COMMUTING-SQUARE SUBFACTORS AND CENTRAL SEQUENCES

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Let $M_0 \subset M_1$ be a finite-index infinite-depth hyperfinite II_1 subfactor and ω a free ultrafilter of the natural numbers. We show that if this subfactor is constructed from a commuting square then the central sequence inclusion $M_0^\omega \cap M_1' \subset (M_1)_\omega$ has infinite Pimnser-Popa index. We will also demonstrate this result for certain infinite-depth hyperfinite subfactors coming from groups.

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1. Introduction

The central sequence subfactor induced by a finite-depth inclusion was described in [9], [10] and [11] (see also [5]). Let $M_0 \subset M_1 \cong R$ be a finite-depth finite-index hyperfinite subfactor, with Jones tower $M_0 \subset M_1 \subset M_2 \subset ... \subset M_\infty$, and ω a free ultrafilter. In this case the von Neumann algebra $M_0^\omega \cap M_1'$ is a factor. It is a finite-index subfactor of $M_1' \cap M_1^\omega = (M_1)_\omega$, and the dual of its standard invariant may be computed from the asymptotic inclusion $M_0 \vee (M_0' \cap M_\infty) \subset M_\infty$.

If the original subfactor is infinite depth then the situation is less clear. $M_0^\omega \cap M_1'$ need not be a factor. We may still define its index in $(M_1)_\omega$ using the generalized index of Pimsner and Popa [12], but it is not known in general whether this index is finite or infinite. V. Jones has conjectured that the index is always infinite in this case, and we will prove this conjecture for infinite-depth subfactors constructed in certain ways. Throughout this paper, we will always take our subfactors to be hyperfinite, so that $(M_1)_\omega$ at least is a II_1 factor.

For some infinite-depth subfactors coming from groups, it is possible to write down $M_0^\omega \cap M_1'$ as the fixed points of the II_1 factor R_ω under the action of an infinite group. The proof of the conjecture follows immediately in these cases, as we will show in section 2. However, it is generally difficult to explicitly describe $M_0^\omega \cap M_1'$; for many examples, it is not clear whether or not this algebra is larger than $\mathbb{C}$.

In certain other cases, the conjecture can be proved by exhibiting projections in $(M_1)_\omega$ whose conditional expectations onto $M_0^\omega \cap M_1'$ have arbitrarily small norm. From [12], this gives infinite index for the inclusion, but this approach presents certain technical difficulties. If the original subfactor does not admit a generating tunnel, it may be hard to specify any elements of $M_0^\omega \cap M_1'$ besides the scalars. While finding many elements of $(M_1)_\omega$ is relatively straightforward, determining their conditional expectations onto $M_0^\omega \cap M_1'$ may be intractable.

Using commuting-square subfactor ameliorates these problems. Here the tower $M_0 \subset M_1 \subset M_2 \subset ...$ may be approximated by a grid of finite-dimensional von Neumann algebras as in [7]. As we will show in section 3, the iterated canonical shift provides projections in $(M_k)_\omega$ whose conditional expectation onto to $M_0^\omega \cap M_1'$ may be bounded in norm. In section 4 some technical lemmas on index of ultrapower factors, combined with Sato's result [14] on the global index of the horizontal and vertical commuting-square subfactors, will give the desired inequalities.

Throughout this paper, we will take ω to be an arbitrary free ultrafilter.

2. Subfactors obtained from groups

2.1. *The diagonal subfactor*

The hyperfinite diagonal subfactor (see [1], [13]) may be constructed from a finitely generated group of automorphisms of the hyperfinite II_1 factor R. Let $\alpha_2, \dots, \alpha_n$ be outer automorphisms of R, with $\alpha_1 = id$. Let $A = M_n(\mathbb{C})$ have matrix units $\{e_{ij}\}$, and $M_1 = R \otimes A$. Each α_i extends to an action on M_1 as $\alpha_i \otimes id$.

Let α be the map from R to M_1 given by $\alpha(y) = \sum_{i=1}^n e_{ii}\alpha_i(y)$. Then $M_0 = \alpha(R)$, and $M_0 \subset M_1$ is the diagonal subfactor.

Let G be the image in $\mathrm{Out}R = \mathrm{Aut}R/\mathrm{Int}R$ of the group generated by the α_i's, and for $g \in G$ let α_g be an arbitrary representative of g in $\mathrm{Aut}R$. Let $J : M_0 \to M_0$ be the antilinear anti-isometry defined by $J(x) = x^*$. Then J extends to $L^2(M_0)$. For each g we may define $L^2(M_0)$ as an $M_0 - M_0$ bimodule, with action $x \cdot \xi \cdot y = xJ\alpha_g(y^*)J(\xi)$ for $x, y \in M_0$ and $\xi \in L^2(M_0)$. All the $M_0 - M_0$ bimodules in the Jones tower of the subfactor are isomorphic to a bimodule of this form. Therefore if the subfactor is infinite depth, then G is infinite.

Now we compute $M_0^\omega \cap M_1'$. If $x = (x_n) \in (M_1)_\omega$ is an element of $M_0^\omega \cap M_1'$, this means that $x_n = \alpha(y_n)$ for some bounded sequence $y = (y_n)$ in R^ω. The sequence x asymptotically commutes with the constant sequence $(1 \otimes e_{i1})$ for each i. Since $(1 \otimes e_{i1})x_n = y_n \otimes e_{i1}$, while $x_n(1 \otimes e_{i1}) = \alpha_i(y_n) \otimes e_{i1}$, it follows that $||\alpha_i(y_n) - y_n||_2$ goes to zero along the ultrafilter for all i. From the definition of α, this means that $||\alpha_i(x_n) - x_n||_2$ goes to zero along the ultrafilter as well.

The automorphism α_i induces a pointwise action $(\alpha_i)_\omega$ on $(M_1)_\omega$ given by $(\alpha_i)_\omega((a_n)) = (\alpha_i(a_n))$. The above limit then means that $(\alpha_i)_\omega$ fixes x, for all $x \in M_0^\omega \cap M_1'$. The same is true of $(\alpha_g)_\omega$ for all $g \in G$, since the α_i's generate G.

From [4], for $\alpha \in \mathrm{Aut}R$, if α is properly outer then α_ω is as well. Since $M_1 \cong R$, $g \to (\alpha_g)_\omega$ defines an outer action of G on $(M_1)_\omega$. The fixed points of this actions contain $M_0^\omega \cap M_1'$.

In [12], the authors define a generalized index on an inclusion of von Neumann algebras $A \subset B \subset M$, where M is a II_1 factor. This index $[B : A]$ is the supremum of $\{\lambda^{-1}\}$, where λ is a scalar such that $E_A(x) \geq \lambda x$ for all positive $x \in B$. Pimsner-Popa index agrees with Jones index on factors.

If G is an infinite group with outer action on a II_1 factor X, then X^G is not necessarily a factor. However the Pimsner-Popa index $[X : X^G]$ is always infinite. This is trivial if the center of X^G is infinite-dimensional or $\mathbb{C}$, and may be shown in other cases by examining the inclusions $p_i X^G \subset p_i X p_i$ for $\{p_i\}$ the minimal central projections of X^G. It follows that $M_0^\omega \cap M_1' \subset (M_1)_\omega$ has infinite index if $M_0 \subset M_1 = R$ has infinite depth.

2.2. *Bisch-Haagerup subfactors*

Let H and K be finite groups with outer actions on R, and $M_0 \subset M_1$ be the subfactor $R^H \subset R \rtimes K$. Let the action of K on R be implemented by unitaries $\{u_k\}$ in the crossed product. Such group-type subfactors were described by Bisch and Haagerup in [2]. We now construct the central sequence subalgebra. This argument was suggested to the author by V. Jones.

First we note that for a group X with an outer action α on R, $R^\omega \cap (R \rtimes X)' = R^\omega \cap \{R, \{u_x\}\}' = R_\omega \cap \{u_x\}'$. The adjoint action of the constant sequences (u_x) on elements of R_ω is just the pointwise action of α_x described in the previous section, so this is R_ω^X. Also every element of $(R^X)^\omega \cap R'$ commutes with the constant sequences coming from the u_x's, so $(R^X)^\omega \cap R'$ is contained in $(R^X)^\omega \cap (R \rtimes X)' \subset R_\omega^X$ as well.

It follows that $(R^H)^\omega \cap (R \rtimes K)' = ((R^H)^\omega \cap R') \cap (R^\omega \cap (R \rtimes K)') \subset R_\omega^H \cap R_\omega^K$. Let G now be the image of the free product $H * K$ in $\mathrm{Out}R$, i.e. $G = H * K/(\mathrm{Int}R \cap H * K)$. Then similarly to the previous section we have $(R^H)^\omega \cap (R \rtimes K)' \subset R_\omega^G$.

From [2], G is infinite if and only if the subfactor is infinite depth. In such cases, $(R^H)^\omega \cap (R \rtimes K)'$ is infinite index in R_ω, and hence in R_ω^K as well, since K is finite. $R_\omega^K = R^\omega \cap (R \rtimes K)' \subset (R \rtimes K)_\omega$, so if $M^H \subset M \rtimes K$ is infinite depth then the associated central sequence subalgebra is infinite index.

3. Towers of finite-dimensional algebras

3.1. *The canonical shift*

We will now describe the central sequence subalgebra induced by a commuting-square subfactor. We begin with a discussion of towers of finite-dimensional algebras. This is taken from [6] and [7].

Let $A_0 \subset A_1$ be a unital inclusion of finite-dimensional von Neumann algebras, with positive trace tr. We may define the Hilbert space $L^2(A_1)$ using the inner product $\langle x, y \rangle = \mathrm{tr}(y^* x)$. Then the conditional expectation E_{A_0} acts on this Hilbert space. A_1 also acts on $L^2(A_1)$, via left multiplication. Let A_2 be the von Neumann algebra on $L^2(A_1)$ generated by A_1 and E_{A_0}. This is the basic construction on the inclusion $A_0 \subset A_1$.

Repeating this process (on $A_1 \subset A_2$, etc.) gives the tower of finite-dimensional

C^* algebras

$$A_0 \subset A_1 \subset A_2 \subset A_3 \subset ...$$

If the original inclusion is connected (i.e. $A_0 \cap A_1' = \mathbb{C}$) and the trace on A_1 is the unique Markov trace, then there is a positive trace on the entire tower. In this case we may apply the GNS construction to $\cup_i A_i$, giving the hyperfinite II_1 factor A_∞. We will label the Jones projections $\{e_i\}$ for this tower according to the convention $A_i = \{A_{i-1}, e_i\}''$.

From [7], these projections have essentially the same properties as the Jones projections for the tower of factors of [8]. We mention some of these properties which we will use in this section.

- each e_i has the same trace τ
- $e_i e_{i\pm 1} e_i = \tau e_i$
- for $x \in A_{i+1}$, $e_{i+2} x e_{i+2} = e_{i+2} E_{A_i}(x)$
- for $x \in A_i$, $e_{i+2} x = 0$ implies $x = 0$
- $A_i = A_{i-1} e_i A_{i-1}$

From [7], if $0 \le i \le j$ then there is an isomorphism from $A_i' \cap A_j$ to $A_{i+2}' \cap A_{j+2}$. We now describe this isomorphism explicitly in terms of the Jones projections.

Lemma 3.1.
Let $A_0 \subset A_1 \subset A_2 \subset ... \subset A_\infty$ be the tower arising from the connected, Markov inclusion $A_0 \subset A_1$, with Jones projections $\{e_i\}$ labeled as above. Let $w_{ij} = \tau^{-\frac{j-i}{2}} e_{i+2} e_{i+3} e_{i+4} ... e_{j+1} e_{j+2}$, where τ is the trace of the Jones projections. Then for all $x \in A_i' \cap A_j$, there is a unique $y \in A_{i+2}' \cap A_{j+2}$ such that $e_{i+2} y = w_{ij} x w_{ij}^*$, and the map $\theta_{i,j} : A_i' \cap A_j \to A_{i+2}' \cap A_{j+2}$ defined by $\theta_{i,j}(x) = y$ is a $*$-isomorphism.

Proof.
Let y be an element of $A_{i+2}' \cap A_{j+2}$. We first note that

$$e_{j+2} A_{j+2} e_{j+2} = e_{j+2} A_{j+1} e_{j+2} A_{j+1} e_{j+2} = e_{j+2} A_j e_{j+2} A_j e_{j+2} = e_{j+2} A_j \qquad (3.1)$$

Therefore there is some $x \in A_j$ such that $e_{j+2} x = w_{ij}^* y w_{ij} \in e_{j+2} A_{j+2} e_{j+2}$; from the above properties of the Jones projections, this x is unique. Define $\rho : A_{i+2}' \cap A_{j+2} \to A_j$ by $e_{j+2} \rho(y) = w_{ij}^* y w_{ij}$. We will show that this map is a $*$-isomorphism into $A_i' \cap A_j$, and compute $\theta_{i,j}$ as its inverse.

It follows from the definition of ρ that $\rho(y) = \tau^{-1} E_{A_j}(w_{ij}^* y w_{ij})$. Therefore ρ respects the $*$ operation. Furthermore, since $A_i \subset A_j$, and both w_{ij} and y commute with A_i, $\rho(y)$ commutes with A_i for all y and $\rho(A_{i+2}' \cap A_{j+2}) \subset A_i' \cap A_j$.

We may compute as well

$$e_{j+2} \rho(y_1) \rho(y_2) = e_{j+2} \rho(y_1) e_{j+2} \rho(y_2) = w_{ij}^* y_1 w_{ij} w_{ij}^* y_2 w_{ij} \qquad (3.2)$$

But $w_{ij}w_{ij}^* = e_{i+2}$ by the properties of the Jones projections. e_{i+2} commutes with y_1, and $w_{ij}^* e_{i+2} = w_{ij}^*$, so this is $w_{ij}^* y_1 y_2 w_{ij} = e_{j+2}\rho(y_1 y_2)$. Therefore ρ is a homomorphism.

Now let y be a positive element of $A'_{i+2} \cap A_{j+2}$. $\rho(y) = 0$ only if $w_{ij}^* y w_{ij} = 0$, and

$$\mathrm{tr}(w_{ij}^* y w_{ij}) = \mathrm{tr}(y w_{ij} w_{ij}^*) = \mathrm{tr}(y e_{i+2}) \tag{3.3}$$

$\mathrm{tr}(ye_{i+2})$ is equal to $\mathrm{tr}(E_{A_{i+2}}(ye_{i+2})) = \mathrm{tr}(E_{A_{i+2}}(y)e_{i+2})$. Since $y > 0$ and y commutes with A_{i+2}, $E_{A_{i+2}}(y)$ is a positive element of the center $Z(A_{i+2})$. e_{i+2} has full central support in A_{i+2}, so for any $a > 0$ in $Z(A_{i+2})$, $\mathrm{tr}(e_{i+2}a) > 0$. So $\mathrm{tr}(e_{i+2}E_{A_{i+2}}(y)) = \mathrm{tr}(e_{i+2}y) = \mathrm{tr}(\rho(y))$ is positive. This means that for any $z \in A'_{i+2} \cap A_{j+2}$, $\mathrm{tr}(\rho(z)^*\rho(z)) = \mathrm{tr}(\rho(z^*z)) > 0$, and ρ is an injective homomorphism from $A'_{i+2} \cap A_{j+2}$ onto $A'_i \cap A_j$.

From [7], the algebras $A'_{i+2} \cap A_{j+2}$ and $A'_i \cap A_j$ are isomorphic, so ρ is an isomorphism. It follows that for $x \in A'_i \cap A_j$, $x = \rho(y)$ for some unique $y \in A'_{i+2} \cap A_{j+2}$.

$x = \rho(y)$ means that $e_{j+2}x = w_{ij}^* y w_{ij}$. Conjugating by w_{ij} and its adjoint shows that this is true if and only if $e_{i+2}y = w_{ij} x w_{ij}^*$, so there is also a unique y obeying this second relation. If we define a map $\theta_{i,j}$ from $A'_i \cap A_j$ to $A'_{i+2} \cap A_{j+2}$ by $\theta_{i,j}(x) = y$, then $\theta_{i,j} = \rho^{-1}$ and is an isomorphism as desired. □

This map θ is the canonical shift on the tower of finite-dimensional algebras (c.f. [11]). We may likewise construct the iterated shift $\theta_{i,j}^k$ as the product

$$\theta_{i,j}^k = \theta_{i+2(k-1),j+2(k-1)}\theta_{i+2(k-2),j+2(k-2)}...\theta_{i+2,j+2}\theta_{i,j} \tag{3.4}$$

This is a $*$-isomorphism from $A'_i \cap A_j$ to $A'_{i+2k} \cap A_{j+2k}$.

The asymptotic properties of the trace on the tower will be important later on, so we mention some results derived from Perron-Frobenius theory, following [7].

For a finite-dimensional von Neumann algebra A with minimal central projections $\{p_1, ..., p_n\}$, we have a trace vector $\mathbf{t}$ and a size vector $\mathbf{s}$; i.e., the matrix subalgebra $p_i A$ is s_i by s_i, and the trace of a minimal projection in this subalgebra is t_i. From normalization of the trace, we must have $\langle \mathbf{s}, \mathbf{t} \rangle = \sum_{i=1}^n s_i t_i = 1$.

Let $\mathbf{s}^{(i)}$ and $\mathbf{t}^{(i)}$ be the size and trace vectors for A_i. From [7], the inclusion matrices of $A_n \subset A_{n+1}$ and $A_{n+1} \subset A_{n+2}$ are transposes of each other for all n. Let Λ be the inclusion matrix of $A_i \subset A_{i+1}$; then this means that there is a labeling of the central projections of the A_i's such that for all $k \geq 0$ we have $\mathbf{s}^{(i+2k+1)} = \Lambda^T \mathbf{s}^{(i+2k)}$ and $\mathbf{s}^{(i+2k+2)} = \Lambda \mathbf{s}^{(i+2k+1)}$. Since $A_i \subset A_{i+1}$ is a connected inclusion, all entries of $(\Lambda\Lambda^T)^k$ are positive for k sufficiently large, and this matrix has a unique Perron-Frobenius eigenvector v with $||v|| = 1$ and $v_i > 0$. The Perron-Frobenius eigenvalue is $||\Lambda\Lambda^T|| = \tau^{-1} = \mathrm{tr}(e_i)^{-1}$.

We likewise have $\mathbf{t}^{(i+2k)} = \Lambda \mathbf{t}^{(i+2k+1)}$, $\mathbf{t}^{(i+2k+1)} = \Lambda^T \mathbf{t}^{(i+2k+2)}$. From [7], the Markov condition implies that $\mathbf{t}^{(i+2k)}$ is a positive scalar multiple of v for all k, so for all $k \geq 0$ we have $\mathbf{t}^{(i+2k+2)} = \tau \mathbf{t}^{(i+2k)}$.

It then follows from Perron-Frobenius theory that $\lim_{k\to\infty} \mathbf{s}^{(i+2k)}\tau^k = \gamma \mathbf{t}^{(i)}$, for some positive scalar γ. The normalization condition and the above expression for $\mathbf{t}^{(i+2k)}$ implies that $\gamma = ||\mathbf{t}^{(i)}||^{-2}$. With this result we can describe the asymptotic behavior of the trace on $A'_{i+2k} \cap A_{j+2k}$.

Lemma 3.2. *Let $A_0 \subset A_1 \subset ... \subset A_\infty$ be a Jones tower as above. Fix $0 \le i \le j$. There exists $\epsilon > 0$ such that for all $k \ge 0$, and all nonzero projections $p \in A'_{i+2k} \cap A_{j+2k}$, we have $\mathrm{tr}(p) \ge \epsilon$.*

Proof. For $k \ge 0$, we take the minimal central projections of A_{i+2k}, A_{j+2k} to be $\{p_x^k\}$, $\{q_y^j\}$. As above, we may use the same set of labels for all k. Every minimal central projection r_{xy}^k in $A'_{i+2k} \cap A_{j+2k}$ is of the form $p_x^k q_y^k$. Let $\tilde{r}_{xy}^k$ be an arbitrary minimal projection in the matrix algebra $r_{xy}^k(A'_{i+2k} \cap A_{j+2k})$. From [7] we may compute $\mathrm{tr}(\tilde{r}_{xy}^k) = s_x^{(i+2k)} t_y^{(j+2k)}$.

Since $\mathbf{t}^{(j+2k)} = \tau^k \mathbf{t}^{(j)}$ and $\mathbf{s}^{(i+2k)}\tau^k$ approaches $\mathbf{t}^{(i)}||\mathbf{t}^{(i)}||^{-2}$, this trace approaches the limit $t_x^{(i)} t_y^{(j)} ||\mathbf{t}^{(i)}||^{-2}$ as k goes to ∞. Then for fixed i and j, the sequence $(\mathrm{tr}(\tilde{r}_{xy}^k))$ consists of positive numbers approaching a positive value, and is therefore bounded away from zero. Since the centers $Z(A_i)$ and $Z(A_j)$ are finite-dimensional, there is some number $\epsilon > 0$ which bounds the traces of these minimal projections away from zero for all x, y. We then have $\epsilon \le \mathrm{tr}(\tilde{r}_{xy}^k)$ for all k, x, y. Since the $\tilde{r}_{xy}^k$'s are minimal projections in every central component of $A'_{i+2k} \cap A_{j+2k}$, it follows that ϵ is less than or equal to the trace of any nonzero projection in any of these algebras. □

3.2. *The grid of finite-dimensional algebras*

Let

$$\begin{array}{ccc} A_{01} & \subset & A_{11} \\ \cup & & \cup \\ A_{00} & \subset & A_{10} \end{array}$$

be a quadrilateral of finite-dimensional von Neumann algebras, with trace. We may construct the Hilbert space $L^2(A_{11})$ as in the previous section. This quadrilateral is a commuting square if $E_{A_{10}}$ commutes with $E_{A_{01}}$ on this Hilbert space.

In [6] (see also [7]), the authors construct a subfactor from such a commuting square. The commuting square must have some additional properties: every constituent inclusion must be connected and have the unique Markov trace. Additionally, the square must be symmetric. Several equivalent conditions for symmetry are discussed in [7]; one of them is $A_{10}A_{01} = A_{11}$, which we will take as our definition.

All of our commuting squares in this paper will be connected, symmetric, and Markov. The following construction is taken from [7].

Starting with a commuting square with these properties, we may build the Jones tower $A_{01} \subset A_{11} \subset A_{21} \subset ... \subset A_{\infty 1} = M_1$, with Jones projections $\{e_i\}$ labeled as in the previous section. Then we may define A_{i0} inductively for $i \ge 2$ by $A_{i0} =$

$\{A_{i-1,0}, e_i\}''$. From [7], the A_{i0}'s form a Jones tower as well, and $M_0 = A_{\infty 0} = \overline{\cup_i A_{0i}}^{st}$ is a subfactor of the hyperfinite II_1 factor M_1.

The index $[M_1 : M_0]$ will be finite, and we may construct the tower of II_1 factors $M_0 \subset M_1 \subset M_2 \subset ....$ We will label the vertical Jones projections for this tower by $M_j = \{M_{j-1}, f_j\}''$, $j \geq 2$.

We may then construct a grid of finite-dimensional algebras, defining $A_{ij} \subset M_j$ by $\{A_{i,j-1}, f_j\}''$. $A_{0j} \subset A_{1j} \subset A_{2j} \subset ... \subset A_{\infty j} = M_j$ is again a Jones tower, and we may use the same Jones projections $\{e_i\}$ for any value of j.

This implies that for $0 \leq j \leq k$, $M_j \subset M_k$ is also a commuting-square subfactor. It may be shown that the quadrilateral

$$\begin{array}{ccc} A_{0k} & \subset & A_{1k} \\ \cup & & \cup \\ A_{0j} & \subset & A_{1j} \end{array}$$

commutes and is symmetric (i.e., $A_{0k}A_{1j} = A_{1k}$) by induction on $k-j$. The inclusion $A_{0k} \subset A_{1k}$ is Markov, so the entire square is Markov [7], and connectedness follows from connectedness of the original commuting square. So this quadrilateral generates a commuting-square subfactor. Since the Jones projections for the tower of $A_{0k} \subset A_{1k}$ are the e_i's, this subfactor may be taken to be $(A_{1j} \cup \{e_i\})'' \subset (A_{1k} \cup \{e_i\})'' = M_j \subset M_k$.

We may likewise consider the vertical tower $A_{i0} \subset A_{i1} \subset ....$ From [7] this is again a Jones tower with the same Jones projections $\{f_j\}$ for any value of i. The vertical limits $P_i = A_{i\infty} = \overline{\cup_j A_{ij}}^{st}$ are II_1 factors, with $P_0 \subset P_1 \subset P_2 \subset ...$ a Jones tower.

This grid may be used to explicitly compute the higher relative commutants of a commuting-square subfactor. Ocneanu's compactness argument, described in [7], shows that $M_0' \cap M_k = A_{10}' \cap A_{0k}$. An analogous statement may be made about the vertical relative commutants $P_0' \cap P_k$.

It follows from Perron-Frobenius arguments similar to those used in Lemma 3.2 that all commuting-square subfactors are extremal. This result wil be necessary for demonstrating the desired inequalities on index in section 4.

Lemma 3.3. *Let the grid of algebras $\{A_{ij}\}$ be as above. Let x be an element of A_{0k}. Then for $j \leq k$, $\lim_{i\to\infty} E_{A'_{ij} \cap A_{ik}}(x)$ exists and equals $E_{M'_j \cap M_k}(x)$.*

Proof.

Fix $x \in A_{0k}$.

We consider the projections $E_{A'_{ij} \cap M_k}$, acting on $L^2(M_k)$. This is a decreasing series of orthogonal projections onto closed subspaces of $L^2(M_k)$, and therefore strongly approaches the orthogonal projection onto the intersection of these subspaces. So $\lim_{i\to\infty} E_{A'_{ij} \cap M_k}(x)$ exists and equals $E_{\cap_i (A'_{ij} \cap M_k)}(x)$. We may compute

$$\cap_i (A'_{ij} \cap M_k) = (\cup_i A_{ij})' \cap M_k = (\overline{(\cup_i A_{ij})}^{st})' \cap M_k = M'_j \cap M_k \qquad (3.5)$$

The quadrilateral

$$\begin{array}{ccc} A_{ik} & \subset & M_k \\ \cup & & \cup \\ A'_{ij} \cap A_{ik} & \subset & A'_{ij} \cap M_k \end{array}$$

commutes, since $E_{A_{ik}}(A'_{ij} \cap M_k)$ commutes with A_{ij}. Therefore $E_{A'_{ij} \cap M_k}(x) = E_{A'_{ij} \cap A_{ik}}(x)$, since $x \in A_{0k} \subset A_{ik}$. It follows that $\lim_{i \to \infty} E_{A'_{ij} \cap A_{ik}}(x)$ exists and equals $E_{M'_j \cap M_k}$, as desired. □

Theorem 3.1. *All commuting-square subfactors are extremal.*

Proof.

Let the grid of algebras $\{A_{ij}\}$ be as above. Let $\tau_h = \mathrm{tr}(e_i)$ and $\tau_v = \mathrm{tr}(f_i)$ be the Markov constants for the horizontal and vertical inclusions respectively, and let A_{ij} have size vector $\mathbf{s}^{(ij)}$ and trace vector $\mathbf{t}^{(ij)}$. From [7] we may label the central projections of the $A'_{ij}s$ so that $\mathbf{t}^{(i+2k,j+2l)} = \mathbf{t}^{(ij)} \tau_v^k \tau_h^l$. From the previous section, with this labeling we have

$$\lim_{k \to \infty} \mathbf{s}^{(i+2k,j+2l)} \tau_h^k = \mathbf{t}^{(i,j+2l)} ||\mathbf{t}^{(i,j+2l)}||^{-2} = \mathbf{t}^{(ij)} ||\mathbf{t}^{(ij)}||^{-2} \tau_v^{-l} \tag{3.6}$$

Now we consider the inclusion of algebras $A_{2k,0} \subset A_{2k,1} \subset A_{2k,2}$. Let the minimal central projections of these three algebras be $\{p_x^{(k)}\}$, $\{q_y^{(k)}\}$, and $\{\tilde{p}_x^{(k)}\}$ respectively, labeled as above. From [7], we then have $p_x^{(k)} f_2 = \tilde{p}_x^{(k)} f_2$. The minimal central projections of $A'_{2k,0} \cap A_{2k,1}$ are all of the form $p_x^{(k)} q_y^{(k)}$, and those of $A'_{2k,1} \cap A_{2k,2}$ are of the form $\tilde{p}_x^{(k)} q_y^{(k)}$. Our choice of labels lets us use the same indices (x, y) for all k.

Let $r_{xy}^{(k)}$ be any minimal projection in $p_x^{(k)} q_y^{(k)} (A'_{2k,0} \cap A_{2k,1})$. Then $\mathrm{tr}(r_{xy}^{(k)}) = s_x^{(2k,0)} t_y^{(2k,1)}$. This means that

$$\lim_{k \to \infty} \mathrm{tr}(r_{xy}^{(k)}) = t_x^{(00)} t_y^{(01)} ||\mathbf{t}^{(01)}||^{-2} \tag{3.7}$$

Similarly, if $\tilde{r}_{xy}^{(k)}$ is an arbitrary minimal projection in $\tilde{p}_x^{(k)} q_y^{(k)} (A'_{2k,1} \cap A_{2k,2})$, then

$$\lim_{k \to \infty} \mathrm{tr}(\tilde{r}_{xy}^{(k)}) = t_x^{(00)} t_y^{(01)} ||\mathbf{t}^{(00)}||^{-2} \tau_v \tag{3.8}$$

If Λ_v is the inclusion matrix for $A_{00} \subset A_{01}$, then

$$||\mathbf{t}^{(00)}||^{-2} = ||\Lambda_v^T \mathbf{t}^{(01)}||^{-2} = \langle \Lambda_v \Lambda_v^T \mathbf{t}^{(01)}, \mathbf{t}^{(01)} \rangle^{-1} = ||\mathbf{t}^{(01)}||^{-2} \tau_v^{-1} \tag{3.9}$$

So the two limits above are the same, and for all x, y and $\epsilon > 0$ there exists k_0 such that $k > k_0$ implies $||\mathrm{tr}(\tilde{r}_{xy}^{(k)}) - \mathrm{tr}(r_{xy}^{(k)})|| < \epsilon$.

From [7], $A_{2k,2}$ acts on the Hilbert space $L^2(A_{2k,1})$, where $A_{2k,1}$ acts by left multiplication and f_2 is the conditional expectation onto $A_{2k,0}$. If J is the order-2 anti-linear anti-isometry given by $J(x) = x^*$, then on this Hilbert space $J A_{2k,1} J = A'_{2k,1}$, $J A'_{2k,0} J = A_{2k,2}$. It follows that $J(A'_{2k,0} \cap A_{2k,1}) J = A'_{2k,1} \cap A_{2k,2}$. If we define

$\phi(x)$ on $A'_{2k,0} \cap A_{2k,1}$ by $\phi(x) = Jx^*J$, then ϕ is a $*$-isomorphism from $A'_{2k,0} \cap A_{2k,1}$ onto $A'_{2k,1} \cap A_{2k,2}$.

With this labeling we have $\phi(p_x q_y) = \phi(p_x)\phi(q_y) = \tilde{p}_x q_y$ [7]. Therefore ϕ sends a minimal projection in $p_x q_y(A'_{2k,0} \cap A_{2k,1})$ to one in $\tilde{p}_x q_y(A'_{2k,1} \cap A_{2k,2})$.

For any $\epsilon > 0$, the above argument on convergence of traces (and Lemma 3.2) gives k_0 such that $k \geq k_0$ implies $|\mathrm{tr}(\phi(r)) - \mathrm{tr}(r)| < \epsilon||r||_2$ for all minimal projections r in $A'_{2k,0} \cap A_{2k,1}$. Since $A'_{2k,0} \cap A_{2k,1}$ is spanned by minimal projections, and is of fixed finite dimension, for k sufficiently large we then have $|\mathrm{tr}(\phi(v)) - \mathrm{tr}(v)| < \epsilon||v||_2$ for all $v \in A'_{2k,0} \cap A_{2k,1}$.

Now choose $x \in A'_{2k,0} \cap A_{2k,1}$, and let z be an arbitrary element of $L^2(A_{2k,1})$. We compute $(xf_2)(z) = xE_{A_{2k,0}}(z)$. x commutes with $A_{2k,0}$, so this is equal to

$$E_{A_{2k,0}}(z)x = (x^*E_{A_{2k,0}}(z)^*)^* = (Jx^*Jf_2)(z) = (\phi(x)f_2)(z) \tag{3.10}$$

Since $\phi(x)f_2$ and xf_2 agree on $L^2(A_{2k,1})$, they are the same element of $A'_{2k,0} \cap A_{2k,2}$.

This means that $\mathrm{tr}(\phi(x)f_2) = \mathrm{tr}(xf_2)$, which is equal to $\mathrm{tr}(x)\tau_v$ by the Markov property of the trace. For k sufficiently large, this is within $\epsilon||x||_2$ of $\mathrm{tr}(\phi(x))\tau_v = \mathrm{tr}(\phi(x)(\tau_v 1))$, for any $\phi(x) \in A'_{2k,1} \cap A_{2k,2}$.

This implies $||E_{A'_{2k,1} \cap A_{2k,2}}(f_2) - \tau_v 1||_2 < \epsilon$ for k sufficiently large, so $\lim_{k\to\infty} E_{A'_{2k,1} \cap A_{2k,2}}(f_2) = \tau_v 1$. From Lemma 3.3, this means that $E_{M'_1 \cap M_2}(f_2) = \tau_v 1$, and so $M_0 \subset M_1$ is extremal. □

4. Commuting-square subfactors and central sequences

4.1. *Remarks on index*

The iterated canonical shift from the previous section allows us to give a bound for the norm of certain central sequence subalgebras.

Lemma 4.1. *Let $M_0 \subset M_1$ be the subfactor obtained by iterating the basic construction on the commuting square*

$$\begin{array}{ccc} A_{01} & \subset & A_{11} \\ \cup & & \cup \\ A_{00} & \subset & A_{10} \end{array}$$

Let the grid of algebras $\{A_{ij}\}$ be as in section 3.2. Let p be a projection in $A'_{0k} \cap A_{ik}$. Then $[(M_k)_\omega : M_0^\omega \cap M_1'] \geq ||E_{A'_{00} \cap A_{i0}}(p)||^{-1}$.

Proof.

The conditions of Lemma 3.1 are satisfied by the Jones tower

$$A_{0k} \subset A_{1k} \subset ... \subset M_k$$

So there are isomorphisms $\{\theta^l_{i,j}\}$ from $A'_{ik} \cap A_{jk}$ to $A'_{i+2l,k} \cap A_{j+2l,k}$. Isomorphisms $\{\psi^l_{i,j}\}$ likewise exist from $A'_{i0} \cap A_{j0}$ to $A'_{i+2l,0} \cap A_{j+2l,0}$, as in the lemma.

For p a projection in $A'_{0k} \cap A_{ik}$, let $\tilde{p}$ be the element of M_k^ω given by $\tilde{p}_l = \theta^l_{0,j}(p)$. This is a projection.

By construction $\tilde{p}$ commutes with $A_{2l,k}$ for all l. Since these algebras generate M_k, $\tilde{p}$ is contained in $(M_k)_\omega$. Since $\theta^l_{0,j}(p) \in A'_{2l,k} \cap A_{j+2l,k}$, from Lemma 3.2 there exists $\epsilon > 0$ such that $\mathrm{tr}(\theta^l_{0,j})(p) \geq \epsilon$ for all l. Therefore $\tilde{p} \neq 0$ as an element of $(M_k)_\omega$.

Let x be an element of $A'_{i,k} \cap A_{j,k}$. From section 3.1, $\theta^1_{i,j}(x) = \theta_{i,j}(x)$ is defined as the unique element of $A'_{i+2,k} \cap A_{j+2,k}$ such that $e_{i+2}\theta_{i,j}(x) = w_{ij}xw^*_{ij}$. But w_{ij} is a word in the horizontal Jones projections, which are contained in M_0. So applying E_{M_0} to both sides, we find that $e_{i+2}E_{M_0}(\theta_{i,j}(x)) = w_{ij}E_{M_0}(x)w^*_{ij}$.

For any $n \geq 0$, it may be shown by induction on k that $E_{M_0}(A_{nk}) = A_{n0}$. Therefore $E_{M_0}(\theta_{i,j}(x)) \in A'_{i+2,0} \cap A_{j+2,0}$ and $E_{M_0}(x) \in A'_{i,0} \cap A_{j,0}$. But for $y \in A'_{i,0} \cap A_{j,0}$, $\psi_{i,j}(y)$ is defined as the unique element of $A'_{i+2,0} \cap A_{j+2,0}$ obeying the relation $e_{i+2}\psi_{i,j}(y) = w_{ij}yw^*_{ij}$. Therefore for all $i \leq j$, and all $x \in A'_{ik} \cap A_{jk}$, $\psi_{i,j}(E_{M_0}(x)) = E_{M_0}(\theta_{i,j}(x))$.

From the definitions of the composite operators $\psi^l_{0,j}$ and $\theta^l_{0,j}$ in section 3.1, it follows that $E_{M_0}(\theta^l_{0,j}(p)) = \psi^l_{0,j}(E_{M_0}(p))$.

We may compute the conditional expectation from M^ω_k onto M^ω_0 by applying E_{M_0} pointwise. Therefore $||E_{M^\omega_0}(\tilde{p})_l|| = ||E_{M_0}(\theta^l_{0,j}(p))|| = ||\psi^l_{0,j}(E_{M_0}(p))||$. Since isomorphisms preserve ∞-norm, this is $||E_{M_0}(p)||$.

All components of $E_{M^\omega_0}(\tilde{p})$ have norm at most $||E_{M_0}(p)||$, so the same is true of $E_{M^\omega_0}(\tilde{p})$ itself. Since $M^\omega_0 \cap M'_1$ is a von Neumann subalgebra of M^ω_0, we have $||E_{M^\omega_0 \cap M'_1}(\tilde{p})|| \leq ||E_{M_0}(p)||$ as well. From [12], since $\tilde{p}$ is a nonzero projection in $(M_1)_\omega$, this gives

$$[(M_1)_\omega : M^\omega_0 \cap M'_1] \geq ||E_{M_0}(p)||^{-1} = ||E_{A'_{00} \cap A_{i0}}(p)||^{-1} \tag{4.1}$$

as desired. □

The above argument only bounds $[(M_k)_\omega : M^\omega_0 \cap M'_1]$, which is not quite what we want. Some additional lemmas on index will allow us to show that this bound also applies to $[(M_1)_\omega : M^\omega_0 \cap M'_1]$.

Lemma 4.2. *Let $X \subset Y$ be a II_1 subfactor, with X hyperfinite. Then $X' \cap Y^\omega$ is a II_1 factor.*

Proof. For $L \subset P \subset Q$ II_1 factors, and L hyperfinite, the central freedom lemma (see [10]) states that $(L' \cap P^\omega)' \cap Q^\omega = L \vee (P' \cap Q)^\omega$.

We apply this lemma to the inclusion $X \subset Y \subset Y$, obtaining $(X' \cap Y^\omega)' \cap Y^\omega = X \vee (Y' \cap Y)^\omega$. Since Y is a factor, this is just X. This means that $(X' \cap Y^\omega)' \cap Y^\omega \cap X' = X \cap X' = \mathbb{C}1$, since X is a factor as well. So $X' \cap Y^\omega$ has trivial center, and is a factor.

This factor is contained in the II_1 factor Y^ω, and contains the II_1 factor X_ω. Therefore $X' \cap Y^\omega$ is of type II_1. □

Lemma 4.3. *Let $Y \subset Z$ be a finite-index II_1 subfactor. Let X be a von Neumann* *subalgebra of* Y*, with finite Pimsner-Popa index* $[Y : X]$*. Then*

$$[Z : X] = [Z : Y][Y : X]$$

Proof. By applying the downward basic construction (c.f. [8]) we may find a projection $e \subset Z$ such that $E_Y(e) = [Z : Y]^{-1}1$ and $Y_0 = e' \cap Y$ is a II_1 factor. Also, from [12], for all ϵ we have a positive element $q \in Y$ such that $E_X(q)$ is not greater than $([Y : X]^{-1} + \epsilon)q$.

Since Y_0 is a II_1 factor, it contains projections of every trace. Therefore every projection in Y is unitarily conjugate to an element of Y_0. The same is true of any countable bounded linear combination of orthogonal projections. Furthermore, for any $\alpha < 1$, we may use the Borel functional calculus for q to find $q_\alpha \in Y$ such that $\alpha q \leq q_\alpha \leq q$ and q_α is a countable linear combination of orthogonal projections.

Now suppose $E_X(q_\alpha) \geq \lambda q_\alpha$ for some scalar λ. $E_X(q) \geq E_X(q_\alpha)$ and $\alpha q \leq q_\alpha$, so $E_X(q) \geq \lambda\alpha q$. This means $\lambda\alpha$ must be less than $[Y : X]^{-1} + \epsilon$, from the properties of q. Choosing α_0 sufficiently close to one, we find $q' = q_{\alpha_0}$ such that $E_X(q')$ is not greater than $([Y : X] + 2\epsilon)q'$, and there exists a unitary $u \in Y$ such that $uq'u^* \in Y_0$.

Since $uq'u^*$ is in Y_0, it commutes with e. This means that u^*eu commutes with q', and u^*euq' is a positive element of Z. Now we compute $E_X(u^*euq')$. This is equal to $E_X(E_Y(u^*euq'))$.

$$E_Y(u^*euq') = u^*E_Y(e)uq' = u^*[Z : Y]^{-1}uq' = [Z : Y]^{-1}q' \tag{4.2}$$

$E_X([Z : Y]^{-1}q') = [Z : Y]^{-1}E_X(q')$, which is not greater than $[Z : Y]^{-1}([Y : X]^{-1} + 2\epsilon)q'$. Therefore $[Z : X] \geq [Z : Y][Y : X]$.

For any positive element $a \in Z$, we have $E_X(a) = E_X(E_Y(a))$. By definition of Pimsner-Popa index, $E_Y(a) \geq [Z : Y]^{-1}a$. So

$$E_X(E_Y(a)) \geq [Y : X]^{-1}E_Y(a) \geq [Z : Y]^{-1}[Y : X]^{-1}a \tag{4.3}$$

Since this is true for all $a \geq 0$ in Z, we have $[Z : X] \leq [Z : Y][Y : X]$ as well. This gives the desired equality. □

Lemma 4.4. *Let*

$$\begin{array}{ccc} C & \subset & D \\ \cup & & \cup \\ A & \subset & B \end{array}$$

be a quadrilateral of von Neumann algebras. Let B*,* C*, and* D *be* II_1 *factors, with* $[D : B] \leq [D : C]$ *and* $[D : B] < \infty$*. Then* $[C : A] \leq [B : A]$*.*

Proof. First we consider the case $[D : A] = \infty$. From [PP], if $X \subset Y \subset Z$ are von Neumann algebras with finite trace, and $[Z : X] = \infty$, then either $[Y : X]$ or $[Z : Y]$ must be infinite. Since $[D : B] < \infty$, we must have $[B : A] = \infty$, giving the desired inequality for any value of $[C : A]$.

Now let $[D : A]$ be finite. This implies that all four inclusions of the above quadrilateral are finite index. From Lemma 4.3, since D and B are factors we have

$[D : A] = [D : B][B : A]$. Likewise $[D : A] = [D : C][C : A]$. Since all indices are positive real numbers here, we may compute $[C : A] = \frac{[D:B][B:A]}{[D:C]}$. $[D : B] \leq [D : C]$ by hypothesis, so $[C : A] \leq [B : A]$ in this case as well. $\square$

With these results, we can start to approximate the indices of various kinds of central sequence inclusions.

Lemma 4.5. *Let $M_0 \subset M_1$ be a finite-index II_1 factor, with X a hyperfinite subfactor of M_0. Then $[X' \cap M_1^\omega : X' \cap M_0^\omega] = [M_1 : M_0]$.*

Proof. We first consider the quadrilateral of von Neumann algebras

$$\begin{array}{ccc} M_0^\omega & \subset & M_1^\omega \\ \cup & & \cup \\ X' \cap M_0^\omega & \subset & X' \cap M_1^\omega \end{array}$$

This is a commuting square since $X \subset M_0^\omega$. So for any $a > 0$ contained in $X' \cap M_1^\omega$ we have $E_{X' \cap M_0^\omega}(a) = E_{M_0^\omega}(a) \geq [M_1^\omega : M_0^\omega]^{-1} a$. This means that $[X' \cap M_1 : X' \cap M_0] \geq [M_1^\omega : M_0^\omega]$, which is equal to $[M_1 : M_0]$ from [12].

From the downward basic construction there exists $e \in M_1$ with $E_{M_0}(e) = [M_1 : M_0]^{-1}$, $e' \cap M_0 = M_{-1}$, where M_{-1} is a II_1 factor.

Let $X = \overline{\{\cup_i X_i\}}^{st}$, where each X_i is a matrix algebra and $X_i \subset X_{i+1}$. There is a matrix algebra $A_i \subset M_{-1}$ of the same size as X_i. Any two matrix algebras of the same size in a II_1 factor are unitarily equivalent, so there is a unitary $u_i \in M_0$ with $u_i X_i u_i^* \subset M_{-1}$.

Since $u_i X_i u_i^*$ commutes with e, it follows that $u_i^* e u_i$ commutes with X_i. The sequence $\tilde{e}$ defined by $\tilde{e}_i = u_i^* e u_i$ gives a projection in M_1^ω. This sequence asymptotically commutes with every X_i. Since the X_i's are dense in X, $\tilde{e} \in X' \cap M_1^\omega$.

As above, $E_{X' \cap M_0^\omega}(\tilde{e}) = E_{M_0^\omega}(\tilde{e})$. We may compute this conditional expectation by applying E_{M_0} pointwise, obtaining $(E_{M_0^\omega}(\tilde{e}))_i = E_{M_0}(u_i^* e u_i) = u_i^* E_{M_0}(e) u_i = [M_1 : M_0]^{-1} 1$. This is a constant sequence, and so $E_{X' \cap M_0^\omega}(\tilde{e}) = [M_1 : M_0]^{-1} 1$ as an element of $X' \cap M_0^\omega$. It follows that $[X' \cap M_1^\omega : X' \cap M_0^\omega] \geq [M_1 : M_0]$. The reverse inequality has been shown above, so the two indices are equal. $\square$

Lemma 4.6. *Let $M_0 \subset M_1$ be an extremal finite-index hyperfinite II_1 subfactor. Then $[M_0' \cap M_1^\omega : (M_1)_\omega] \geq [M_1 : M_0]$.*

Proof. Let $X \subset Y \subset Z$ be an inclusion of II_1 factors, with X hyperfinite. We consider the quadrilateral of von Neumann algebras

$$\begin{array}{ccc} Y' \cap Z^\omega & \subset & X' \cap Z^\omega \\ \cup & & \cup \\ Y' \cap Z & \subset & X' \cap Z \end{array}$$

$X' \cap Z^\omega$ is a II_1 factor by Lemma 4.2, so there is a unique trace on this quadrilateral, and we may compute conditional expectations. Since $Y \subset Z$, $E_Z(Y' \cap Z^\omega)$ is contained in $Y' \cap Z$, and so $E_{X' \cap Z}(Y' \cap Z^\omega)$ is as well. Therefore the quadrilateral is a commuting square, and in general for $a \in X' \cap Z$, $E_{Y' \cap Z^\omega}(a) = E_{Y' \cap Z}(a)$.

Let M_2 be obtained by applying the basic construction to $M_0 \subset M_1$, with Jones projection $e \in M_2$. We consider the quadrilateral of II_1 factors

$$\begin{array}{ccc} M_1' \cap M_2^\omega & \subset & M_0' \cap M_2^\omega \\ \cup & & \cup \\ (M_1)_\omega & \subset & M_0' \cap M_1^\omega \end{array}$$

We may embed e in $M_0' \cap M_2^\omega$ as a constant sequence. The above argument shows that $E_{M_1' \cap M_2^\omega}(e) = E_{M_1' \cap M_2}(e)$. This is $[M_2 : M_1]^{-1}1$ since $M_0 \subset M_1$ is extremal by hypothesis. Therefore $[M_0' \cap M_2^\omega : M_1' \cap M_2^\omega] \geq [M_2 : M_1]$.

From Lemma 4.5, we have $[M_0' \cap M_1^\omega : M_0' \cap M_2^\omega] = [M_2 : M_1]$. We have $[M_2 : M_1] = [M_1 : M_0]$ [8], which is finite by hypothesis. So all the assumptions of Lemma 4.4 are satisfied, implying that $[M_0' \cap M_1^\omega : (M_1)_\omega] \geq [M_1' \cap M_2^\omega : (M_1)_\omega]$. Again by Lemma 4.5, $[M_1' \cap M_2^\omega : (M_1)_\omega] = [M_2 : M_1]$, giving the desired inequality. □

4.2. *Infinite-depth central sequence subfactors*

We have not yet used the fact that our initial subfactor is of infinite depth. A result of Sato allows us to use this to describe the asymptotic behavior of $[(M_k)_\omega : M_0^\omega \cap M_1']$.

Lemma 4.7. *Let $M_0 \subset M_1$ be a commuting-square subfactor of infinite depth, with Jones tower $M_0 \subset M_1 \subset M_2 \subset ....$ Then $\lim_{k \to \infty}[(M_k)_\omega : M_0^\omega \cap M_1'] = \infty$.*

Proof. Let $M_0 \subset M_1$ be generated by the commuting square

$$\begin{array}{ccc} A_{01} & \subset & A_{11} \\ \cup & & \cup \\ A_{00} & \subset & A_{10} \end{array}$$

with the grid of algebras $\{A_{ij}\}$ as in section 3.2. The centers of the A_{ij}'s have bounded dimension; let this bound be L. Then $\dim Z(A_{ij}' \cap A_{kl}) < L^2$ for any $0 \leq i \leq k, 0 \leq j \leq l$.

Choose $\epsilon > 0$.

Since $M_0 \subset M_1$ is of infinite depth, the vertical subfactor $A_{0\infty} \subset A_{1\infty} = P_0 \subset P_1$ is also of infinite depth [14]. Therefore the central dimension of $P_0' \cap P_i$ increases without limit as i goes to infinity. By Ocneanu compactness, the same statement is true of the algebras $A_{01}' \cap A_{i0}$. Specifically there is some i with $\dim Z(A_{01}' \cap A_{i0}) > 2L^2/\epsilon$.

For any $k > 1$, we consider the inclusion $A_{01}' \cap A_{i0} \subset A_{0k}' \cap A_{ik}$. There must be a minimal central projection q_k of $A_{0k}' \cap A_{ik}$ such that $\dim Z(q_k(A_{01}' \cap A_{i0})) > 2/\epsilon$.

Suppose $q_k(A_{01}' \cap A_{i0})$ has minimal central projections $r_1, ..., r_n$, $n > 2/\epsilon$. Let s be a projection in $q_k(A_{0k}' \cap A_{ik})$ such that $sr_a = s_a$ is a projection for all $1 \leq a \leq ne$, and s_a is minimal in $q_k(A_{0k}' \cap A_{ik})$. All the s_a's are equivalent in $q_k(A_{0k}' \cap A_{ik})$, so the projections $\{s_1, ..., s_n\}$ may be extended to a system of matrix units $\{t_{ab}\}$, i.e. $t_{aa} = s_a$, $t_{ab}t_{cd} = \delta_{b,c}t_{ad}$, $t_{ab} = t_{ba}^*$. Necessarily $E_{q_k(A'_{01}\cap A_{i0})}(t_{ab}) = 0$ for $a \neq b$.

Then $p_k = \frac{1}{n}\sum_{a,b=1}^n t_{ab}$ is a projection in $q_k(A'_{0k}\cap A_{ik})$, with $E_{q_k(A'_{01}\cap A_{i0})}(p_k) = \frac{1}{n}\sum_a t_{aa} = s/n$. This means that $||E_{q_k(A'_{01}\cap A_{i0})}(p_k)|| = 1/n < \epsilon/2$. Since $E_{A'_{01}\cap A_{i0}}(x) = E_{A'_{01}\cap A_{i0}}(E_{q_k(A'_{01}\cap A_{i0})}(x))$ for any x in $q_k(A'_{0k}\cap A_{ik})$, we have $||E_{A'_{01}\cap A_{i0}}(p_k)|| < \epsilon/2$ as well.

For any $y \in A'_{00}\cap A_{i0}$, from Lemma 3.3 and Ocneanu compactness

$$\lim_{k\to\infty} \operatorname{tr}(E_{A'_{0k}\cap A_{ik}}(y)p_k) - \operatorname{tr}(E_{A'_{01}\cap A_{i0}}(y)p_k) = 0 \tag{4.4}$$

$p_k \in A'_{0k}\cap A_{ik}$, so this is

$$\lim_{k\to\infty} \operatorname{tr}(yp_k) - \operatorname{tr}(yE_{A'_{01}\cap A_{i0}}(p_k)) \tag{4.5}$$

Since $||E_{A'_{01}\cap A_{i0}}(p_k)||_2 < ||E_{A'_{01}\cap A_{i0}}(p_k)|| < \epsilon/2$, this means that $|\operatorname{tr}(yp_k)| < \epsilon||y||_2$ for k greater than some k_y. $A'_{00}\cap A_{i0}$ is finite dimensional; let its dimension be d. We pick an orthonormal basis for for this vector space, and let k_0 be the supremum of k_b for b in the basis. Then we have $|\operatorname{tr}(xp_k)| < d\epsilon||x||_2$ for all $x \in A'_{00}\cap A_{i0}$, $k > k_0$. This means that $||E_{A'_{00}\cap A_{i0}}(p_k)||_2 < d\epsilon$ for such k. Taking τ to be the smallest trace of a nonzero projection in $A'_{00}\cap A_{i0}$, we may bound the operator norm of $E_{A'_{00}\cap A_{i0}}(p_k)$ as well: $||E_{A'_{00}\cap A_{i0}}(p_k)|| < d\tau^{-1}\epsilon$ for $k > k_0$. d and τ do not depend on k.

From Lemma 4.1, we then have $[(M_k)_\omega : M_0^\omega \cap M_1'] > d^{-1}\tau\epsilon^{-1}$ for all $k > k_0$. ϵ is arbitrary, and d and τ are fixed. Therefore $\lim_{k\to\infty}[(M_{2k})_\omega : M_0^\omega\cap M_1'] = \infty$. □

The lemmas from section 4.1 allow us to show that $[(M_1)_\omega : M_0^\omega\cap M_1'] \geq [(M_k)_\omega : M_0^\omega\cap M_1']$. Combining this fact with Lemma 4.7 above gives the main result of this paper.

Theorem 4.1. *Let $M_0 \subset M_1$ be an commuting-square subfactor of infinite depth. Then the induced central sequence subalgebra $M_0^\omega\cap M_1' \subset (M_1)_\omega$ has infinite index.*

Proof. Let the subfactor $M_0 \subset M_1$ have Jones tower $M_0 \subset M_1 \subset M_2 \subset ...$

We choose $k \in \mathbb{N}$, and consider the quadrilateral of von Neumann algebras

$$\begin{array}{ccc} (M_k)_\omega & \subset & M_1'\cap M_k^\omega \\ \cup & & \cup \\ M_0^\omega\cap M_1' & \subset & (M_1)_\omega \end{array}$$

Since $M_1 \subset M_k$ is a hyperfinite subfactor, from Lemma 4.5 we know that $[M_1'\cap M_k^\omega : (M_1)_\omega] = [M_k : M_1]$. $M_1 \subset M_k$ is itself a commuting-square subfactor from section 3.2, and so is extremal by Theorem 3.1. This means that Lemma 4.6 applies, and $[M_1'\cap M_k^\omega : (M_k)_\omega] \geq [M_k : M_1]$.

From Lemma 4.2, all the algebras in the above quadrilateral are factors, except possibly $M_0^\omega\cap M_1'$ in the lower left. We have $[M_1'\cap M_k^\omega : (M_1)_\omega] < \infty$ and $[M_1'\cap M_k^\omega : (M_1)_\omega] \leq [M_1'\cap M_k^\omega : (M_k)_\omega]$, so the conditions of Lemma 4.4 are satisfied and we have $[(M_1)_\omega : M_0^\omega\cap M_1'] \geq [(M_k)_\omega : M_0^\omega\cap M_1']$ for all k.

The original commuting-square subfactor $M_0 \subset M_1$ is of infinite depth, so by Lemma 4.7 the sequence of Pimsner-Popa indices $([(M_k)_\omega : M_0^\omega \cap M_1'])$ goes to infinity with k. Therefore the index of the central sequence subalgebra $M_0^\omega \cap M_1' \subset M_1^\omega$ must be infinite. □

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