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G-spectra are spectral Mackey functors joint w/ Peter May
|a| < ∞.

GTop: f: X → Y is w.e. (fib) if X^H → Y^H is w.e. (fib) ∀ H
(can choose favorite family of H's; focus on all H).

X^H ≅ GMap(G/H, X)

orbit category O_G: objects G/H

morphisms G/H → G/K ≡ O_G(G/H, G/K) = GMap(G/H, G/K)

so full subcat. of GTop

X ∈ GTop ⇔ ∃ X(G/H → X^H) is GMap(-, X).

Thm (Dwyer-Kan): Fun(O_G^op, Top) ⇌ GTop ^{eval} is a Quillen equiv
w/ proj model structure on left.

Naive G-spectra: E ↪ G

Fun(O_G^op, X) ⇌ GX ^{naive G-spectra}

or

Fun(Σ^∞ O_G^op, X) ⇌ GX

spectrally enriched functors, so this is just an easy rephrasing.

Defect: naive G-spectra missing transfers; don't have duality theory

Duality: M ↔ R^N w normal bundle S^N → TD DM_+ ≅ T_2 N S^{-N}

If M G-manifold,

M → R^N ↪ G ^{requires} representation, not just Euclidean space.

∃ E_v? G-spaces, to get genuine spectra

Genuine G-spectra:

~~π_n^H(E)~~ π_n^H(E) = π_n(colim ∂^V E_v)^H

π_n^{(-)}(E) gives a Mackey functor.

Burnside category B_G: obj G/H morph B_G(G/H, G/K) = ([Gset_{G/H, G/K} / N])^{group complete}

Mackey functor is additive functor out of B_G^*

M: B_G^* → Ab

Ex: Burnside ring Mackey functor: A(H) = B_G^*(G/H, *)

since $G\text{Set}/G/H \cong H\text{Set}$

$\text{Ho}(G\text{-Agen})(S^n \wedge^G/H, E) \cong \pi_n^H(E)$ (NB: thinking of orthogonal spectra)

$\text{Ho}(G\text{-Agen})(\sum_n^{\infty} G/H, \sum_n^{\infty} G/K) \cong \mathcal{B}_G(G/H, G/K)$, so $\pi_0 S^0$ is Burnside ring.

$\mathcal{B}_G \subseteq G\text{-Agen}$ full \mathcal{A} -left subcat on $\sum_n^{\infty} G/H$ (fibrant versions)

This is the spectral Burnside category

Thm $\text{Fun}_\mathcal{A}(\mathcal{B}_G^{\text{op}}, \mathcal{A}) \iff G\text{-Agen}$ (w/ proj model structure on left) (Schwede-Shikey)

(up to here we could use compact Lie groups)

Point: a new model of $\mathcal{B}_G^{\text{op}}$ that doesn't require prior knowledge of $G\text{-Agen}$.

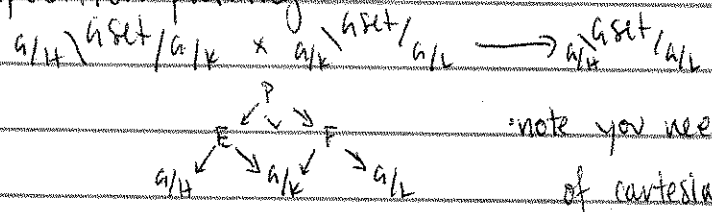
New model for \mathcal{B}_G :

$\mathcal{B}_G(G/H, G/K) \cong K(G/H \setminus G\text{Set}/G/K)$ Equivariant Barratt-Priddy-Quillen, version I

$G/H = * : F(S^0, \sum_n^{\infty} G/K) = (\sum_n^{\infty} G/K)^G$

(Recall: BPO says $K(\text{sets}/X) \cong \sum^{\infty} X$, so $K(G\text{Set}/(S_n)^G$)

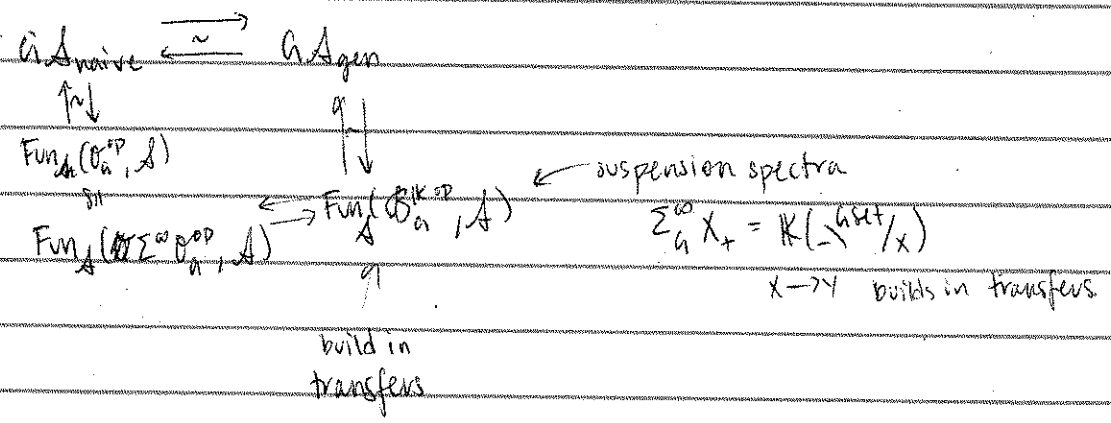
• Composition pairing:



• note you need to worry about non-associativity of cartesian product, but you can rectify.

• feed this into K theory machine, such as Elmendorf-Mandell to get spectral category: $K(G/H \setminus G\text{Set}/G/K)$ give a spectral category

Thm (Gvillen-May) $\mathcal{B}_G^K \cong \mathcal{B}_G$ as spectral categories



Sketch of proof of thm:

(i) Both $(-)^G$ of G -spectral categories.

$$\begin{array}{ccc} \mathcal{B}_G^K & \mathcal{B}_G & \mathcal{B}_G(a/H, a/k) = F(\sum_a^{\infty} a/H, \sum_a^{\infty} a/k) \\ \downarrow \cong & \downarrow \cong & \\ (K_a)^G & (F_a)^G & = F_a(\sum_a^{\infty} a/H, \sum_a^{\infty} a/k)^G \end{array}$$

G -cat: $K_a(a/H, a/k) := K_a(\overset{G\text{-set}}{a/H}, \overset{G\text{-set}}{a/k})$

" E_∞^G -category" \leftarrow operad over E_∞^G -operad.

Examples: 1. E_∞^G -operad

\mathcal{U} discrete complete G -universe

(countable G -set w/ countable copies of each fin. G -set)

$Q(j) = \text{Fin}(j, \mathcal{U}, \mathcal{U})$ (take contractible groupoid on the $\text{Fin}(j, \mathcal{U}, \mathcal{U})$)

\rightarrow separates things that overlap on disjoint union.

objects have a G -action: finite subsets of \mathcal{U} .

this gives an E_∞^G -model of $G\text{-set}$

2. E_∞^G -operad

$\mathcal{O}_a(j) = \text{Ehom}(G, \Sigma_j)$ \leftarrow set maps, take contractible groupoid

$\cong \text{hom}_{\text{cat}}(EG, E\Sigma_j)$ \leftarrow if $G = *$, Barrett-Eccles operad

E^∞ version of $G\text{-sets}$:

$\text{hom}_{\text{cat}}(EG, \text{Set})$ (functor category)

\leftarrow then use Barrett-Eccles operad

$(\text{hom}_{\text{cat}}(EG, \text{Set}))^H \cong \text{hom}_H(EG, \text{Set}) \cong \text{hom}(BH, \text{Set}) = H\text{-set}$

Back to sketch:

$$\begin{array}{ccc} \mathcal{B}_G^K & \mathcal{B}_G & \\ \downarrow \cong & \downarrow \cong & \\ (K_a)^G & (F_a)^G & \end{array}$$

\leftarrow as models for $\Sigma_a^{\infty} G/H$

$K_a \rightarrow F_a$ full G -spectral subcat. on $K_a(a\text{-set}/a/H)$

$K_a(a\text{-set}/a/H, a/k) \rightarrow F_a(K_a(a\text{-set}/a/H), K_a(a\text{-set}/a/k)) \leftarrow$ adjoint to comp. map.

by Equiv BPO
rv. 2
(w/ fixed pts)

$$\begin{array}{ccc} \uparrow & & \downarrow \\ K_a(a\text{-set}/a/H \times a/k) & \xrightarrow{\sim} & F_a(\sum_a^{\infty} a/H, \sum_a^{\infty} a/k) \rightarrow \\ \uparrow & & \uparrow \\ \sum_a^{\infty} (a/H \times a/k)_+ & \xrightarrow{\sim} & F_a(\sum_a^{\infty} a/H, \sum_a^{\infty} a/k) \end{array}$$

\leftarrow since orbits are self-dual