Bert Guillou: 2/14/2012

G-spectra are spectral Mackey functors   joint w/ Peter May

$|G| < \infty$.

GTop: $f: X \to Y$ is w.e. (fib) if $X^H \to Y^H$ is w.e. (fib) $\forall H$

(can choose favorite family of H's; focus on all H).

$X^H \cong G\mathrm{Map}(G/H, X)$

orbit category $\mathcal{O}_G$: objects G/H

morphisms $G/H \to G/K$ in $\mathcal{O}_G(G/H, G/K) = G\mathrm{Map}(G/H, G/K)$

so full subcat. of GTop

$X \in G\mathrm{Top} \rightsquigarrow \Phi X(G/H \mapsto X^H)$ is $G\mathrm{Map}(-, X)$.

Thrm (Dwyer-Kan): $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top}) \underset{\Phi}{\overset{ev_{G/e}}{\underset{\sim}{\rightleftarrows}}} G\mathrm{Top}$ is a Quillen equiv w/ proj model structure on left.

Naive G-spectra: $E \circlearrowleft G$

$\mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{S}) \underset{\Phi}{\overset{\sim}{\rightleftarrows}} G\mathcal{S}$ = naive G-spectra

or

$\mathrm{Fun}_{\mathcal{S}}(\Sigma^\infty \mathcal{O}_G^{op}, \mathcal{S}) \rightleftarrows G\mathcal{S}$

spectrally enriched functors, so this is just an easy rephrasing.

Defect: naive G-spectra missing transfers; don't have duality theory

Duality: $M \hookrightarrow \mathbb{R}^N$ ν normal bundle $S^N \to T\nu$ $DM_+ \simeq T\nu \wedge S^{-N}$

If M G-manifold,

$M \hookrightarrow \mathbb{R}^N \circlearrowleft G$ requires representation, not just Euclidean space.

$\{E_V\}$ G-spaces, to get genuine spectra

Genuine G-spectra:

$\pi_n^H(E) = \pi_n(\operatorname{colim}_V \Omega^V E_V)^H$

$\pi_n^{(-)}(E)$ gives a Mackey functor.

Burnside category $\mathcal{B}_G$: obj G/H morph $\mathcal{B}_G(G/H, G/K) = \left([G/H \backslash G\mathrm{Set}/_{G/K}]/\sim \right)^{\text{group compl}}$

Mackey functor is additive functor out of $\mathcal{B}_G^*$

$M: \mathcal{B}_G^{op} \to \mathrm{Ab}$.

Ex: Burnside ring Mackey functor: $A(H) = \mathcal{B}_G(G/H, \underset{G/G}{*})$

since $GSet/_{G/H} \simeq HSet$

$Ho(G\mathcal{S}_{gen})(S^n \wedge G/H_+, E) \cong \pi_n^H(E)$ (NB: thinking of orthogonal G-spectra)

$Ho(G\mathcal{S}_{gen})(\Sigma^\infty_G G/H_+, \Sigma^\infty_G G/K_+) \cong \mathcal{B}_G(G/H, G/K)$, so $\pi_0 S^0$ is Burnside ring.

$\mathcal{B}_G \subseteq G\mathcal{S}_{gen}$ full $\mathcal{S}$-~~cat~~ subcat on $\Sigma^\infty_G G/H_+$ (fibrant versions)

This is the spectral Burnside category.

<u>Thrm</u> $Fun_{\mathcal{S}}(\mathcal{B}_G^{op}, \mathcal{S}) \rightleftarrows G\mathcal{S}_{gen}$ (w/ proj model structure on left)

(Schwede-Shipley)

(up to here we could use compact Lie groups)

<u>Point</u>: a new model of $\mathcal{B}_G^{op}$ that doesn't require prior knowledge of $G\mathcal{S}_{gen}$.

New model for $\mathcal{B}_G$:

- $\mathcal{B}_G(G/H, G/K) \cong \mathbb{K}(G/H \backslash GSet/_{G/K})$ Equivariant Barratt-Priddy-Quillen, version I.

  $G/H = *$: $F(S^0, \Sigma_G G/K_+) = (\Sigma^\infty_G G/K_+)^G$

  (Recall: BPQ says $\mathbb{K}(sets/X) \simeq \Sigma^\infty X_+$, so $\mathbb{K}(GSet) \simeq (S_G)^G$.

- Composition pairing:

$$G/H\backslash GSet/_{G/K} \times G/K\backslash GSet/_{G/L} \longrightarrow G/H\backslash GSet/_{G/L}$$

$$\begin{tikzcd} & & P \arrow[dl] \arrow[dr] & & \\ & E \arrow[dl] \arrow[dr] & & F \arrow[dl] \arrow[dr] & \\ G/H & & G/K & & G/L \end{tikzcd}$$

- note you need to worry about non associativity of cartesian product, but you can rectify

- feed this into $\mathbb{K}$ theory machine, such as Elmendorf-Mandell to get spectral category: $\mathbb{K}(G/H\backslash GSet/_{G/K})$ give a spectral category

<u>Thrm</u> (Guillou-May) $\mathcal{B}_G^{\mathbb{K}} \simeq \mathcal{B}_G$ as spectral categories

$$\begin{tikzcd} G\mathcal{S}_{naive} \arrow[r, shift left] \arrow[d, shift left] & G\mathcal{S}_{gen} \arrow[l, shift left, "\sim"] \arrow[d, shift left] \\ Fun_{\mathcal{S}}(\mathcal{O}_G^{op}, \mathcal{S}) \arrow[u, shift left, "\sim"] \arrow[d, phantom, "\cong" rotate=90] & Fun_{\mathcal{S}}(\mathcal{B}_G^{\mathbb{K}\,op}, \mathcal{S}) \arrow[u, shift left, "g"] \\ Fun_{\mathcal{S}}(\Sigma^\infty \mathcal{O}_G^{op}, \mathcal{S}) \arrow[ur, shift left] & \end{tikzcd}$$

$\leftarrow$ suspension spectra

$\Sigma^\infty_G X_+ = \mathbb{K}(-\backslash GSet/X)$

$X \to Y$ builds in transfers.

build in transfers

Sketch of proof of thrm:

1. Both $(-)^G$ of G-spectral categories.

$$\mathcal{B}_G^{\mathbb{K}} \cong (\mathbb{K}_G)^G \qquad \mathcal{B}_G \cong (F_G)^G \qquad \mathcal{B}_G(G/H, G/K) = F(\Sigma_G^\infty G/H_+, \Sigma_G^\infty G/K_+) = F_G(\Sigma_G^\infty G/H_+, \Sigma_G^\infty G/K_+)^G$$

$G\mathcal{S}$-cat: $\mathbb{K}_G(G/H, G/K) := \mathbb{K}_G(\text{"}{}_{G/H}{}^{G\mathrm{Set}}/_{G/K}\text{"})$

"E_∞^G-category" ← operad over E_∞^G-operad.

Examples: 1. E_∞^G-operad

$\mathcal{U}$ discrete complete G-universe

(countable G-set w/ countable copies of each fin. G-set)

$Q(j) = E\mathrm{Inj}(\coprod_j \mathcal{U}, \mathcal{U})$ (take contractible groupoid on the Inj.)

→ separates things that overlap on disjoint union.

objects have a G-action: finite subsets of $\mathcal{U}$.

this gives an E_∞^G-model of $G\mathrm{Set}$

2. E_∞^G-operad

$O_G(j) = E\hom(G, \Sigma_j)$ ← set maps; take contractible groupoid

$\cong \hom_{Cat}(EG, E\Sigma_j)$ ← if $G = *$, Barrett-Eccles operad.

E^∞ version of GSets:

$\hom_{Cat}(EG, \mathrm{Set})$. (functor category)

& then use Barrett-Eccles operad

$(\hom(EG, \mathrm{Set}))^H \cong \hom_H(EG, \mathrm{Set}) \cong \hom(BH, \mathrm{Set}) = H\mathrm{Set}$.
EH

Back to sketch: $\mathcal{B}_G^{\mathbb{K}} \simeq (-)^G \quad \mathcal{B}_G \simeq (-)^G$

$\mathbb{K}_G \longrightarrow F_G$ full G-spectral subcat. on $\mathbb{K}_G(G\mathrm{Set}/_{G/H})$ ← as models for $\Sigma^\infty G/H$

$$\begin{array}{ccc}
\mathbb{K}_G({}_{G/H}{}^{G\mathrm{Set}}/_{G/K}) & \longrightarrow & F_G(\mathbb{K}_G(G\mathrm{Set}/_{G/H}), \mathbb{K}_G(G\mathrm{Set}/_{G/K})) \\
\text{SII} & & \downarrow \\
\mathbb{K}_G(G\mathrm{Set}/_{G/H \times G/K}) & & \\
\Uparrow & & \\
\Sigma_G^\infty (G/H \times G/K)_+ & \xrightarrow{\sim} & F_G(\Sigma_G^\infty G/H_+, \Sigma_G^\infty G/K_+) \longrightarrow
\end{array}$$

← adjoint to comp. map

by Equiv BPQ ver. 2 (w/o fixed pts) →

↑ since orbits are selfdual