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jt. w/ Ib Madsen

$(\mathcal{C}, w\mathcal{C}, D, \eta)$: pointed exact category w/ weak equiv. + strict duality

$KR(\mathcal{C}, w\mathcal{C}, D, \eta)$: real algebraic K-theory spectrum

real symmetric spectrum i.e. $G = Gal(\mathbb{C}/\mathbb{R})$-symm. spectrum in sense of Mandell

variation of Waldhausen construction

Compare with $KR^{\oplus}(\mathcal{C}, w\mathcal{C}, D, \eta)$: real direct sum K-theory spectrum

real symmetric spectrum, variant of Segal construction

natural map $KR^{\oplus}(\mathcal{C}, w\mathcal{C}, D, \eta) \xrightarrow{\phi^*} KR(\mathcal{C}, w\mathcal{C}, D, \eta)$

(because of description in terms of diagrams)

Main Theorem: If $\mathcal{C}$ is split-exact, then $KR^{\oplus}(\mathcal{C}, i\mathcal{C}, D, \eta) \xrightarrow{\phi^*} KR(\mathcal{C}, i\mathcal{C}, D, \eta)$ is a level weak equiv. of real symm. spectra.

Defn: A (strong) duality structure on a category $\mathcal{C}$ is a pair (D, η) with

$\mathcal{C}^{op} \xrightarrow{D} \mathcal{C}$ $\quad id_{\mathcal{C}} \overset{\eta}{\Longrightarrow} D \circ D^{op}$ s.t.

$(D, D^{op}, \eta, \eta^{op})$ is adjoint equiv. of categories.

Ex: A = ring (unital, assoc),

$\mathcal{P}(A)$ = cat. of fin gen. proj. right A-modules

An antistructure on A is a pair (L, α) with $L = L_{12}$ = a right $A \otimes A$-module

$\alpha: L_{12} \longrightarrow L_{21}$ isom. of $A \otimes A$-modules $\qquad$ 2 right A-mod. structures

s.t. 1.) $L_1, L_2 \in ob\,\mathcal{P}(A)$.

2.) $\alpha \circ \alpha = id_L$

3.) $A_2 \longrightarrow {}_1Hom_A(L_2, L_1)_2 \quad 1 \longmapsto \alpha$ is an isomorphism

"line bundle condition" on L.

Associated duality structure on $\mathcal{P}(A)$:

$DP^{op} = Hom(P, L_1)_2 \qquad \mathcal{P}(A)^{op} \xrightarrow{D} \mathcal{P}(A) \qquad P^{op} \in \mathcal{P}(A)^{op}$

$\eta_P(x)(f) = \alpha(f(x)) \qquad id \overset{\eta}{\Longrightarrow} D \circ D^{op}$

Every duality structure on $\mathcal{P}(A)$ is of this form.

Ex: 1.) A commutative. $L = \mu^* A$, $A \otimes A \xrightarrow{\mu} A$ $\quad \underset{\text{orthog K-thry}}{\alpha = +\mathrm{id}}$ (or $\underset{\text{symplectic K-thry}}{\alpha = -\mathrm{id}}$)

2.) Γ group. $A = \mathbb{Z}[\Gamma]$ group ring (Hopf algebra)

$L = \mathbb{Z} \underset{\overset{\nwarrow}{\varepsilon}\, A\, \overset{\to}{\Delta}}{\otimes} (A \otimes A)$ $\quad \alpha = \mathrm{id} \otimes (\text{twist})$

Rmk: Can replace every cat. w/ duality $(\mathcal{C}, D, \eta)$ by an equiv. cat w/ strict duality $(\tilde{\mathcal{C}}, \tilde{D})$ (where $\tilde{\eta} = \mathrm{id}$).

$\mathrm{ob}\,\tilde{\mathcal{C}}$: (c, c', f) $\quad c \xrightarrow[\sim]{f} Dc'^{\,op}$

mor $(c_1, c_1', f_1) \xrightarrow{(g, g')} (c_2, c_2', f_2)$ s.t.
$\begin{array}{ccc} c_1 & \xrightarrow{f_1} & Dc_1'^{\,op} \\ g \downarrow & & \downarrow g'^{\,op} \\ c_2 & \xrightarrow{f_2} & Dc_2'^{\,op} \end{array}$ commutes.

$\tilde{D}(c, c', f) = (c', c, \underbrace{Df^{op} \circ \eta_c}_{\text{adjoint of } f})$

Fix $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ $\sigma \in G$ generator (cx conj.)

Real Set: closed symm. monoidal category

obj: real set = left G-set; morph real maps = G-equiv maps

Internal hom: $\underline{\mathrm{Hom}}(X, Y) = \{\underline{\text{all}} \text{ maps } f: X \to Y\}$ w/ conj. action: $(\sigma f)(x) = \sigma f(\sigma^{-1} x)$

A Real category is a cat. enriched in Real set (obs: no action on objects)

Real functor = enriched functor

Ex: Real simplicial index category $\Delta\mathbb{R}$

obj $[n] = 0 \leftarrow 1 \leftarrow \cdots \leftarrow n$ $\quad (n \geq 0)$

$\underline{\mathrm{Hom}}_{\Delta\mathbb{R}}([m], [n]) = \mathrm{Hom}_{\Delta}([m], [n])$ with action

$\begin{array}{c} [m] \\ \downarrow \theta \\ [n] \end{array} \overset{\sigma}{\longmapsto} \begin{array}{ccc} [m]^{op} & \xrightarrow{\cong} & [m] \\ \downarrow \theta^{op} & & \downarrow \sigma\cdot\theta \\ [n]^{op} & \xrightarrow{\cong} & [n] \end{array}$ $\leftarrow$ unique isomorphisms!

$[n-1] \xrightarrow{d^i} [n]$ $\quad \sigma \cdot d^i = d^{n-i} : [n-1] \to [n]$.

Defn: A real symplicial set is a real functor $\Delta\mathbb{R}^{op} \xrightarrow{X[-]} \text{Real Set}$

("same" as the ΔG-sets of Fiedorowicz – Loday)

Ex: (Real nerve) $(\mathcal{C}, D)$: cat. w/ strict duality

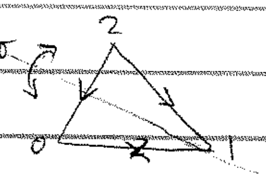
$\Delta\mathbb{R}^{op} \xrightarrow{N(\mathcal{C}, D)[-]} \text{Real Set}$

$N(\mathcal{C}, D)[n] \longmapsto \{\underline{\text{all}} \text{ functors } [n] \to \mathcal{C}\}$ action: $\begin{array}{c} [n] \\ \downarrow c \\ \mathcal{C} \end{array} \overset{\sigma}{\longmapsto} \begin{array}{ccc} [n]^{op} & \xrightarrow{\cong} & [n] \\ \downarrow c^{op} & & \downarrow \sigma\cdot c \\ \mathcal{C}^{op} & \xrightarrow{D} & \mathcal{C} \end{array}$

Geometric realization: $\text{Real Simplicial Set} \underset{\mathrm{Sin}_{\mathbb{R}}(-)[-]}{\overset{|-|_{\mathbb{R}}}{\rightleftarrows}}$ ~~Real Spaces~~ Real Top $|-|_{\mathbb{R}}$ enriched coend.

Ex: $S^{2,1}[-] = \Delta\mathbb{R}[2][-] / \partial\Delta\mathbb{R}[2][-]$

$S^{2,1} := |S^{2,1}[-]|_{\mathbb{R}} \xrightarrow{\sim} S^{\mathbb{C}}$



• not simplicial action
• this is simplicial structure.

Also: adjunction $\quad \text{Real Simplicial Set} \underset{ex}{\overset{sd}{\rightleftarrows}} \text{Simplicial Real Set}$

$\Delta\mathbb{R} \longleftarrow \Delta$

$[2n+1] \longleftrightarrow [n]$

so subdivision gives  (but try not to subdivide so as not to lose control.)

real symmetric spectrum = symm. sp. in Real Top_* w.r.t. $S^{2,1}$

Real Waldhausen construction:

$(\mathcal{C}, w\mathcal{C}, D, O)$ ptd exact cat. w/ weak equivalences & strict duality

$\downarrow$

$(S^{2,1}\mathcal{C}[-], wS^{2,1}\mathcal{C}[-], D[-], O[-])$ real simplicial ptd exact cat w/ w-equiv & strg. duality

(why ordinary Waldhausen construction doesn't work

$|Nw\mathcal{C}[-]| \wedge_{\mathbb{R}} S^1 \longrightarrow |NwS\,\mathcal{C}[-]|_{\mathbb{R}}$

$S^{1,1} = S^{i\mathbb{R}}$ not equivariant for triv. action on S^1)

The category $S^{1,1}\mathcal{C}[n] \subset Cat(Cat([1],[n]), \mathcal{C})$ w/ some conditions.

now: change 1 to 2

$S^{1,2}\mathcal{C}[n] \subset Cat(Cat([2],[n]), \mathcal{C})$ is the full subcat of functors

$A: Cat([2],[n]) \to \mathcal{C}$ s.t.

1.) for all $\mu: [1] \longrightarrow [n]$ $\quad A(s_0\mu) = A(s_1\mu) = 0$

2.) for all $\sigma: [3] \longrightarrow [n]$, the sequence

$A(d_0\sigma) \to A(d_1\sigma) \to A(d_2\sigma) \to A(d_3\sigma)$ is a 4-term exact seq.

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\text{push out}"] & B \arrow[r] \arrow[d] & C \arrow[r, two heads] & D \\ 0 \arrow[r, tail] & X \arrow[ur] \arrow[r] & 0 \arrow[ur] \arrow[u, phantom, "\text{pullback}"] & \end{tikzcd}$$

, 4 term exact

(nb $A \rightarrowtail B \to C$ triv.)

Rmk: $S^{2,1}\mathcal{C}[0] = \{0[0]\}$; $S^{2,1}\mathcal{C}[1] = \{0[1]\}$

$S^{2,1}\mathcal{C}[2] \xrightarrow{\sim} \mathcal{C} \quad A \longmapsto A(\mathrm{id}_{[2]})$

Defn The real algebraic K-thry spectrum of $(\mathcal{C}, w\mathcal{C}, D, O)$ is the real symmetric spectrum with r^{th} space

$$KR(\mathcal{C}, w\mathcal{C}, D, O)_r = |N(wS^{2r,r}\mathcal{C}[-], D[-])[-]|_{\mathbb{R}}$$

and with structure maps

$$KR(\mathcal{C}, w\mathcal{C}, D, O)_r \wedge S^{2s,s} \xrightarrow{\sigma_{r,s}} KR(\mathcal{C}, w\mathcal{C}, D, O)_{r+s}$$

induced from the inclusion of the 2-skeleton in the last s real simplicial directions

The real algebraic K-groups:

$$KR_{p,q}(\mathcal{C}, w\mathcal{C}, D, O) = [S^{p,q}, KR(\mathcal{C}, w\mathcal{C}, D, O)]_{\mathbb{R}}$$

$$S^{1,0} = \Sigma^{\infty}_{\mathbb{Z}/2} S^{\mathbb{R}} \;; \quad S^{1,1} = \Sigma^{\infty}_{\mathbb{Z}/2} S^{i\mathbb{R}} \quad \leftarrow \text{sign rep.}$$

$$S^{p,q} = (S^{1,0})^{\wedge p-q} \wedge (S^{1,1})^{\wedge q}$$

Lewis-Mandell: RO(G) htpy grps for G-sym. spectra

" $KR^{\oplus}(\mathcal{C}, w\mathcal{C}, D, O)_r = |N(w\{\text{ptd } \mathcal{C}\text{-valued sheaves on } S^{2r,r}[-]\}, D[-])[-]|$ "

In the pipeline: use real Waldhausen thry to prove additivity, etc.

Main Thrm implies:

Cor If $\mathcal{C}$ is split-exact, then:

1.) For all $r \geq 1$, $KR(\mathcal{C}, i\mathcal{C}, D, O)_r \xrightarrow[\sim]{\tilde{\sigma}_{r,1}} \Omega^{2,1} KR(\mathcal{C}, i\mathcal{C}, D, O)_{r+1}$ is a weak equiv. of pointed real spaces

2.) For every subgroup $H \subset G$,

$H_*(KR(\mathcal{C}, i\mathcal{C}, D, O)_0^H)[\pi_0 KR(\mathcal{C}, i\mathcal{C}, D, O)_0^H)^{-1}]$ is isomorphic to

$H_*((\Omega^{2,1} KR(\mathcal{C}, i\mathcal{C}, D, O)_1)^H)$

By Segal subdivision:

$$KR(\mathcal{C}, i\mathcal{C}, D, O)_0^G = |N(i\mathcal{C}, D)[-]|_{\mathbb{R}}^G \xleftarrow{\simeq} |N\mathrm{Sym}(i\mathcal{C}, D)[-]|$$

where $\mathrm{Sym}(i\mathcal{C}, D)$ is the cat. of symmetric spaces in $(i\mathcal{C}, D)$:

obj: $(c, c \xrightarrow[\simeq]{f} Dc^{op})$ s.t. $f = Df^{op}$ (symm/skew symm matrice for $\alpha = \mathrm{id}/-\mathrm{id}$)

morph:
$$\begin{tikzcd} c_0 \\ c_1 \arrow[u, "g"] \end{tikzcd} \qquad \begin{tikzcd} c_0 \arrow[r, "f_0"] \arrow[dr, phantom, "\circlearrowright"] & Dc_0^{op} \arrow[d, "Df^{op}"] \\ c_1 \arrow[u] \arrow[r, "f_1"] & Dc_1^{op} \end{tikzcd}$$

Cor: $KR_{0,0}(\mathcal{C}, i\mathcal{C}, D, 0) = (\pi_0 \, Sym(i\mathcal{C}, D), \perp)^{gr}$ ← orthog. sum

group completion

Final remarks:

1) $KR_{p,0}(\mathcal{C}, w\mathcal{C}, D, 0) = K\,Herm_p(\mathcal{C}, w\mathcal{C}, D, 0)$

$KR_{p,-1} = U_p$ (Karoubi) $\quad KR_{p,1} = V_{p+1}$ (Karoubi)

$KP_{p,-2} = E_p$ " $\quad KR_{p,2} = D_{p+2}$ "

2) $KR(\mathcal{C}, w\mathcal{C}, D, 0)^{gr}$ is going to be L-theory of some kind (if 2 invertible, Ranicki L-groups.)

3.) this is connective theory but should be made nonconnective Petersen-Wiebel methods.