BASIC NOTIONS OF EQUIVARIANT STABLE HOMOTOPY THEORY

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Abstract. These notes provide an introduction to some of the basic constructions in equivariant stable homotopy theory. We begin with a construction of equivariant spectra and then discuss equivariant spheres, fixed point spectra and splitting isotropy groups. The notes conclude with a simple definition of $Ro(G)$-graded homology and cohomology.

1. Introduction

We start with the fundamental question: what is equivariant stable homotopy theory? Equivariant stable homotopy theory is the branch of algebraic topology that is both "equivariant" and "stable." By "equivariant," we mean that all spaces in sight have an action of a particular compact Lie group G, and we wish to do algebraic topology taking into account this action. By "stable," we mean that (motivated perhaps by the Freudenthal suspension isomorphism) we want suspension to be an "invertible" operation—we should be able to desuspend any object. More formally, loops and suspension should give a self-equivalence of the (equivariant) stable homotopy category.

To get this stability property, we have to pass from working with topological spaces to working with spectra. The first section of these notes discuss spectra and their properties. After that, we discuss other important constructions in equivariant stable homotopy theory, such as equivariant spheres, fixed point functors and splittings.

From now on all G-spaces will be based, with a G-fixed point basepoint. For further references on this subject, consult the "Alaska Notes", [1].

2. Spectra

Nonequivariantly, we think of a prespectrum X as a sequence of spaces $\{X_n\}$ for $n \geq 0$ with structure maps

$$\sigma_n \colon \Sigma X_n \to X_{n+1}.$$

If the adjoint structure maps $\tilde{\sigma}_n \colon X_n \to \Omega X_{n+1}$ are homeomorphisms, then X is a *spectrum.*

Equivariantly, we could just try the same things, but this doesn't really capture all of the equivariant structure we want. Rather than thinking about spectra as being indexed on the natural numbers, $\{0, 1, 2, \dots\}$, we can think of them as being indexed on the spheres $S^0, S^1, S^2, \dots$. But equivariantly, these spheres have a fixed G-action, so they can't account for all the structure. We need to introduce spheres with G-actions and to have some way of keeping track of how these spheres are related to each other.

Consider an n-dimensional real representation λ of G. If we forget about the G-action, λ is just $\mathbb{R}^n$, so its one-point compactification is nonequivariant just S^n. As a G-space, the one-point compactification of λ is a new sphere S^λ. We want a G-spectrum to be a collection of G-spaces, one for each λ, that fit together in the right way. To understand what we mean by "fit together," we introduce the idea of a G-universe.

Definition 2.1. A *G-universe* $\mathcal{U}$ is a countably infinite dimensional representation of G with an inner product such that

(1) $\mathcal{U}$ contains the trivial representation
(2) $\mathcal{U}$ contains countably many copies of each finite dimensional subrepresentation.

We can think of $\mathcal{U}$ as the direct sum of $(V_i)^\infty$ where $\{V_i\}$ is a set of distinct irreducible representations of G. We call a G-universe *complete* if it contains every irreducible representation of G, up to isomorphism. A G-universe is called *trivial* if it contains only the trivial representation.

Fix a G-universe $\mathcal{U}$. Let V be a finite dimensional subrepresentation of $\mathcal{U}$ and let S^V be its one-point compactification. Denote $\Sigma^V(-) = S^V \wedge -$ and $\Omega^V(-) = F(S^V, -)$, the function space.

Definition 2.2. A *G-prespectrum* indexed on $\mathcal{U}$ is a collection of G-spaces $\{EV\}$ for each finite dimensional subrepresentation of $\mathcal{U}$ together with G-maps

$$\sigma_{V,W} \colon \Sigma^{W-V} EV \to EW$$

whenever $V \subset W$. Here $W - V$ is the orthogonal complement of V in W. We also require the appropriate transitivity diagram to commute when $V \subset W \subset X$.

A *G-spectrum* indexed on $\mathcal{U}$ is a G-prespectrum where we require the adjoint structure maps

$$\tilde{\sigma}_{V,W} \colon EV \to \Omega^{W-V} EW$$

to be homeomorphisms.

Example 1. If $\mathcal{U} = \mathbb{R}^\infty$ and G is the trivial group, we basically get the normal definition of spectrum.

Example 2. For any G-space X, we can take $EV = S^V \wedge X$ to get the suspension prespectrum of X. This can be "spectrified" to get the suspension spectrum of X.

A G-spectrum indexed on a complete universe is called *genuine*; a G-spectrum indexed on a trivial universe is called *naive*.

We get a category $G\mathcal{S}\mathcal{U}$ of G-spectra indexed on the universe $\mathcal{U}$ by defining morphisms $D \to E$ to be maps $DV \to EV$ that commute with the structure maps. This is a nice category to work in and G-spectra are nice objects. In particular, we have the following useful facts:

Useful Facts.

(1) There is a "spectrification" functor that turns prespectra indexed on $\mathcal{U}$ into spectra indexed on $\mathcal{U}$. This functor is analogous to the "sheafification" of a presheaf and comes from the adjoint functor theorem.
(2) Naive G-prespectra are equivalent to sequences E_n of G-spaces with structure maps $\Sigma E_n \to E_{n+1}$ that are G-maps.

(3) A map of spectra $D \to E$ can be defined to be a weak equivalence if every $DV \to EV$ is an equivariant weak equivalence. This doesn't work for prespectra. Note that a map $X \to Y$ is an equivariant weak equivalence if it induces an isomorphism $\pi_*(X^H) \to \pi_*(Y^H)$ for each closed subgroup $H \leq G$.
(4) There is a nice (symmetrical monoidial) smash product on the category of G-spectra over a universe $\mathcal{U}$. The homotopy category of G-spectra (by which we mean "formally invert weak equivalences") is triangulated. When $\mathcal{U}$ is complete, this is what is meant by the "stable equivariant homotopy category."
(5) We can suspend or desuspend by any representation sphere S^V and these functors are inverse equivalences of categories. Furthermore, the suspension spectrum functor Σ^∞ from G-spaces to G-spectra has an adjoint Ω^∞ functor that takes a spectrum to its zeroth space $E\{0\} = E_0$.
(6) Given a (complete) G-universe $\mathcal{U}$, we can take its G fixed points to get a trivial universe $\mathcal{U}^G$ with an inclusion $\mathcal{U}^G \to \mathcal{U}$. This inclusion gives a pair of adjoint functors

$$G\mathcal{S}\mathcal{U}(i_*E, E') \cong G\mathcal{S}\mathcal{U}^G(E, i^*(E'))$$

for E a naive $\mathcal{U}^G$ spectrum and E' a (genuine) $\mathcal{U}$ spectrum. If $V \subset \mathcal{U}^G$ and , then $i^*(E')$ has Vth space $(i^*E')V = E'(iV')$ and if $V' \subset \mathcal{U}$, then i_*E has V'th space $(i_*E)V' = EV \wedge S^{V'-iV}$ where $V = i^{-1}(V' \cap i(\mathcal{U}^G))$.

3. Equivariant Homotopy Groups

Consider a G-space X. How do we define $\pi_n(X)$ equivariantly? Homotopy classes of G-maps $[S^n, X]^G$ won't do, because since S^n has a fixed G-action, any map $f\colon S^n \to X$ has to land in the fixed set X^G. Again, we need to consider spheres with G actions, but this time we get G-actions by smashing a sphere with a G-orbit. Since G-orbits give all possible transative G-actions, this is enough to see all kinds of G-actions. Thus, nonstably, we define equivariant homotopy groups

$$\pi_n^H(X) = [S^n \wedge (G/H)+, X]^G.$$

Since each S^n is fixed, we see that a map $S^n \wedge G/H_+ \to X$ must land in the H fixed points of X.

Passing into the stable world, we make an analogous definition. Let $\underline{S}^n = \Sigma^\infty S^n$, and consider $\underline{S}^n_H = G/H_+ \wedge \underline{S}^n$ for all closed $H \leq G$.

Definition 3.1. The homotopy groups of a G-spectrum E are

$$\pi_n^H(E) = [\underline{S}^n_H, E]^G$$

where here we mean maps in the homtopy category of G-spectra.

These homotopy groups fit together to form what's called a *Mackey functor*; that is, a contravariant functor from $\mathcal{B}_G\mathcal{U}$ to abelian groups, where $\mathcal{B}_G\mathcal{U}$ is the full subcategory of the homotopy category on the $\underline{S}^n_H$s.

The following theorem tells us that our definition of weak equivalence of spectra in Section 2 makes sense.

Theorem 3.2. *If $f\colon E \to E'$ is a map of G-spectra, then each component map $fV\colon EV \to E'V$ is a weak equivalence of G-spaces if and only if $f_*\colon \pi_n^H(E) \to \pi_n^H(Y)$ is an isomorphism for all closed $H \leq G$ and all n.*

In other words, these G-spheres from orbits are enough to detect weak equivalences; in this sense, we now have "enough" homotopy groups.

4. Fixed Points

Given a G-spectrum E, we want to find a nonequivariant spectrum that behaves as the fixed points of E. But what do we mean by "fixed points?" Consider the case of G-spaces.

If X is a nonequivariant space and Y is a G-space with fixed point space Y^G, we have an adjunction

$$G\text{-maps}(X, Y) \cong \text{maps}(X, Y^G)$$

where on the left side we give X the trivial G-action. We want fixed point spectra to fit into a similiar adjunction: Consider a G-universe $\mathcal{U}$ with fixed points $\mathcal{U}^G$. For a naive G-spectrum D indexed on $\mathcal{U}^G$, define a nonequivariant spectrum D^G indexed on $\mathcal{U}^G$ by $(D^G)V = (DV)^G$ for $V \subset \mathcal{U}^G$. If C is a nonequivariant spectrum indexed on $\mathcal{U}^G$, we then have an adjunction

$$G\mathcal{S}\mathcal{U}^G(C, D) \cong \mathcal{S}\mathcal{U}^G(C, D^G).$$

To make sense of the left-hand side, we regard C as a naive G-spectrum with the trivial action. Thus D^G gives us the fixed point spectrum of the naive G-spectrum D.

For a general G-spectrum E indexed on $\mathcal{U}$, we just take the fixed points of its underlying naive spectrum. Explicitly, $E^G = (i^*E)^G$ where $i^*\colon G\mathcal{S}\mathcal{U} \to G\mathcal{S}\mathcal{U}^G$ comes from the inclusion $i\colon \mathcal{U}^G \to \mathcal{U}$. We can compose the (i_*, i^*) adjunction with the fixed points adjunction to get

$$G\mathcal{S}\mathcal{U}(i_*(C), E) \cong \mathcal{S}\mathcal{U}^G(C, (i^*E)^G) = \mathcal{S}\mathcal{U}^G(C, E^G).$$

This construction gives us a good notion of fixed points for an arbitary G-spectrum, in the sense that we have the right kind of adjunctions.

However, this notion of fixed points is not particularly intuitive. For genuine G-spectra, there is a map

$$E^G \wedge (E')^G \to (E \wedge E')^G,$$

but this map is not usually an equivalence. Moreover, taking fixed points does not commute with taking suspension spectra: for a G-space X, $(\Sigma^\infty X)^G \neq \Sigma^\infty(X^G)$.

We have a different notion of fixed points, called *geometric fixed points*, that behaves better in this light. This is a functor $\Phi^G\colon G\mathcal{S}\mathcal{U} \to \mathcal{S}\mathcal{U}^G$ with the properties

- $\Sigma^\infty(X^G) \simeq \Phi^G(\Sigma 6\infty X)$ and
- $\Phi^G(E) \wedge \Phi^G(E') \simeq \Phi^G(E \wedge E')$.

There is also a comparision map $\Phi^G \to (-)^G$. To construct the geometric fixed points functor, we need the concepts introduced in the next section.

5. Universal spaces and splitting isotropy types

We begin with a definition.

Definition 5.1. A *family* $\mathcal{F}$ of subgroups of G is a set subgroups that is closed under subconjugacy: if $H \in \mathcal{F}$ and $g^{-1}Kg \subset H$ for some $g \in G$, then $K \in \mathcal{F}$.

Some basic examples are the set $\{1\}$ consisting of just the trivial group, the set of all subgroups of G, the set of all finite subgroups of G, and the set of all proper subgroups of G.

For a given family $\mathcal{F}$, define an $\mathcal{F}$-space to be a G-space all of whose isotropy groups are in $\mathcal{F}$. Each family $\mathcal{F}$ has a universal $\mathcal{F}$-space $E\mathcal{F}$ which is defined (as a homotopy type) by

$$(E\mathcal{F})^H = \begin{cases} * & \text{if } H \in \mathcal{F} \\ \emptyset & \text{if } H \notin \mathcal{F} \end{cases}$$

The space $E\mathcal{F}$ is universal in the sense that if X is an $\mathcal{F}$-space with the homotopy type of a CW-complex, there is a unique homotopy class of G-maps $X \to E\mathcal{F}$.

Example 3. If $\mathcal{F} = \{1\}$, then

$$E\{1\}^H = \begin{cases} * & \text{if } H = 1 \\ \emptyset & \text{else.} \end{cases}$$

Hence $E\{1\}$ is a free G-space that is nonequivariantly contractible; that is $E\{1\} = EG$ and $E\{1\}/G = BG$ is the classifying space for G.

Example 4. If $\mathcal{F} = \mathcal{A}\mathrm{ll}$, then $E\mathcal{A}\mathrm{ll}^H = *$ for all H, so $E\mathcal{A}\mathrm{ll} = *$.

Given a family $\mathcal{F}$, we use the universal space $E\mathcal{F}$ to construct an "isotropy splitting cofibration"

$$E\mathcal{F}_+ \to S^0 \to \tilde{E}\mathcal{F}$$

where the first map is just the map $E\mathcal{F} \to *$ with a disjoint basepoint added. Note that this cofibration implies

$$(\tilde{E}\mathcal{F})^H = \begin{cases} * & \text{if } H \in \mathcal{F} \\ S^0 & \text{if } H \notin \mathcal{F} \end{cases}.$$

We can smash any G-space X (or any G-spectrum E) with this cofibration to split X into an $\mathcal{F}$-space $E\mathcal{F}_+ \wedge X$ and an $\mathcal{F}$-contractible space $\tilde{E}\mathcal{F} \wedge X$. This allows us to understand general G-spaces by understanding $\mathcal{F}$-spaces and $\mathcal{F}$-contractible spaces separately.

One application of this cofiber sequence is in defining the geometric fixed points functor Φ^G.

Definition 5.2. Let $\mathcal{P}$ be the family of all proper subgroups of G; then

$$\Phi^G(E) = (E \wedge \tilde{E}\mathcal{P})^G.$$

Notice that $E \wedge \tilde{E}\mathcal{P}$ is H-trivial (i.e. its H-fixed points are contractible) for all proper subgroups H, so that, in some sense, only the G-fixed points of E are left in $(E \wedge \tilde{E}\mathcal{P})$. We could also define geometric fixed points essentially by just taking the fixed points of each indexing space of a spectrum, but one must be careful about the indexing and spectrify, so it's not as nice of a definition.

Another place the isotropy splitting comes up is in thinking about free spectra: A G-spectrum E is *free* if and only if the canonical map $EG_+ \wedge E \to E$ coming from the isotropy splitting cofibration is a G-equivalence. This definition works no matter what universe we work over, and even works for G-spaces.

6. The tom Dieck splitting theorem

In Section 4, we noted that for a G-space X, $(\Sigma^\infty X)^G \neq \Sigma^\infty(X^G)$. The following theorem tells us how to find the fixed points of a suspension G-spectrum in term of the underlying space.

Theorem 6.1 (tom Dieck)**.** *For a based G-CW complex X, there is a natural equivalence*

$$(\Sigma^\infty X)^G \simeq \bigvee_{\text{conj. classes} H \leq G} \Sigma^\infty(EWH_+ \wedge_{WH} \Sigma^{\mathrm{Ad}(WH)} X^H)$$

where $WH = NH/H$ is the Weyl group of H in G and $\mathrm{Ad}(WH)$ is the adjoint representation.

For example, let X be a two sphere with an action of C_2 by rotation by π around a central axis. Then

$$\begin{aligned}(\Sigma^\infty X)^{C_2} &\simeq \Sigma^\infty(E1_+ \wedge X^{C_2}) \vee \Sigma^\infty(EC_{2+} \wedge_{C_2} X)\\ &= \Sigma^\infty(X^{C_2}) \vee \Sigma^\infty(EC_{2+} \wedge_{C_2} X)\\ &= S^1 \vee (\text{the Borel construction on } X).\end{aligned}$$

Thus the fixed point spectrum of X is the suspension spectrum of the fixed points of X together with an extra Borel construction term.

7. $RO(G)$-graded homology and cohomology

Our construction of spectra to include representation allows us to define homology and cohomology theories that are not just integer graded, but are in fact graded on the real representation ring $RO(G)$ of G. For any virtual representaion $\nu = W - V$, we form a geniune G-spectrum $S^\nu = \Sigma^W S^{-V}$. Of course, one must be careful about what universe one works over and expressing ν in terms of subrepresentations. But once we have constructed S^ν, we can then define the homology and cohomology groups represented by a geniune G-spectrum E by

$$\begin{aligned}E^G_\nu(X) &= [S^\nu, E \wedge X]^G\\ E^\nu_G(X) &= [S^{-\nu} \wedge X, E]^G = [S^{-\nu}, F(X,E)]^G.\end{aligned}$$

These homology and cohomology theories satisfy sensible $RO(G)$-graded versions of the usual axioms for homology or cohomology.

References

[1] J.P. May with contributions by M. Cole, G. Comezaña, S. Costenoble, A.D. Elmendorf, J.P.C. Greenlees, L.G. Lewis, R.J. Piacenza, G. Triantafillou, and S. Waner. Equivariant homotopy and cohomology theory. Regional conference series in mathematics 91. AMS 1996.