alpha(x))$. So α commutes with ρ_{ij} for all i, j. It follows that α commutes with each Θ_{ij} as well. Since inner automorphisms act trivially on central sequences, this gives us a test for outerness of compatible automorphisms.

Lemma 2.4. *Let the tower of B_i's be as above. If α is a compatible automorphism of B_∞, and α does not act trivially on $B_0' \cap B_i$ for all i, then α is outer.*

Proof. Let x be an element of $B_0'\cap B_i$, for some $i \geq 0$. Suppose that $\alpha(x) \neq x$. Then $\alpha(x)-x$ is a nonzero element of $B_0'\cap B_i$, and so by lemma 2.3 $\tilde{\Theta}(\alpha(x)-x)$ is a nonzero element of the central sequence algebra $(B_\infty)_\omega$.

α has a pointwise action on $(B_\infty)^\omega$ which restricts to $(B_\infty)_\omega$. Since α commutes with Θ and Θ^n, we have

$$\alpha(\tilde{\Theta}(x)) = (\alpha(\Theta^n_{0i}(x))) = (\Theta^n_{0i}(\alpha(x))) = \tilde{\Theta}(\alpha(x))$$

$\tilde{\Theta}$ is injective, so we have $\alpha(\tilde{\Theta}(x)) - \tilde{\Theta}(x)) \neq 0$ as well. This means that the induced action of α on central sequences is nontrivial. Inner automorphisms act trivially on central sequences, so with the above assumption α is outer. □

We may conclude that if α is not outer, i.e., $\alpha = \mathrm{Ad}u$ for some unitary $u \in B_\infty$), it must fix $B_0' \cap B_i$ for all i. This means that u commutes with $B_0' \cap B_i$ for all i, and hence with the strong closure $\overline{\cup_i^\infty B_0' \cap B_i}^{st}$.

Lemma 2.5. *Let the tower of B_i's be as above. Then $\overline{\cup_i^\infty B_0' \cap B_i}^{st} = B_0' \cap B_\infty$.*

Proof. The following square commutes:

$$\begin{array}{ccc} B_i & \subset & B_\infty \\ \cup & & \cup \\ B_0' \cap B_i & \subset & B_0' \cap B_\infty \end{array}$$

The B_i's are dense in B_∞, so $||x - E_{B_i}(x)||_2$ goes to zero as i goes to infinity. $E_{B_i}(x) = E_{B_0'\cap B_i}(x)$, so x is in the 2-norm closure of $\cup_i B_0' \cap B_i$. A sequence of elements in B_∞ converges strongly if it converges in 2-norm, implying that x is in the strong closure of $\cup_i B_0' \cap B_i$. □

These results imply that if $\mathrm{Ad}u$ is compatible inner, then u must commute with $B_0' \cap B_\infty$. Finite-dimensional algebras in a II_1 factor have the bicommutant property, so u must be in B_0 if $\mathrm{Ad}u$ is compatible. Since compatible automorphisms are determined by their restriction to B_1, we may make a slightly stronger statement, as follows:

Theorem 2.3. *Let the tower of B_i's be as above. If α is a compatible automorphism of B_∞, then α is inner if and only if $\alpha|_{B_1} = \mathrm{Adu}|_{B_1}$ for some unitary $u \in B_0$.*

Proof. First suppose that α is compatible, and $\alpha|_{B_1} = \mathrm{Ad}u|_{B_1}$ for some unitary $u \in B_0$. Then α agrees with $\mathrm{Ad}u$ on B_1, and both automorphisms fix the e_i's. B_1 and the e_i's generate B_∞, so in this case $\alpha = \mathrm{Ad}u$ and is inner.

Alternatively, let α be inner and compatible. Then $\alpha = \mathrm{Ad}u$ for some unitary $u \in B_0$, and $\alpha|_{B_1} = \mathrm{Ad}u|_{B_1}$. □

This theorem reduces determining outerness of a compatible automorphism to a purely computational problem.

3 Commuting-square subfactors and group actions

In [3], Bisch and Haagerup discuss group type subfactors of the form $M^H \subset M \rtimes K$, where H and K are finite groups with outer actions on M. The principal and dual principal graphs of such subfactors may be computed by finding the quotient $G = H * K/\mathrm{Int}M$. This requires being able to determine whether a specified word $w \in H * K$ produces an outer automorphism. In general this may be difficult, even if M is hyperfinite.

We will apply this technique to the commuting-square subfactors mentioned in the introduction. We will give conditions for a commuting square subfactor to be of fixed-point or crossed-product type, and describe how to compose two such subfactors to obtain a Bisch-Haagerup subfactor. As we will see, in this case the action of H and K is compatible with the Jones tower of the intermediate subfactor. This will allow us to use the results of the previous section to classify many previously intractable examples.

We briefly review the basic construction on commuting squares, following [16]. A quadrilateral of von Neumann algebras (with trace)

$$\begin{array}{ccc} B_0 & \subset & B_1 \\ \cup & & \cup \\ A_0 & \subset & A_1 \end{array}$$

is a commuting square if E_{B_0} and E_{A_1} commute as operators on $L^2(B_1)$. If we use the Markov trace on $B_0 \subset B_1$, we may iterate the basic construction on $B_0 \subset B_1$ as in the previous section to obtain a hyperfinite II_1 factor. We obtain a tower of A_i's as well, given by $A_i = \{A_{i-1}, e_i\}''$.

e_i implements the conditional expectation from A_{i-1} to A_{i-2}. By [16], for the inclusion $A_i \subset A_{i+1} \subset A_{i+2}$ to be isomorphic to the basic construction on $A_i \subset A_{i+1}$ it is then sufficient for the ideal $A_{i-1}e_iA_{i-1}$ to include the identity. From the properties of the Jones projections, this is true if the ideal $A_1e_2A_1$ includes the identity as operators on $L^2(B_1)$. This is the symmetry property of [16]; Jones and Sunder give several descriptions of this property, which they show are all equivalent to the following.

Definition 3.1. *A commuting square*

$$\begin{array}{ccc} B_0 & \subset & B_1 \\ \cup & & \cup \\ A_0 & \subset & A_1 \end{array}$$

is **symmetric** *if* $1 \in A_1 E_{B_0} A_1$ *as operators on* $L^2(B_1)$.

The Markov trace on $B_0 \subset B_1$ extends to B_∞, and then restricts to $A_\infty \subset B_\infty$ with no additional assumptions, producing a hyperfinite II_1 subfactor. The index $[B_\infty : A_\infty]$ of this inclusion is the squared norm of the inclusion matrix for the algebras $A_0 \subset B_0$. Every symmetric connected commuting square admits a unique Markov trace, so from now on we will assume that this is the trace we use for any such commuting square.

In order for B_∞ and A_∞ to be factors, the horizontal inclusions in the above square must be connected. However, we will not require the vertical inclusions to be connected, since we are not concerned here with the vertical basic construction.

Theorem 3.1. *Consider the symmetric, horizontally connected commuting square*

$$\begin{array}{ccc} A_{01} & \subset & A_{11} \\ \cup & & \cup \\ A_{00} & \subset & A_{10} \end{array}$$

generating a subfactor $N \subset M$ *via horizontal iteration of the basic construction, with Jones projections* $\{e_i\}$*. Suppose that there exist intermediate algebras* B_0*,* B_1*, as follows:*

$$\begin{array}{ccc} A_{01} & \subset & A_{11} \\ \cup & & \cup \\ B_0 & \subset & B_1 \\ \cup & & \cup \\ A_{00} & \subset & A_{10} \end{array}$$

Assume $B_0 \subset B_1$ is connected, and the quadrilateral

$$\begin{array}{ccc} A_{01} & \subset & A_{11} \\ \cup & & \cup \\ B_0 & \subset & B_1 \end{array}$$

commutes. Then there is an intermediate subfactor P obtained by iterating the basic construction on the B_i's, and both $N \subset P$ and $P \subset M$ arise from sub-commuting-squares of the original diagram.

Proof. The ideal $B_1 E_{A_{10}} B_1$ necessarily contains the identity, since the original commuting square is symmetric and $A_{10} \subset B_1$. Therefore the upper commuting square

$$\begin{array}{ccc} A_{01} & \subset & A_{11} \\ \cup & & \cup \\ B_0 & \subset & B_1 \end{array}$$

is symmetric, and is Markov by hypothesis. Let $B_{i+1} = \{B_i, e_{i+1}\}''$; then by the symmetric property all inclusions $B_i \subset B_{i+1} \subset B_{i+2}$ are standard. The Markov trace on the A_{i1}'s restricts to one on the B_i's [16], and we obtain an intermediate subfactor $N \subset P = \overline{\cup_i B_i}^{st} \subset M$.

The lower quadrilateral

$$\begin{array}{ccc} B_0 & \subset & B_1 \\ \cup & & \cup \\ A_{00} & \subset & A_{10} \end{array}$$

automatically commutes, since $E_{A_{10}}(B_0) \subset E_{A_{10}}(A_{01}) = A_{00}$. To compute $A_{10} E_{B_0} A_{10}$ as operators on $L^2(B_1)$, we note that $1 \in A_{10} E_{A_{01}} A_{10}$ as operators on $L^2(A_{11})$. Multiplying both sides by E_{B_1}, we find that $E_{B_1} \in A_{10} E_{B_1} E_{A_{01}} A_{10}$, since E_{B_1} commutes with A_{10}. Since the upper quadrilateral commutes by assumption, we have $E_{B_1} E_{A_{01}} = E_{B_0}$. Therefore $E_{B_1} \in A_{10} E_{B_0} A_{10}$, as operators on $L^2(A_{11})$. If we restrict to $L^2(B_1)$, then E_{B_1} is the identity, showing that $1 \in A_{10} E_{B_0} A_{10}$ on $L^2(B_1)$. Also we have already shown that the trace on B_1 is the Markov trace for the inclusion $B_0 \subset B_1$. So the lower quadrilateral is symmetric Markov, and we can obtain the subfactor $N \subset P$ by iterating the basic construction on it.

We conclude that $N \subset P$ and $P \subset M$ are both commuting-square subfactors, generated by

$$\begin{array}{ccc} B_0 & \subset & B_1 \\ \cup & & \cup \\ A_{00} & \subset & A_{10} \end{array}$$

and

$$\begin{array}{ccc} A_{01} & \subset & A_{11} \\ \cup & & \cup \\ B_0 & \subset & B_1 \end{array}$$

respectively. ☐

In [18] Landau showed that if a commuting-square subfactor $N \subset M$ has an intermediate subfactor P, then intermediate algebras $B_0 \subset B_1$ exist, with upper and lower symmetric commuting squares as above. The above theorem may be thought of as the converse of this result.

As we will see later in this section, certain assumptions on the small commuting squares will allow us to describe the subfactor $N \subset M$ as a composition of depth-2 subfactors $P^H \subset P \rtimes K$. Such subfactors were studied by Bisch and Haagerup in [3]. From this paper, in order to find the principal and dual principal graphs of these group-type subfactors, it is sufficient to find the group generated by H and K in $\mathrm{Out}P$. While this task can be complicated in general, it is relatively simple when the actions of H and K on are compatible with the tower of the B_i's.

Theorem 3.2. *Let P be the II_1 factor obtained by iterating the basic construction on an inclusion of finite-dimensional C^*-algebras $B_0 \subset B_1$. Let H and K be finite groups with outer actions on P, with both actions compatible with the tower of the B_i's. Let ρ be the representation of $H * K$ obtained by combining these actions. Then $G = H * K/\mathrm{Int}P$ may be computed by considering only $\rho|_{B_1}$.*

Proof. To find $G = H * K/\mathrm{Int}P$, it is sufficient to be able to determine whether ρ_w is outer for an arbitary word $w \in H * K$. But since H and K map into the group of compatible automorphisms, ρ_w is compatible as well. It follows that ρ_w is inner if and only if $\rho_w|_{B_1} = \mathrm{Ad}u|_{B_1}$ for some unitary $u \in B_0$. ☐

This can be computed rapidly for any particular w, assuming that B_1 is a reasonable size. If we have a bit more information about the structure of G it may only be necessary to evaluate a few thousand words, or even fewer. In some cases further simplifications occur, and this computation can be done by hand.

A Hadamard matrix is a matrix with orthogonal columns whose entries are all ± 1 [11]. For our purposes, such matrices are incorrectly scaled: we will define a complex Hadamard matrix to be an $n \times n$ unitary matrix whose entries all have the same complex modulus, namely $n^{-1/2}$. If H is an n by n complex Hadamard matrix, then from [15] it is the biunitary connection for a commuting square of the form

$$\begin{array}{ccc} \mathbb{C}^n & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} & \subset & H\mathbb{C}^n H^* \end{array}$$

Likewise if such a quadrilateral commutes, then H must be a Hadamard matrix.

These are Hadamard commuting squares. They are symmetric and connected, and so with their Markov trace they give subfactors via iteration of the basic construction [16].

Definition 3.2. *A* **Hadamard subfactor** *is a subfactor obtained by iterating the basic construction on the commuting square coming from a complex Hadamard matrix.*

Two Hadamard commuting squares are isomorphic if their matrices are Hadamard equivalent, i.e., if the matrices can be obtained from each other by the operations of permuting rows and columns, and multiplying rows and columns by scalars of modulus 1 [15]. In this case the corresponding Hadamard subfactors are the same. The index of a Hadamard subfactor is equal to the size of the matrix.

If G is a finite abelian group, with $|G| = n$, then its left regular representation on $l^\infty(G)$ gives rise to the commuting square

$$\begin{array}{ccc} \mathbb{C}[G] & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} & \subset & l^\infty(G) \end{array}$$

by taking $M_n(\mathbb{C}) = l^\infty(G) \rtimes G$.

Any two maximal abelian subalgebras of $M_n(\mathbb{C})$ are unitarily equivalent, so there exists $H_G \in M_n(\mathbb{C})$ with $\mathrm{Ad}H_G(\mathbb{C}[G]) = l^\infty(G)$. Since the above square commutes, H_G must be a complex Hadamard matrix. We construct H_G as follows.

For an abelian group G, $\mathrm{Hom}(G, \mathbb{C})$ is isomorphic to G. Specifically, G has n 1-dimensional representations $\{\rho_g\}$, with $\rho_g(x)\rho_h(x) = \rho_{gh}(x)$ for $g, h, x \in G$. Indexing the rows and columns by elements of G, we then let

$(H_G)_{ij} = \rho_j(i)$ (different indexing of the representations deos not change the Hadamard equivalence class). This is the discrete Fourier transform of the group G [11], and is known as the Fourier matrix when G is cyclic.

Theorem 3.3. *The discrete Fourier transform H_G of a finite abelian group G gives the commuting square*

$$\begin{array}{ccc} \mathbb{C}[G] & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} & \subset & l^\infty(G) \end{array}$$

and the corresponding Hadamard subfactor is $R \subset R \rtimes G$.

Proof. Let $\mathbb{C}[G]$ to be the diagonal subalgebra of $M_n\mathbb{C}$, spanned by minimal projections $\{e_x | x \in G\}$. In this case we may take $u_g \in \mathbb{C}[G]$ to be $\sum_{x \in G} \rho_g(x) e_x$, where ρ is as above. The minimal projections $\{f_x | x \in G\}$ of $l^\infty(G)$ are then given as $f_x = He_xH^*$, $H = H_G$. To show that we have found the correct connection, we should have the u_g's acting on the f_x's via permutation.

We compute the entries of f_x as $(f_x)_{ij} = H_{ix}(H^*)_{xj} = H_{ix}\overline{H_{jx}}$. From the definition of H_G, this is $\rho_x(i)\overline{\rho_x(j)}$. Conjugating by the diagonal unitary u_g gives $(u_g f_x u_g^*)_{ij} = \rho_g(i)\rho_x(i)\overline{\rho_x(j)\rho_g(j)}$. The group is abelian, so by the properties of ρ this is $\rho_{gx}(i)\overline{\rho_{gx}(j)} = (f_{gx})_{ij}$. So the u_g's act by the regular representation on $H\mathbb{C}[G]H^*$, and we have indeeed found the connection for the group type Hadamard commuting square.

Let $M_0 \subset M_1$ be the Hadamard subfactor induced by this commuting square. Then each $u_g \subset \mathbb{C}[G]$ normalizes $l^\infty(G)$ and commutes with the horizontal Jones projections, hence normalizes M_0. Adu_g is thus a compatible automorphism of M_0, and from theorem 2.3 is outer for any $g \neq 1$ (in this case we have $B_0 = \mathbb{C}$, so any nontrivial compatible automorphism is outer).

This means that $\{\{u_g\}, M_0\}''$ is isomorphic to $M_0 \rtimes G$. The u_g's span $\mathbb{C}[G]$, and M_1 is generated by $\mathbb{C}[G]$ and M_0, so in fact $M_1 = M_0 \rtimes G$. ☐

Let H_1 and H_2 be complex Hadamard matrices, of respective sizes m and n. Then their tensor product $H = H_1 \otimes H_2$ is unitary. If we take $i, k \in \{1...m\}$, $j, l \in \{1...n\}$, then the matrix entry $H_{ij,kl}$ is equal to $(H_1)_{ik}(H_2)_{jl}$. Since H_1 and H_2 are Hadamard, the complex modulus of this entry does not depend on i, j, k, l, and H is Hadamard as well. The twisted tensor product is the matrix $H_{ij,kl} = (H_1)_{ik}(H_2)_{jl}\lambda_{il}$, where each λ_{il} is an arbitrary complex number of modulus 1. Each matrix entry of this twisted tensor product has the same complex modulus, namely $(nm)^{-\frac{1}{2}}$. To see that H is still unitary, we take T to be the unitary element of the diagonal algebra $\Delta_m \otimes \Delta_n$ with components $T_{gh,gh} = \lambda_{gh}$. Then H is equal to the matrix product $(1 \otimes H_2)T(H_1 \otimes 1)$, which is unitary.

Theorem 3.4. *Let $H_1 = H_H^*$ and $H_2 = H_K$, for finite abelian groups H and K. Let $T \in \mathbb{T}^{|H||K|}$ be a twist. Then the twisted tensor product $H = (1 \otimes H_2)T(H_1 \otimes 1)$ induces a Hadamard subfactor of Bisch-Haagerup type.*

Proof. Applying the Hadamard matrix H to the tower of algebras

$$\mathbb{C} \subset \Delta_m \otimes 1 \subset \Delta_m \otimes \Delta_n \subset M_m(\mathbb{C}) \otimes \Delta_n \subset M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$$

gives the following diagram:

$$\begin{array}{ccccc}
\Delta_m \otimes \Delta_n & \subset & M_m(\mathbb{C}) \otimes \Delta_n & \subset & M_m(\mathbb{C}) \otimes M_n(\mathbb{C}) \\
 & & \cup & & \\
\cup & & M_m(\mathbb{C}) \otimes 1 & & \cup \\
 & \subset & & \subset & \\
\Delta_m \otimes 1 & & & & H(M_m(\mathbb{C}) \otimes \Delta_n)H^* \\
 & \subset & & \subset & \\
\cup & & H(M_m(\mathbb{C}) \otimes 1)H^* & & \cup \\
 & & \cup & & \\
\mathbb{C} & \subset & H(\Delta_m \otimes 1)H^* & \subset & H(\Delta_m \otimes \Delta_n)H^*
\end{array}$$

There are now two inclusions which are not obviously correct, namely $\Delta_m \otimes 1 \subset H(M_m(\mathbb{C}) \otimes 1)H^*$ and $M_m(\mathbb{C}) \otimes 1 \subset H(M_m(\mathbb{C}) \otimes \Delta_n)H^*$.

Note first that $H_1 \otimes 1$ normalizes $(M_m(\mathbb{C}) \otimes 1$, so $H(M_m(\mathbb{C}) \otimes 1)H^* = (1 \otimes H_2)T(M_m(\mathbb{C} \otimes 1)T^*(1 \otimes H_2)^*$. $\Delta_m \otimes 1$ is contained in $M_m(\mathbb{C}) \otimes 1$ and commutes with both T and $1 \otimes H_2$, so $\Delta_m \otimes 1 \subset H(M_m(\mathbb{C}) \otimes 1)H^*$.

Now we consider $H(M_m(\mathbb{C}) \otimes \Delta_n)H^*$. $M_m(\mathbb{C}) \otimes \Delta_n$ is the commutant in $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ of $1 \otimes \Delta_n$, so $H(M_m(\mathbb{C}) \otimes \Delta_n)H^* = (H(1 \otimes \Delta_n)H^*)'$ as well. But $H_1 \otimes 1$ and T commute with $1 \otimes \Delta_n$, so this is the commutant of $(1 \otimes H_2)(1 \otimes \Delta_n)(1 \otimes H_2^*)$. We conclude that

$$H(M_m(\mathbb{C}) \otimes \Delta_n)H^* = M_m(\mathbb{C}) \otimes (H_2 \Delta_n H_2^*)$$

which includes $M_m(\mathbb{C}) \otimes 1$.

So all inclusions in the above diagram are correct.

Next we compute the conditional expectation

$$E_{M_m(\mathbb{C})\otimes\Delta_n}(\Delta_m \otimes \Delta_n) = \Delta_m \otimes E_{H_2\Delta_n H_2^*}(\Delta_n)$$

Since

$$\begin{array}{ccc} \mathbb{C}^n & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} & \subset & H_2\mathbb{C}^n H_2^* \end{array}$$

commutes, the above conditional expectation is $\Delta_m \otimes 1$, implying that

$$\begin{array}{ccc} \Delta_m \otimes \Delta_n & \subset & M_m(\mathbb{C}) \otimes M_n(\mathbb{C}) \\ \cup & & \cup \\ \Delta_m \otimes 1 & \subset & H(M_m(\mathbb{C}) \otimes \Delta_n)H^* \end{array}$$

commutes as well. Furthermore, $\Delta_m \otimes 1 \subset H(M_m(\mathbb{C})\otimes\Delta_n)H^*$ is a connected inclusion, since $(\Delta_m\otimes 1)\cap(M_m(\mathbb{C})\otimes 1)' = \mathbb{C}$. From theorem 3.1, we then have an intermediate subfactor P, obtained by iterating the basic construction on

$$\Delta_m \otimes 1 \subset H(M_m(\mathbb{C}) \otimes \Delta_n)H^*$$

From the definition of the group Hadamard matrix, we have $\{v_h\} \subset \Delta_m$ such that $H_1 v_h H_1^*$ acts via the left regular representation of H on $\Delta_m = l^\infty(H)$. Let $u_h = H(v_h \otimes 1)H^*$. By the definition of H this acts via the left regular representation on $(1 \otimes H_2)T(\Delta_m \otimes 1)T^*(1 \otimes H_2)^* = \Delta_m \otimes 1$.

So u_h is an element of $H(M_m(\mathbb{C})\otimes\Delta_N)H^*$ which normalizes $\Delta_m\otimes 1$. $\mathrm{Ad}u_h$ therefore induces a compatible action on P. This action is outer for $h \neq 1$, since any element of $\Delta_m \otimes 1$ acts trivially by conjugation on $\Delta_m \otimes 1$, while u_h acts on this algebra by nontrivial permutation of the minimal projections. We therefore obtain an outer compatible action of H on P, induced by $\mathrm{Ad}(u_h)$.

Since $u_h \in H(\Delta_m \otimes \Delta_n)H^*$, and this algebra is abelian, P^H contains $H(\Delta_m\otimes\Delta_n)H^*$. The action of H is compatible, so $e_i \in P^H$ as well, implying that $N \subset P^H$. The index $[P : P^H] = m = [P : N]$, so in fact $N = P^H$ with the action of H induced by $\mathrm{Ad}u_h$.

Likewise, we have $v_k \in \Delta_n$ acting via the left regular representation of K on $H_2\Delta_n H_2^*$. Taking $u_k = 1 \otimes v_k$, this gives $\{u_k\}$ as elements of $\Delta_m \otimes \Delta_n$, which act via conjugation on $M_m(\mathbb{C}) \otimes (H_2\Delta_n H_2^*) = H(M_m(\mathbb{C}) \otimes \Delta_n)H^*$ and normalize (in fact act trivially on) $\Delta_m \otimes 1$. u_k commutes with the horizontal Jones projections $\{e_i\}$, so $\mathrm{Ad}u_k$ fixes these projections and acts compatibly on P. $\mathrm{Ad}u_k$ acts nontrivially on the center of $H(M_m(\mathbb{C})\otimes\Delta_n)H^*$, so the induced action is outer. This means that we have the factor $P \rtimes K$ embedded in M as the algebra generated by P and $\{u_k\}$, with the action given by $\mathrm{Ad}(u_k)$. Since $[M : P] = n = [P \rtimes K : P]$, in fact $M = P \rtimes K$.

Therefore the subfactor is $P^H \subset P \rtimes K$, and it may be analyzed using [3]. □

We now give a bit more detail on the role of the twist in producing the actions of H and K for a twisted tensor product Hadamard subfactor, with all notation as above.

Let $\tilde{T} = (1 \otimes H_2)T(1 \otimes H_2^*)$. Then $\tilde{T} \in B_0' \cap B_1$. Since $H_1 \otimes 1$ and $1 \otimes H_2$ normalize $M_m(\mathbb{C}) \otimes 1$,

$$\mathrm{Ad}(\tilde{T})(M_m(\mathbb{C}) \otimes 1) = \mathrm{Ad}(\tilde{T}(H_1 \otimes H_2))(M_m(\mathbb{C}) \otimes 1) = \mathrm{Ad}H(M_m(\mathbb{C}) \otimes 1)$$

So $\tilde{T}$ gives the difference between these two matrix algebras.

We have $u_h \in H(M_m(\mathbb{C}) \otimes 1)H^*$, while $1 \otimes u_k$ commutes with $M_m \otimes 1$. It follows that if $\tilde{T}$ normalizes $M_m \otimes 1$, then the two group actions commute with each other and we are again in the depth 2 case. Conversely, for general T, the two algebras $M_m \otimes 1$ and $\mathrm{Ad}T(M_m \otimes 1)$ are different, and we will not expect the actions to commute.

As before, let $a(h) = (H_1 v_h H_1^*) \otimes 1$, $a(k) = 1 \otimes v_k$. This still provides an induced action α of $H \oplus K$ on P, but $a(h)$ does not commute with N, so this direct sum no longer describes the structure of the subfactor. Instead, note that $u_h = H(v_h \otimes 1)H^* = \mathrm{Ad}\tilde{T}(a(h))$, since $1 \otimes H_2$ commutes with $a(h)$. $\tilde{T}$ itself induces a compatible automorphism τ on P, since $\mathrm{Ad}\tilde{T}$ normalizes B_1 and B_0. It follows from the properties of compatible automorphisms that u_h induces the compatible automorphism $\tau\alpha_h\tau^{-1}$.

This allows us to describe the correct action β of the free product $H * K$. Let $b(h) = \tilde{T}a(h)\tilde{T}^*$, $b(k) = a(k)$. b induces the compatible action β, defined by $\beta_h = \tau\alpha_h\tau^{-1}$, $\beta_k = \alpha_k$. We have $N = P^H$, $M = P \rtimes K$ for this action, so the properties of the subfactor $N \subset M$ may be determined by examining $b(H * K)$. For general T, $\tau\alpha_h\tau^{-1}$ will not commute with α_k, even as elements of $\mathrm{Out}P$. This construction will therefore provide Hadamard subfactors of depth greater than 2.

Theorem 3.5. *Take β a representation of $H * K$ as above. Let N_1 be the first commutator subgroup of $H * K$, N_2 the second commutator subgroup. Let $\tilde{S} = \ker\beta$, $S = \{g|\beta(g) \in \mathrm{IntR}\}$. Then $N_2 \subset \tilde{S} \subset S \subset N_1$.*

Proof. We have $\tilde{S} \subset S$ by definition.

$\mathrm{Ad}b(h)$ acts on $B_0 = \Delta_m \otimes 1$ via permutation, but acts trivially on the center of B_1, i.e. $1 \otimes (H_2 \Delta_n H_2^*)$; $\mathrm{Ad}b(k)$ does the opposite. If $x \in H * K$ but $x \notin N_1$, then $b(x)$ must nontrivially permute the minimal projections of $B_0' \cap B_1$. Therefore such $b(x)$ are outer, since $\mathrm{Ad}u$ acts trivially on this set for $u \in B_0$. This gives $S \subset N_1$.

Now let x be in N_1. $\mathrm{Ad}b(x)$ fixes $B_0' \cap B_1$, since the induced permutations of B_0 and $Z(B_1)$ are trivial. This means that $B(x)$ commutes with this algebra.

But $\Delta_m \otimes (H_2 \Delta_n H_2^*)$ is maximal abelian in $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$, so $b(x)$ must be contained in $B_0' \cap B_1$. Since this algebra is abelian, $b(x)$ must commute with $b(n)$ for every other element n of N_1. This means that the induced actions on P commute as well, and $b(N_1)$ is abelian. So $b(x)$ is the identity for all $x \in N_2$, and $N_2 \subset \tilde{S}$.

□

Each element of any group with the above properties may be written uniquely as hkn, $h \in H$, $k \in K$, $n \in N_1$. We therefore write $G = H * K/\mathrm{Int} = HKN$. N is an abelian group generated by the $(|H|-1)(|K|-1)$ elements of the form $hkh^{-1}k^{-1}$, $h \neq 1 \in H$, $k \neq 1 \in K$. N is a normal subgroup of $H * K$. We may determine $N = N_1/\mathrm{Int}$, and therefore G itself, by determining which elements n of N_1 have β_n outer.

From section 2, β_n is inner if and only if $\mathrm{Ad}b(n)|_{B_1} = \mathrm{Ad}u|_{B_1}$ for some unitary $u \in B_0$, with B_0 and B_1 as above. This will be true if and only if $b(n) = uv$, where u and v are unitary elements of B_0, $Z(B_1)$ respectively. This allows us to readily determine the order in $OutP$ of each generator of N_1, and hence the structure of the group N_1/Int.

To find $H * K/\mathrm{Int}$, we must also know how H and K act on N; i.e., hnh^{-1} and knk^{-1} for $h \in H$, $k \in K$, $n \in N$. To find these, we consider $B_0' \cap B_1 = l^\infty(H) \otimes l^\infty(K) = l^\infty(H \oplus K)$. In this representation, two elements of $B_0' \cap B_1$ induce the same local automorphism up to inner perturbation if they differ by a unitary u with coordinates $u(h,k) = f(h)g(k)$, where f and g are functions from H and K respectively to the complex scalars of modulus 1. We may therefore put each element of N in a unique standard form, with $n_{std}(1,k) = 1 = n_{std}(h,1)$ With this description of N, $\mathrm{Ad}b(H)$ and $\mathrm{Ad}b(K)$ act via the left regular representation ρ on the appropriate component. We know that hnh^{-1} must be equivalent to some $n' \in N$, and we can readily determine which one by putting hnh^{-1} in standard form. The same holds true for the action of K.

The above description of N provides a particularly good way of writing the generators. For $x = b(hkh^{-1}k^{-1}) \in b(N)$, we know that

$$x = \tilde{T}a(h)\tilde{T}^*a(h)\tilde{T}a(g^{-1})\tilde{T}^*a(h^{-1})$$

$\tilde{T}$ is itself an element of $B_0' \cap B_1 = l^\infty(H \oplus K)$. Ad$a(h)$ and Ad$a(k)$ act on $l^\infty(H \oplus K)$ via the left regular representation ρ, so we have

$$x = \tilde{T}\rho_h(\tilde{T}^*)\rho_{hk}(\tilde{T})\rho_{hkh^{-1}}(\tilde{T}^*)a(h)a(k)a(h^{-1})a(k^{-1})$$

$[\rho_h, \rho_k] = 0 = [a(h), a(k)]$, so we have

$$b(hkh^{-1}k^{-1}) = T\rho_h(T*)\rho_k(T^*)\rho_{hk}(T) \tag{1}$$

This gives us $x = b(hkh^{-1}k^{-1})$ as an element of $B_0' \cap B_1$.

This allows the complete computation of $G = H * K/\mathrm{Int} = HKN$ for any twisted tensor product Hadamard subfactor. Using the methods of [3], we may then obtain the principal graphs.

Since B_0 is fixed by K, multiplying $\tilde{T}$ by an inner $z \in B_0$ will not affect any of the generators of N: the change to $b(hkh^{-1}k^{-1})$ will be multiplication by

$$z\rho_h(z^*)\rho_k(z^*)\rho_{hk}(z) = z\rho_h(z^*)z^*\rho_h(z) = 1$$

for any $h \in H$, $k \in K$. The same is true of any perturbation coming from $Z(B_1)$.

We may therefore put $\tilde{T}$ itself in standard form without affecting the action of $\beta(N)$. The size of the group G is determined by $\beta|_N$; as we will see in section 5, the 3-cocycle obstruction associated with the action of G is as well. The group G and its 3-cocycle determine the standard invariant of a Bisch-Haagerup subfactor for groups with the above characteristics (see section 4, [2]) so we only need to consider twists in standard form. This will give us a better idea of the size of the space of examples obtained from this construction.

4 Classification of Bisch-Haagerup subfactors

Before giving specific examples, we will discuss the classification up to subfactor isomorphism of the subfactors obtained from this twisted tensor product, in the finite depth case.

In this section we summarize some results from [13], which we will use to classify our group actions.

Let $\tilde{G}$ be a finite group acting via ρ on the hyperfinite II_1 factor R, with inner subgroup S. Let S be implemented by unitaries u_S, i.e. $\rho_s = \mathrm{Ad}u_s$ for $s \in S$. For $g \in \tilde{G}$, $s, s' \in S$, we have $\alpha_g(u_{g^{-1}sg}) = \lambda(g,s)u_s$, $u_s u_{s'} = \mu(s,s')u_{ss'}$. Here λ is a function from $\tilde{G} \times N$ to the complex scalars of modulus 1, and μ is a similar function on $N \times N$.

Jones defines the characteristic invariant $\Lambda_{\tilde{G},S}$ as the set of all such pairs (λ, μ) which are allowable, in the sense that they can actually arise from some action of $\tilde{G}$ on R.

Jones also defines the inner invariant, which is determined from the restriction of the trace to $\mathbb{C}[S]$.

Two actions of $\tilde{G}$ with the same inner subgroup are conjugate if and only if their characteristic invariants and inner invariants are the same.

An action of $\tilde{G}$ on R provides a representation of the *kernel* $G = \tilde{G}/S$ in $OutR$. This representation lifts to an action of G on R if and only if the associated obstruction $\omega \in H^3(G)$ is zero. ω may be computed from the characteristic invariant. In fact, from [13] there is an exact sequence $H^2(\tilde{G}) \to \Lambda(\tilde{G}, S) \to H^3(G)$.

Now we will show that outer conjugacy of two actions of $H * K$ gives isomorphism of the corresponding Bisch-Haagerup subfactors, with some conditions on the action.

Definition 4.1. *A triplet isomorphism of Bisch-Haagerup subfactors $M^H \subset M \rtimes K$ and $P^H \subset P \rtimes K$ is a subfactor isomorphism which additionally sends M to P.*

Lemma 4.1. *Suppose ρ and σ are conjugate actions of $H * K$ on the hyperfinite II_1 factor R. Then they induce triplet isomorphic Bisch-Haagerup subfactor.*

Proof. Let α be an automorphism of R with $\alpha\rho\alpha^{-1} = \sigma$. Then $\alpha(R^{\rho(H)}) = R^{\sigma(H)}$. Moreover α extends to an isomorphism from $R \rtimes_\rho K$ to $R \rtimes_\sigma K$, by sending the u_k's in the first crossed product to the u_k's in the second. This provides a triplet isomorphism. □

Lemma 4.2. *Let ρ be an action of $H * K$ on the hyperfinite II_1 factor R. Let u_H, u_K be unitaries in R. Let σ be an action of $H * K$ on R, defined by $\sigma_{h,k} = \mathrm{Ad}v_{H,K}\rho_{h,k}\mathrm{Ad}v^*_{H,K}$ for $h \in H$, $k \in K$ respectively. Then ρ and σ induce triplet isomorphic Bisch-Haagerup subfactors.*

Proof. By the previous lemma, conjugating σ by $\mathrm{Ad}v_H^*$ induces a triplet isomorphism. So we may take $v_H = 1$ without loss of generality. Then we define $\alpha : R \rtimes_\sigma K \to R \rtimes_\rho K$ by $\alpha_R = id$, $\alpha(u_k) = v_K u_k v_K^*$ for $k \in K$. This is a triplet isomorphism. □

Theorem 4.1. *Let $\tilde{G}$ be a finite group, generated by two subgroups H and K. Let ρ and σ be outer conjugate representations of $\tilde{G}$ on the hyperfinite II_1 factor R, with inner subgroup $\rho(\tilde{G} \cap \mathrm{Int}R) = \sigma(\tilde{G} \cap \mathrm{Int}R) = N$. Let there be homomorphisms θ_H, θ_K from G to H and K respectively, with $\theta_H|_H$ and $\theta_K|_K$ equal to the identity and $\theta_{H,K}|_N$ trivial. Then the Bisch-Haagerup subfactors induced by ρ and σ are isomorphic.*

Proof. We consider the representation of $\tilde{G}$ on $R \otimes B(L^2(H)) \otimes B(L^2(K))$ given by $\rho \otimes 1 \otimes 1$. This representation of $\tilde{G}$ has the same inner subgroup, inner invariant and characteristic invariant as ρ, so these two representations are conjugate by [13], and induce the same Bisch-Haagerup subfactor by lemma 4.1. The same is true of σ and $\sigma \otimes 1 \otimes 1$.

Since ρ and σ are outer conjugate, they differ by at most a unitary 1-cocycle, possibly with scalar 2-cohomology. That is, there exist unitaries $\{u_g\}$ for each $g \in \tilde{G}$ such that $\sigma(g) = \mathrm{Ad}u_g \rho(g)$, with some scalar 2-cocycle $\mu : G \times G \to \mathbb{C}$ such that $u_{g_1 g_2} = \mu(g_1, g_2) u_{g1} \rho_{g_1}(u_{g_2})$.

Now let $\{v_h\}$ be unitaries in $B(L^2(H))$ such that $v_{h_1 h_2} = \overline{\mu(h_1, h_2)} v_{h_1} v_{h_2}$ for $h_{1,2} \in H$, and $\{v_k\} \subset B(L^2(K))$ likewise obeying $v_{k_1 k_2} = \overline{\mu(k_1, k_2)} v_{k_1} v_{k_2}$ for $k_{1,2} \in K$. Define a represention α of $\tilde{G}$ by $\alpha(g) = \sigma(g) \otimes \mathrm{Ad}v_{\theta_H(g)} \otimes \mathrm{Ad}v_{\theta_K(g)}$.

This is an inner perturbation of $\sigma \otimes 1 \otimes 1$, so it has the same inner subgroup. Since θ_H and θ_K are trivial on N, $\alpha(n) = \sigma(n) \otimes 1 \otimes 1$ for $n \in N$ and the representations have the same inner invariant. The unitaries implementing the inner subgroup of $\tilde{G}$ for the representation α may be taken to be of the form $u \otimes 1 \otimes 1$, and α and $\sigma \otimes 1 \otimes 1$ agree on all such elements, so the characteristic invariants of α and $\sigma \otimes 1 \otimes 1$ are also the same. Therefore $\sigma \otimes 1 \otimes 1$ and α are conjugate, and induce the same Bisch-Haagerup subfactor by lemma 4.1.

From the definition of the v_H's, $\{u_h \otimes v_h \otimes 1\}$ is a unitary 1-cocycle without cohomology for the representation $\rho \otimes 1 \otimes 1$ restricted to H. That is, for $h_{1,2} \in H$ we have $u_{h_1 h_2} \otimes 1 \otimes v_{h_1 h_2} = u_{h_1} \rho(u_{h_2}) \otimes 1 \otimes v_{h_1} v_{h_2}$, with no additional scalars. By stability of finite group actions on R, this means there is some unitary x_H in $R \otimes B(L^2(H)) \otimes B(L^2(K))$ with $u_h \otimes v_h \otimes 1 = x_H(\rho_h \otimes$

$1 \otimes 1)(x_H^*)$. The same argument gives us $x_K \in R \otimes B(L^2(H)) \otimes B(L^2(K))$ with $u_h \otimes 1 \otimes v_k = x_K(\rho_h \otimes 1 \otimes 1)(x_H^*)$ for $k \in K$.

This means that

$$\alpha|_H = \mathrm{Ad}x_H(\rho \otimes 1 \otimes 1)\mathrm{Ad}x_H^*$$

and

$$\alpha|_K = \mathrm{Ad}x_K(\rho \otimes 1 \otimes 1)\mathrm{Ad}x_K^*$$

So by lemma 4.2, α and $\rho \otimes 1 \otimes 1$ induce the same Bisch-Haagerup subfactor. Therefore ρ and σ do so as well. □

Additionally, we may freely apply automorphisms to H and K separately (or their group algebras) without affecting the triplet isomorphism class.

We will now give a converse of the proceeding theorem. As before, let H and K be finite groups, ρ and σ actions of $H * K$ on M, P respectively with outer restrictions to H and K.

Theorem 4.2. *Let $M^H \subset M \rtimes K$ be triplet isomorphic to $P^H \subset P \rtimes K$ via $\alpha : P \rtimes K \to M \rtimes K$, i.e. $\alpha(P) = M$, $\alpha(P^H) = M^H$. Then ρ and σ are outer conjugate.*

Proof. Take $\tilde{\sigma} = \alpha\sigma\alpha^{-1}$. This gives actions of H and K on $\alpha(P) = M$. Since $\alpha(P^H) = M^H$, $\tilde{\sigma}|_H$ commutes with left and right multiplication by M^H. Any such linear operator on $\mathcal{B}(L^2(M))$ is contained in the relative commutant $(M^H)' \cap M \rtimes H$; this is equal to $\mathbb{C}[H]$, where the action of H is implemented by ρ. So $\tilde{\sigma}$ and ρ give the same H-action, up to group algebra automorphism of H.

Let N be the fixed-point algebra of M under the action of $\tilde{\sigma}|_K$. Since $\alpha(P) = M$ and $\alpha(P \rtimes K) = M \rtimes K$, we have $N \subset M \subset M \rtimes K$ isomorphic to the basic construction on $N \subset M$. Therefore N differs from M^H by an inner automorphism, i.e. $N = uM^K u^*$ for some unitary $u \in M$. It follows as above that $\tilde{\sigma}_k = u\rho_k u^*$ for $k \in K$, up to group algebra automorphism of K.

We conclude that up to inner perturbation, $\tilde{\sigma}$ and ρ agree on H and K, and hence on the free product. So triplet isomorphism of the corresponding subfactors implies that ρ and σ are outer conjugate, up to separate automorphisms of the individual group algebras. □

Summarizing the results of this section, let ρ be an action of a finite group $H * K$ on the hyperfinite II_1 factor R, with inner subgroup S and with

$\tilde{S} = \rho^{-1}(id) \subset S$. Assume $G = H * K/\tilde{S}$ is a finite group, and that S is contained in the first commutator subgroup N_1 of $H * K$. For such actions, there is a homomorphism from G to $H \oplus K$, namely the quotient of G by the normal subgroup $N_1/\tilde{S}$, and the two components of this homomorphism satisfy the condition of theorem 4.1.

Therefore in such cases, outer conjugacy of the action is equivalent to triplet subfactor isomorphism $M^H \subset M \subset M \rtimes K \cong P^H \subset P \subset P \rtimes K$.

In [6], Bisch, Nicoara and Popa considered subfactors $M^H \subset M \rtimes K$ where H is abelian and K is prime-order cyclic. They showed that the normalizer of M^H in $M \rtimes K$ is equal to M for any such subfactors. Since normalizers are preserved by isomorphism, this again means that subfactor isomorphism implies triplet isomorphism in all such cases, and is equivalent to outer conjugacy (up to automorphism of the group algebras of H and K) given the above restriction on the action.

5 Applications to Hadamard Subfactors

We now take the outer actions of H and K on the hyperfinite II_1 factor P to come from the twisted tensor product of two group Hadamard matrices, as described in section 3.

In this case, the condition in theorem 4.1 on the action of $H*K$ is always true. From theorem 3.5, the inner subgroup of $H*K$ is contained in the first commutator subgroup N_1. Therefore the quotient map $H*K \to H*K/N_1 = H \oplus K$ factors through $H*K/\text{Int}$, and may be defined on $H*K/\text{Int} = G$. This quotient map (p_H, p_K) has the properties in the assumption of the theorem. It follows that outer conjugacy of actions implies subfactor isomorphism in the Hadamard case.

Furthermore, H and K must be abelian, so the result of [6] will frequently apply; if H or K is prime-order cyclic, then subfactor isomorphism implies triplet isomorphism, and is therefore equivalent to outer conjugacy.

Let $G = H * K/\text{Int}P$ be finite. Let $\tilde{G} \subset \text{Aut}P$ be a finite group with $\tilde{G}/\text{Int}P = G$ and $\tilde{G} \cap \text{Int}P = S$. From [13], such a finite $\tilde{G}$ always exists. We will take $\tilde{G}$ to act on P via the representation ρ. The inner subgroup S is in general nontrivial. As above, we know that S must be contained in the first commutator subgroup $\tilde{N}$ of $\tilde{G}$. Since $\tilde{G}$ has a compatible action, from theorem 2.3 each element of S may be implemented by some $u_s \in B_0 = \Delta_m = l^\infty(H)$, where the tower of B_i's is as in section 3.

We now compute the characteristic invariant of $\tilde{G}$. $B_0 = l^\infty(H)$, so we may consider u_s to be a vector with components labeled by elements of H. Since the u_s's are only determined up to scalars, we may require $(u_s)_1 = 1$. It follows that if $u_s u'_s = \mu u_{ss'}$ for some scalar μ, then $\mu = 1$. Therefore μ is trivial for these actions.

Each element g of $\tilde{G}$ may be written as $g = hkn$, $h \in H$, $k \in K$, $n \in \tilde{N} = N_1/\ker(\rho)$. Since for the Hadamard action K and $\tilde{N}$ act trivially on B_0, $\rho_{kn}(u_s) = u_s$ for k, n as above and $s \in S$. This means we have $\lambda(kn, s) = 1$ and $\lambda(hkn, s) = \lambda(h, s)$ for k, n, s as above and $h \in H$. Therefore the characteristic invariant is determined by $\lambda|_{H\times S}$.

From the definition of λ we have $\rho_h(u_{h^{-1}sh}) = \lambda(h, s)u_s$. We may determine this scalar by examining the first component. Since $(\lambda(h, s)u_s)_1 = 1$ from our choice of u_s, and ρ_h acts via the left regular representation on the minimal projections of B_0, we may compute $\lambda(h, s) = \overline{\rho_h(u_{h^{-1}sh})_1} = \overline{(u_{h^{-1}sh})_{h^{-1}}}$. So the coordinates of the u_s's determine the characteristic invariant, and vice versa.

It follows that the existence of any nontrivial inner subgroup implies a nonzero characteristic invariant, although the induced element of $H^3(G)$ may sometimes still be a coboundary. In addition, if ρ and σ give actions of the same group $\tilde{G}$ on P, with the same inner subgroup $S \subset \tilde{G}$, and the inner elements $\{u_s\}$ have the same coordinates, then their characteristic invariants are the same. In such a case the 3-cocycle obstructions are the same, the actions are outer conjugate, and the subfactors are isomorphic.

Applying [3] to find the principal graph for these Hadamard group actions is in some ways easier than in the general case. The local freeness condition of [3] ($hgk = x$ for $h \in H, k \in K, g \in G$ only if $h = k = 1$) will always apply, so the odd vertices of the principal graph correspond to the $H - K$ double cosets $\{HnK | n \in N\}$. Even vertices in the principal graph correspond to $H - H$ double cosets HgH. We find the edges of the graph by decomposing $HnKH$ into such double cosets. From the above description of G, if $k \neq k'$ then $HnkH$ and $Hnk'H$ are disjoint, so there is always one such double coset for each element of K. Finding the number of single H-cosets in each $HnkH$ (taking advantage of the relatively simple multiplication table of G) then allows us to complete the principal graph. The dual principal graph is computed similarly.

First we consider Hadamard subfactors of index 4. Let $H = K = \mathbb{Z}_2$. In this case H_1 and H_2 are both $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and their tensor product is the unique real 4 by 4 Hadamard matrix

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

We may write the twist as $(\alpha, \beta, \gamma, \delta) \in l^\infty(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$. Putting the twist into standard form sends all the parameters to 1 except δ. Applying a twist of $T = (1, 1, 1, \delta)$ gives the twisted tensor product

$$H = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & \delta & -1 & -\delta \\ 1 & -\delta & -1 & \delta \end{pmatrix}$$

All size-4 complex Hadamard matrices are contained in a single 1-parameter family [11]. As δ takes on values in the torus, H varies over this entire family. In other words, all 4 by 4 Hadamard matrices are twisted tensor products.

We know that the group $G = H * K/\text{Int}$ is of the form HKN, N a normal subgroup of G, from theorem 3.5. H and K are each generated by a single order-2 element, respectively h and k.

N has a single generator $n = hkhk$. From equation (1) in section 3, this compatible automorphism is induced by $\text{Ad}b(n)$, where

$$b(n) = \tilde{T}\rho_h(\tilde{T}^*)\rho_k(\tilde{T}^*)\rho_{hk}(\tilde{T}) \in B_0' \cap N_1$$

For convenience we write elements of $N \subset B_0' \cap B_1$ in array form, since this abelian algebra has one minimal projection for each pair (h, k), $h \in H$, $k \in K$. These are not matrices–multiplication is still pointwise. We will consider rows to be labeled by elements of H, columns by elements of K. So

$$b(n) = \begin{bmatrix} \delta & \overline{\delta} \\ \overline{\delta} & \delta \end{bmatrix}$$

This element of $B_0' \cap B_1$ will induce an inner automorphism when it is constant along each row. This occurs when $\overline{\delta^{2l}} = \delta^{2l}$, i.e. when $\delta^{2l} = \pm 1$.

Suppose that δ is a rational rotation. Then let l be the smallest natural number such that $\delta^{4l} = 1$. This means that $\delta^{2l} = \pm 1$, and $N = \mathbb{Z}_l$, generated by $n = hkhk$. $hnh = knk = n^{-1}$, implying that $G = H * K/\text{Int} = HKN/\text{Int}$

is the dihedral group D_{2l}. Generators are $s = h$ and $t = hk$, with $t^{2k} = s^2 = stst = 1$.

If δ is an irrational rotation, this group is D_∞.

Now we consider cohomology. If δ is rational, we take the two cases $\delta^{2l} = 1$ and $\delta^{2l} = -1$.

If $\delta^{2l} = 1$, then $(hkhk)^n$ is the identity, and we have a true outer action of $H * K = D_{2l}$. Any two outer actions of a finite group are conjugate [13], so for any choice of δ with $\delta^{2l} = 1$, the corresponding Hadamard subfactors are isomorphic (see section 4).

Now suppose $\delta^{2l} = -1$. In this case $(hkhk)^n$ is a nontrivial inner $u = (1, -1) \in B_0 = \Delta_m \otimes 1$, with $u^2 = 1$. This allows us to extend the representation of $H * K = D_{2l}$ in $OutP$ to an action of D_{4l} on P with inner subgroup $\{1, u\} = \mathbb{Z}_2$. u does not depend on the particular choice of root, so the characteristic invariant and associated 3-cocycle of $G = D_{2n}$ are the same for any such δ. It follows that the group actions are outer conjugate, and all such subfactors are again isomorphic (again, see section 4).

This means that there are at most 2 nonisomorphic Hadamard subfactors with each graph $D^{(1)}_{2l+1}$ at index 4. In [12] Izumi and Kawahigashi found that there are $n-2$ subfactors with principal graph $D^{(1)}_n$ for any n. This means that many of these subfactors cannot be constructed from Hadamard commuting squares.

It remains to show that the case $\delta^{2l} = \pm 1$ give distinct subfactors. To do this we show that the associated 3-cocycles are different.

We consider the cyclic subgroup of D_{4l} generated by $a = hk$ in the case $\delta^{2l} = -1$. a has order $2l$ in $OutP$, with $a^l = \mathrm{Ad}u$. Now, $\lambda(a, a^l) = -1$, since $hk(u) = -u$. This is a nontrivial characteristic invariant, but cyclic groups have trivial 2-cohomology, so it does not come from a 2-cocycle on the subgroup. Therefore from the exact sequence of [13] the kernel in Out R of this subgroup has nontrivial associated 3-cocycle, implying the kernel of the full group does as well. Since the cocycle is trivial in the case $\delta^{2l} = 1$, the two corresponding Hadamard subfactors are not isomorphic.

This completes the classification of the index-4 Hadamard subfactors.

Now we will describe some index-6 examples. Let $H = \mathbb{Z}_2$, $K = \mathbb{Z}_3$. Both of these groups are prime-order cyclic, so from section 4 and [6] outer conjugacy of group actions is equivalent to subfactor isomorphism, up to automorphism of the two small groups. H has one nontrivial automorphism, so this will be relevant. H is generated by h, K by k. We construct twisted tensor product of the depth-2 Hadamard matrices corresponding to H and K.

The first commutator subgroup for any twisted tensor product will be an abelian group with two generators, namely $x = hkhk^2$ and $y = hk^2hk$. Each one of these generators may be represented as a unitary in $\mathbb{C}[H \oplus K]$ as above.

Let the twist be $(1, 1, 1, 1, \chi, \xi)$, in standard form. In this case we compute from equation (1) in section 3

$$b(x) = \begin{bmatrix} \xi & \overline{\chi} & \overline{\xi}\chi \\ \overline{\xi} & \chi & \xi\overline{\chi} \end{bmatrix}, b(y) = \begin{bmatrix} \chi & \overline{\chi}\xi & \overline{\xi} \\ \overline{\chi} & \chi\overline{\xi} & \xi \end{bmatrix}$$

Multiplying a column by a scalar is trivial, so x and y are respectively induced by

$$\begin{bmatrix} 1 & 1 & 1 \\ \overline{\xi}^2 & \chi^2 & \xi^2\overline{\chi}^2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ \overline{\chi}^2 & \chi^2\overline{\xi}^2 & \xi^2 \end{bmatrix}$$

The first thing we want is the principal graph. To find this we may freely perturb x and y by inners. Specifically we multiply the second row of the above elements of $l^\infty(H) \otimes l^\infty(K)$ by ξ^2, χ^2 respectively. This gives the first commutator subgroup N of $G * H/\text{Int} = K$ as the subgroup of $(S^1)^2$ generated by $(\chi^2\xi^2, \overline{\chi}^2\xi^4)$ and $(\chi^4\overline{\xi}^2, \chi^2\xi^2)$.

We may describe these elements in additive notation, taking $\chi = e^{2\pi a}$, $\xi = e^{2\pi b}$. Then in $\mathbb{R}/\mathbb{Z}$, these generators are $(s, s+t)$ and $(s+t, t)$ in for $s = 4a - 2b$, $t = 4b - 2a$. The group will be finite if and only if s and t are both rational, or equivalently if a and b are.

In this finite case, N will be some finite subgroup of $\mathbb{Z} \oplus \mathbb{Z}$, which may be directly computed without difficulty for any particular χ and ξ. Computations with the generators give $hxh = x^{-1}$, $hyh = y^{-1}$, $kxk^2 = x^{-1}y$, $kyk^2 = x^{-1}$. These relations provide a complete multiplication table for the group $G = HKN$, and so we can use [3] to find the principal graph.

Since we have local freeness, odd vertices will be indexed by double cosets HnK. Each double coset HgH containing p single cosets gH will correspond to a cluster of $|H|/p$ even vertices, all connecting to the same odd vertices. Connections on the graph are determined by breaking up $HnKH$ as a sum of double cosets HgH; the vertex $HnKH$ connects to every vertex or cluster represented in this sum, with multiplicities determined by the number of times each HgH appears.

To compare two different twists, we pick some l sufficiently large so that all components of both twists are lth roots of unity. We then have two actions of $\tilde{G}$, with subfactor isomorphism being equivalent to conjugacy of G-kernels (up to the automorphism of $K = \mathbb{Z}_3$). We can compute the characteristic invariant, which will sometimes allow us to assert that certain subfactors with a given principal graph are isomorphic.

Since $(hk)^6$ and $(hk^2)^6$ act trivially for any choice of twist (i.e., not via an inner automorphism), restriction to cyclic subgroups does not usually provide nontrivial 3-cohomology. We will not in general be able to assert that two of these subfactors with the same G are different–even if the characteristic invariants are different, the 3-cocycles might be the same.

We now give principal graphs for a few simple twists of $H_{\mathbb{Z}_2} \otimes H_{\mathbb{Z}_3}$.

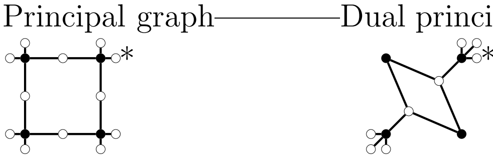
$H = \mathbb{Z}_3$, $K = \mathbb{Z}_2$, $T = (1, 1, 1, 1, 1, -1)$: $G = \mathbb{Z}_6$ and the subfactor is depth 2, so the associated 3-cocycle is trivial. However, the characteristic invariant is nontrivial. The constraints on the action of $\tilde{G}$ could imply that every nontrivial characteristic invariant induces a nontrival 3-cocycle; this example shows that this is not the case.

$H = \mathbb{Z}_2$, $K = \mathbb{Z}_3$, $T = (1, 1, 1, 1, 1, e^{2\pi i/3})$: We put x and y in standard form as elements of $(S^1)^2$ to find $H * K/\mathrm{Int}$. Then $x = (e^{\frac{2}{3}2\pi i}, e^{\frac{1}{3}2\pi i/3}), y = (e^{\frac{1}{3}2\pi i}, e^{\frac{2}{3}2\pi i/3})$. We conclude that $x^2 = y$ in Out, so $N = \mathbb{Z}_3$. G is a non-abelian group of order 18.

Principal graph————Dual principal graph

$H = \mathbb{Z}_2$, $K = \mathbb{Z}_3$, $T = (1, 1, 1, 1, 1, i)$: $|G| = 24$. Here we find $N = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where x and y are the two generators.

Principal graph————Dual principal graph



Now let ξ be a primitive 15th root of unity, and consider $T = (1, 1, 1, 1, 1, \xi)$. We wish to consider cohomology in this case, so we do not perturb the generators by inner automorphisms (multiplying a row by a scalar). However multiplying columns by scalars is still trivial.

We then have $b(x) = \begin{bmatrix} 1 & 1 & 1 \\ \xi^{-2} & 1 & \xi^2 \end{bmatrix}$, $b(y) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \xi^{-2} & \xi^2 \end{bmatrix}$. We have

$x^{15} = y^{15} = 1$, but there is an additional relation: $b(x^5y^5) = \begin{bmatrix} 1 & 1 & 1 \\ \xi^{-10} & \xi^{-10} & \xi^{20} \end{bmatrix}$. Since $\xi^{20} = \xi^{-10}$, this element of $l^\infty(H) \otimes l^\infty(K)$ induces the inner automorphism $\mathrm{Ad}u$, $u = (1, \xi^{-10}) \in \Delta_m$. It follows that $N = \mathbb{Z}_5 \oplus \mathbb{Z}_{15}$, with generators $x + y$ and x, and $|G| = 450$. The principal graph may be computed using the same methods as above.

In this case $\tilde{G}$ is an order-1350 group with inner subgroup $\mathbb{Z}_3$, since u^3 is the identity. The characteristic invariant is completely determined by u, as discussed above. If we let $\xi = e^{\frac{a}{15}2\pi i}$, then for ξ to be a primitive 15th root of unity we must have $a \in \{1, 2, 4, 7, 8, 11, 13, 14\}$. Choosing $a \in \{1, 4, 7, 13\}$ gives the same value for ξ^{-10}. It follows that the corresponding four group actions have the same characteristic invariant, and therefore the same 3-cocycle. This means that the subfactors are isomorphic. Likewise choosing a from $\{2, 8, 11, 14\}$ gives isomorphic subfactors. Applying the automorphism of K sending k to k^2 does not change the characteristic invariant in either case.

These two kinds of roots give different characteristic invariants, but unlike the index-four case it is not possible to detect 3-cohomology on cyclic subgroups. Some appropriate abelian subgroups might allow us to detect 3-cohomology, but for now it is not clear if these two types of subfactor are isomorphic.

There are five real 16 by 16 Hadamard matrices. Numerical computations give the dimensions of the first relative commutants as 16, 7, 4, 3, 3. The first one is depth 2, and the last 3 have excessively sparse intermediate subfactor lattices to be Bisch-Haagerup. Hadamard 16-7, however, may be obtained as the twisted tensor product of the unique real 4×4 Hadamard matrix with itself, using the twist $T = (1, 1, 1, ..., 1, -1)$.

In this case we have $H = K = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The first commutator subgroup N is induced by unitaries in $B_0' \cap B_1$. However, all coefficients are real (i.e. ± 1) so every element in N must be order 2.

We will show that N is in fact $\mathbb{Z}_2^4$, and that $H * K/\mathrm{Int}$ is therefore a non-abelian group of order 256. We will be able to compute the principal graph for the subfactor as well.

N is generated by the nine elements of the form $hkhk$, for h and k nontrivial elements of H, K respectively. Each such element n is induced by $b(n) \in l^\infty(H) \otimes l^\infty(k)$. We write these unitaries in array form, as in the previous section. We take $H = \{1, w, x, wx\}$ and $K = \{1, y, z, yz\}$, with the rows and columns numbered in that order. All coefficients will be ± 1, and we label them by sign. Again rows are indexed by H, columns by K.

$$b(1) = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \quad b(wywy) = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & - & - \\ + & + & - & - \end{bmatrix}$$

$$b(wzwz) = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & - & + & - \\ + & - & + & - \end{bmatrix} \quad b(wyzwyz) = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ - & + & + & - \\ - & + & + & - \end{bmatrix}$$

$$b(xyxy) = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & + & + \\ + & + & - & - \end{bmatrix} \quad b(xzxz) = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & + & + \\ + & - & + & - \end{bmatrix}$$

$$b(xyzxyz) = \begin{bmatrix} + & + & + & + \\ - & + & + & - \\ + & + & + & + \\ - & + & + & - \end{bmatrix} \quad b(wxywxy) = \begin{bmatrix} + & + & - & - \\ + & + & + & + \\ + & + & + & + \\ + & + & - & - \end{bmatrix}$$

$$b(wxzwxz) = \begin{bmatrix} + & - & + & - \\ + & + & + & + \\ + & + & + & + \\ + & - & + & - \end{bmatrix} \quad b(wxyzwxyz) = \begin{bmatrix} - & + & + & - \\ + & + & + & + \\ + & + & + & + \\ - & + & + & - \end{bmatrix}$$

Clearly every element of N is order 2.

Multiplying a row of some $b(n)$ by -1 corresponds to an inner automorphism, while multiplying a column by -1 is trivial. We can rewrite the above generators in standard form, and ignore the first row and column (since these are always 1's). This gives us each generator as having four minus signs, with an even number in each row and column. This parity condition gives five relations, and any product of generators will still have this property, so these elements span a subspace of at most dimension 4 in $\mathbb{Z}_2^9$. In fact $\{wywy$, $wzwz$, $xyxy$, $xzxz\}$ is a minimal generating set for N, with the other elements obeying the relations

$(wywy)(wzwz) = wyzwyz$, $(xyxy)(xzxz) = xyzxyz$,
$(wywy)(xyxy) = wxywxy$, $(wzwz)(xzxz) = wxzwxz$,
and $(wywy)(wzwz)(xyxy)(xzxz) = wxyzwxyz$.

All of these relations are valid in $OutP$, but some require nontrivial inner adjustment.

Next we note that (in Out) N is central. We compute

$$yhkhky = (yhyh)(hykhyk)$$

for $h \in H$, $k \in K$. This will be $(hyhy)(hyhy)(hkhk) = hkhk$ for any h and k. So $hkhk$ commutes with y. It may be simililarly shown that $hkhk$ commutes with every other element of H and K, and hence with their entire free product.

We know that $G = H * K/\mathrm{Int}$ for Hadamard subfactors will be of the form HKN. N is order 16, so $|G| = |H||K||N| = 4 \cdot 4 \cdot 16 = 256$. We have enough data to determine the multiplication table for G. Let $h_1, h_2 \in H$, $k_1, k_2 \in K$, and $n_1, n_2, h_2k_1h_2k_1 \in N$.

$$(h_1k_1n_1)(h_2k_2n_2)$$

$$= (h_1k_1)(h_2k_2)(n_1n_2)$$
$$= (h_2h_2)(h_2k_1h_2k_1)(k_1k_2)(n_1n_2)$$
$$= (h_1h_2)(k_1k_2)(n_1n_2)(h_2k_1h_2k_1)$$

The above relations will always allow us to express $h_2k_1h_2k_1$ in terms of our four generators of N, providing the multiplication table for the group. We can identify this group as number 8935 of its order in the MAGMA small-group catalog.

We may now use the methods of [3] to find the principal graph. Let $h \in H$, $k \in K$.

For $g \in G$, $hgk = g$ only if $h = k = 1$, so the group is locally free. This means that the odd vertices of the graph correspond to the 16 elements of N, i.e. to the $H - K$ double cosets HnK.

Even vertices are divided into classes according to the double coset structure HGH. A double coset HgH will contain 4 elements if g is in HN (and hence commutes with H). If $g = kn$ for $k \neq 1$, then HgH contains the 16 distinct elements of the form $hkh'kh'kn$ for $h, h' \in H$. A 4-element double coset corresponds to a cluster of four even vertices, each representing an irreducible bimodule of H-dimension 1. A 16-element double coset corresponds to a single bimodule of size $4^2 = 16$. We have 12 single vertices and 16 clusters, for a total of $64 + 12 = 76$ even vertices.

An odd vertex HnK is connected to an even vertex HgH once for each time that the bimodule HgH occurs in the product $HnKH$. Every even vertex in a cluster is connected to the same odd vertices.

Since $n \in N$ is central, $HnKH = HKHn$. HKH decomposes as $HyH \cup HzH \cup HyzH \cup H1H$. So for any $n \in N$, HnK is connected to the vertices $HyHn = HynH$, $HznH$, $HyznH$ and the four-vertex cluster Hn. The cluster Hn connects only to HnK, and the vertex $HknH$ connects to the four vertices $H(hkhk)nH$, $h \in H$. This fully describes the principal graph.

We now obtain a new infinite-depth Hadamard subfactor. Let $H = \mathbb{Z}_2$, $K = \mathbb{Z}_3$. If the two twist parameters are mutually irrational, then N is equal to $\mathbb{Z}_2^2$, and the group is $G_{2,3,6}$ of [3]. The corresponding Hadamard subfactor is of infinite depth. The principal graph for this subfactor is given in [3].

The same construction gives a family of infinite-depth Hadamard subfactors. For any two finite abelian groups H, K, we may take a generic twist with all entries mutually irrational. We will likewise obtain $N = \mathbb{Z}^{(|H|-1)(|K|-1)}$, and find an infinite-depth Hadamard subfactor of index $|H||K|$.

For all of these subfactors, $G = H * K/Int$ has a finite-index abelian subgroup, namely N. Therefore G is always amenable, and from [3] these these subfactors are amenable as well. In fact G displays polynomial growth in its generators, so the entropy conditions of [3] apply, and the subfactors are strongly amenable. No examples of nonamenable Hadamard subfactors are currently known.

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References

[1] D. Bisch, P. Das, and S. Ghosh. The planar algebra of group-type subfactors. *arxiv.org*, arXiv:0704.1128 [math.OA]:18 pp, 2008.

[2] D. Bisch, P. Das, and S. Ghosh. The planar algebra of group-type subfactors with cocycle. Work in progress.

[3] D. Bisch and U. Haagerup. Composition of subfactors: new examples of infinite depth subfactors. *Ann. Sci. École Norm. Sup. (4)*, 29(3):329–383, 1996.

[4] D. Bisch. A note on intermediate subfactors. *Pacific J. Math.*, 163(2):201–216, 1994.

[5] D. Bisch. Bimodules, higher relative commutants, and the fusion algebra associated to a subfactor. In *Operator algebras and their applications (Waterloo, On 1994/1995)*, 13–16. Amer. Math. Soc., Providence, 1997.

[6] D. Bisch, R. Nicoara, and S. Popa. Continuous families of hyperfinite subfactors with the same standard invariant. *Internat. J. Math.*, 18(3):255–267, 2007.

[7] R. D. Burstein. *Hadamard Subfactors of Bisch-Haagerup Type*. PhD dissertation, University of California, Berkeley, Department of Mathematics, 2008.

[8] W. Camp and R. Nicoara. Subfactors and Hadamard matrices. *arxiv.org*, arXiv:math/0704.1128 [math.OA]:18 pp, 2007.

[9] D. E. Evans and Y. Kawahigashi. *Quantum symmetries on operator algebras*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1998. Oxford Science Publications.

[10] F. M. Goodman, P. de la Harpe, and V. F. R. Jones. *Coxeter graphs and towers of algebras*, volume 14 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1989.

[11] K. J. Horadam. *Hadamard matrices and their applications*. Princeton University Press, Princeton, NJ, 2007.

[12] M. Izumi and Y. Kawahigashi. Classification of subfactors with the principal graph $D_n^{(1)}$. *J. Funct. Anal.*, 112(2):257–286, 1993.

[13] V. F. R. Jones. Actions of finite groups on the hyperfinite type II_1 factor. *Mem. Amer. Math. Soc.*, 28(237):v+70, 1980.

[14] V. F. R. Jones. Index for subfactors. *Invent. Math.*, 72:1–25, 1983.

[15] V. F. R. Jones. Planar algebras, I. *arxiv.org*, arXiv:9909027 [math.OA]:122 pp, 1999.

[16] V. F. R. Jones and V. S. Sunder. *Introduction to subfactors*, volume 234 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1997.

[17] U. Krishnan and V. S. Sunder. On biunitary permutation matrices and some subfactors of index 9. *Trans. Amer. Math. Soc.*, 348(12):4691–4736, 1996.

[18] Z. A. Landau. *Intermediate Subfactors*. PhD dissertation, University of California, Berkeley, Department of Mathematics, 1998.

[19] P. H. Loi. On automorphisms of subfactors. *J. Funct. Anal.*, 141(2):275–293, 1996.

[20] S. Popa. Classification of subfactors: the reduction to commuting squares. *Invent. Math.*, 101(1):19–43, 1990.

[21] S. Popa. Classification of amenable subfactors of type II. *Acta Math.*, 172(2):163–255, 1994.

[22] A. L. Svendsen. Automorphisms of subfactors from commuting squares. *Trans. Am. Math. Soc.* 358(6):2515-2543, 2004.

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