Peter May: 2/14/2012

$A = \Sigma^\infty_G A_+ \qquad B \wedge A \to B \wedge F_G(A, S_G) \to F_G(A, B) \qquad A, B, C$ finite G-sets

$$\begin{array}{ccc}
\Sigma^\infty_G (C\times B)_+ \wedge \Sigma^\infty_G (B\times A)_+ & \xrightarrow{\alpha\wedge\alpha} & \mathbb{K}_G \mathcal{E}_G (C\times B) \wedge \mathbb{K}_G \mathcal{E}_G (B\times A) \\
\cong\downarrow & & \downarrow \\
\Sigma^\infty_G (C\times B\times B\times A)_+ & \xrightarrow{\alpha} & \mathbb{K}_G \mathcal{E}_G (C\times B\times B\times A)_+ \\
\Sigma^\infty_G t \downarrow & & \downarrow \mathbb{K}_G(id\times\Delta\times id) \\
\Sigma^\infty_G (C\times B\times A)_+ & \xrightarrow{\alpha} & \mathbb{K}_G \mathcal{E}_G (C\times B\times A)_+ \\
\Sigma^\infty \pi \downarrow & & \downarrow \mathbb{K}_G(\pi_1) \\
\Sigma^\infty_G (C\times A)_+ & \xrightarrow{\alpha} & \mathbb{K}_G \mathcal{E}_G (C\times A)
\end{array}$$

Atiyah duality gives moral understanding of equiv. in Bert's talk

t is "inverse" to diagonal map

modelling composition with the ∞ loopspace machine. BPQ thm compatible

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tom Dieck:

$$(\Sigma^\infty_G X_+)^G \simeq \bigvee_{(H)} \Sigma^\infty (EWH \times_{WH} X^H)_+$$

$$(O_G X_+)^G \simeq \prod_{(H)} O(EWH \times_{WH} X^H)_+ \qquad O_G(X_+) \cong |N\mathrm{Cat}(\tilde{G}, O(X_+))|$$

$$(\Sigma^\infty_G X_+)^G \to (\mathbb{E}_G O_G X_+)^G \leftarrow \mathbb{E}_G (O_G X_+)^G$$

impt pt about wedges & products agree after Ω^∞ machine.

$$\begin{array}{ccc}
\Sigma^\infty_G (X\vee Y) & \xrightarrow{\alpha} & \mathbb{E}_G C_G (X\vee Y) \\
\simeq\downarrow & & \downarrow\simeq \\
\Sigma^\infty_G X \vee \Sigma^\infty_G Y & & \mathbb{E}_G (C_G X \times C_G Y) \\
\cap\downarrow\simeq & & \downarrow\simeq \\
\Sigma^\infty_G X \times \Sigma^\infty_G Y & \longrightarrow & \mathbb{E}_G C_G X \times \mathbb{E}_G C_G Y
\end{array}$$

Notation: $EG =: \tilde{G}$ contractible groupoid generated by a set. (generally: $\tilde{X}$).

$\mathcal{C}at_G(\mathcal{A},\mathcal{B})$ ← category $G\mathcal{C}at(\mathcal{A},\mathcal{B}) = \mathcal{C}at_G(\mathcal{A},\mathcal{B})^G$

Interested in ~~Cat~~ $\mathcal{C}at_G(\tilde{G}, -)$. Thomason: $\mathcal{C}at_G(\tilde{G}, -)^G$ as homotopy limit.

$$1 \longrightarrow \Pi \longrightarrow \underset{\Pi \rtimes G}{\overset{\|}{\Gamma}} \longrightarrow G \longrightarrow 1$$

G acts on Π.

$$\begin{tikzcd}
\mathcal{C}at_G(\tilde{G}, \tilde{\Pi}) \arrow[r] \arrow[d] & \mathcal{C}at_G(\tilde{G}, \tilde{\Pi}/\Pi) \\
\mathcal{C}at_G(\tilde{G}, \tilde{\Pi})/\Pi \arrow[ur, "\cong"'] &
\end{tikzcd}$$

give models for universal princ. (G,Π) bundle after taking classifying spaces.

$$\begin{array}{c} E(G;\Pi_G) \\ \downarrow \\ B(G;\Pi) \end{array}$$

principal Π-bundle, G acting everywhere.
Π acts freely.

$\mathcal{F}$: family $\{\Lambda \subset \Gamma$ s.t. $\Lambda \cap \Pi = e\}$

condition for univ. princ. (G,Π) bundle: $E(G,\Pi_G)^\Lambda = \begin{cases} \text{contr. if } \Lambda\cap\Pi = e \\ \emptyset \text{ else.} \end{cases}$

→ we're using that geom. realization passes to orbits, and in this case nerve also passes to orbits (to get $\cong$ I guess).

A E_G^∞-operad: is $\mathcal{C}_G(j) = E(G;\Sigma_j)$ Σ_j w/ trivial G-action.
→ connects to Bert's talk here.

Why do we need generalization where G acts on Π?
Equiv. alg. K-theory, where G acts on ring R.
ex: Galois extension $\begin{array}{c} K \\ G\downarrow \\ k \end{array}$ $\coprod_n GL(n,R)$.

Hilbert's thm 90: $H^1(G; GL(n,R)) \cong *$ (Serre's version of Hilbert's thm 90.)

π_0 ~~Obj~~ $\mathcal{C}at_G(\tilde{G}, \Pi)^G = H^1(G,\Pi)$. $f: G \to \Pi$ crossed homomorph. $f(gh) = f(g)(g\cdot f(h))$.
↓ isomorphism classes of crossed homomorphisms
isom when conj. in Π?

$$G \to * \quad \Pi = \mathcal{C}at(*, \Pi) \xrightarrow{i} \mathcal{C}at_G(\tilde{G}, \Pi)^G$$ (since $*$ fixed

this functor is an equiv. of cat. iff $H^1(G;\Pi) = *$

$\tilde{\Sigma}_j =: E\Sigma_j$ in Bert's notation. This is an operad

Permutative category: $a \times a \to a$.

isomorphism of cat. b/w permutative categories & $\mathcal{O}$ cats

(although can recognized w/ just $\mathcal{O}(0), \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3)$)

Naive permutative G-category is an $\mathcal{O}$-Gcat.

$\to$ action of $\mathcal{O}$ on object of GCat

The functor $\mathcal{C}at_G(\tilde{G}, -)$ is right adjoint & thus commutes w/ products.

Hence gives operad in GCat when applied to any operad in cat.

Defn. $\mathcal{O}_G(j) = \mathcal{C}at_G(\tilde{G}, \tilde{\Sigma}_j)$

A genuine permutative G-cat is a G cat w/ action of $\mathcal{O}_G$

Notation: $\mathcal{O}$-GCat a gives $\mathcal{C}at_G(\tilde{G}, a) \equiv \mathcal{G}a$ an $\mathcal{O}_G$ structure.

This gives ~~a~~ functor $\mathcal{O}$-GCat $\to \mathcal{O}_G$-Cat

$\to$ difference b/w E_G^∞-cat & genuine perm. G-cat

$Ass \subset \mathcal{O} \xrightarrow{i} \mathcal{O}_G(j)$

Note we don't "know" what a genuine symmetric monoidal category

$\Rightarrow$ in category theory regard $\mathcal{O}_G$ ~~as~~ as 2-monad, rectify pseudoalgebras

Mike Mandell: algebras over operad of indiscrete cat on binary trees.

$\to$ G-ify this.

Mike Hill: representation ring gives examples??

Infinite loop space machine:

$B(\Sigma_G^\infty, C_G, X) = \mathbb{E}_G X$ n.b. used Steiner operad in C_G here to get action on Σ_G^∞

$\mathcal{C}_G = |N\mathcal{O}_G| \times \mathcal{K}_G \leftarrow$ Steiner

$\Omega_G^\infty \mathbb{E}_G X \leftarrow X$ group completion

$\mathbb{K}_G a = \mathbb{E}_G(Ba)$ a gen perm G-cat, Ba is usual classifying space

BPQ thm: $\Sigma_G^\infty X \simeq \mathbb{E}_G \mathbf{O}_G X$, where $\mathbf{O}_G = |N\mathcal{O}_G|$

standard 2-sided bar construction

$B(\Sigma^\infty_G, C_G, O_G X) \xleftarrow{\simeq} \Sigma^\infty_G X$

Tom Dieck splitting thm: entirely cat. proof, so you can see how Burnside ring acts on the splitting. (see page 1.)

Equivariant alg. K-theory. G-ring R. $\coprod GL(n,R)$ is naive perm. cat.

$\mathbb{K}_G(R) = \mathbb{E}_G \mathcal{B}\mathcal{GL}_G(R)$ ← + construction gives gen. G-spectrum

there's also a Q-construction, & Mona proved + = Q

Galois extension: $\begin{array}{c} K \\ G\downarrow \\ k \end{array}$ $\mathbb{K}(k) \cong \mathbb{K}_G(K)^G$ (thm!).

→ comes from Hilbert thm 90.