The S^1 Equivariant Generating Hypothesis

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Abstract

These slides are slightly modified from a talk presented at the October 2009 AMS Special Session on New Trends in Triangulated Categories at Penn State. We give a generalization of the generating hypothesis to triangulated categories with multiple generators and then discuss the generating hypothesis for the category of rational S^1-spectra. In particular, we prove that the generating hypothesis fails in this category via the construction of a counterexample.

Freyd's Generating Hypothesis

Let S be the sphere spectrum and $\mathrm{Ho}\mathcal{S}$ be the homotopy category of spectra. Then

- $\mathrm{Ho}\mathcal{S}$ is a triangulated category
- the thick subcategory generated by S is the category of finite spectra

Recall that $\pi_*(X) = [S, X]_*$ and $\pi_*(S)$ is a graded ring.

Freyd's Generating Hypothesis

Freyd's Generating Hypothesis is the conjecture:

Conjecture (Freyd 1966)

The functor $\pi_*(-) = [S, -]_* : \mathrm{Ho}\mathcal{S} \to \pi_*(S)$-modules is faithful on restriction to finite spectra. That is, if X and Y are finite spectra and $f : X \to Y$ induces the zero map $\pi_*(X) \to \pi_*(Y)$, then f is homotopic to zero.

and the Strong Generating Hypothesis is the conjecture:

Conjecture (Freyd 1966)

The functor $\pi_*(-) : \mathrm{Ho}\mathcal{S} \to \pi_*(S)$-modules is full and faithful on restriction to finite spectra.

The Generalized Generating Hypothesis

Let $\mathcal{T}$ be a triangulated category and consider

- an object $B \in \mathcal{T}$
- the full subcategory $\mathcal{B} \subset \mathcal{T}$ on $\{\Sigma^n B\}$
- the thick subcategory $\mathcal{C} \subset \mathcal{T}$ generated by $\mathcal{B}$
- the category $\mathcal{PB}$ of Abelian presheaves on $\mathcal{B}$, i.e. additive contravariant functors $\mathcal{B}^{\mathrm{op}} \to \mathcal{A}b$

Conjecture (Generating Hypothesis)

The functor $\mathcal{C} \to \mathcal{PB}$ given by $C \mapsto [-, C]$ is faithful.

Conjecture (Strong Generating Hypothesis)

The functor $\mathcal{C} \to \mathcal{PB}$ given by $C \mapsto [-, C]$ is full and faithful.

The Generalized Generating Hypothesis

The generalized generating hypothesis reduces to Freyd's generating hypothesis where

- $B = S$ and $\mathcal{B} = \{\Sigma^n S\}$
- $\mathcal{C}$ is the category of finite spectra
- the functor $\mathcal{C} \to \mathcal{P}\mathcal{B}$ sends $X \in \mathcal{C}$ to the presheaf given on $\Sigma^n S$ by $[\Sigma^n S, X] = \pi_n(X)$.

The Generating Hypothesis for Rings

Let R be a ring and $\mathcal{T} = \mathcal{D}(R)$.

- Set $B = R$, as a chain complex concentrated in degree zero.
- Our functor takes a chain complex X to $[\Sigma^* R, X] = H_*(X)$.

Theorem (Hovey–Lockridge–Puninski)

The strong generating hypothesis holds in the derived category of a ring R if and only if R is von Neumann regular.

In fact, Hovey et al characterize rings R for which the generating hypothesis holds in $\mathcal{D}(R)$, and these are "almost von Neumann regular".

The Generating Hypothesis for a Finite Group

Let G be a finite group and $\mathcal{T}$ be the stable module category of G over a field k of characteristic p.

- Take B to be the trivial module
- Our functor takes a kG-module to its Tate cohomology

Theorem (Benson, Carlson, Chebolu, Christensen, Minác)

The generating hypothesis holds for stable module category of kG if and only if the Sylow p-subgroup of G is C_2 or C_3.

In fact, their proof shows that the strong generating hypothesis holds in these cases as well.

A Further Generalization

Let $\mathcal{T}$ be a triangulated category and consider

- a set of objects $\{B \in \mathcal{T}\}$
- the full subcategory $\mathcal{B} \subset \mathcal{T}$ on $\{\Sigma^n B \mid n \in \mathbb{Z}\}$
- the thick subcategory $\mathcal{C} \subset \mathcal{T}$ generated by $\mathcal{B}$
- the category $\mathcal{PB}$ of Abelian presheaves on $\mathcal{B}$, i.e. additive contravariant functors $\mathcal{B} \to \mathcal{A}\mathrm{b}$.

Conjecture (Generating Hypothesis)

The functor $\mathcal{C} \to \mathcal{PB}$ given by $C \mapsto [-, C]$ is faithful.

Conjecture (Strong Generating Hypothesis)

The functor $\mathcal{C} \to \mathcal{PB}$ given by $C \mapsto [-, C]$ is full and faithful.

A Further Generalization

We need this level of generality in the equivariant world.

- We can think about categories that don't have a single generating object.
- In the world of equivariant algebraic topology, we have to consider the action of all subgroups of the group of equivariance, which gives us many "sphere spectra."

The S^1–Equivariant Generating Hypothesis

Consider the compact Lie group S^1 and

- $\mathrm{Ho}S^1\mathcal{S}$, the homotopy category of S^1-spectra
- Let $\{B \in \mathrm{Ho}S^1\mathcal{S}\}$ be $\{S \wedge S^1/H_+\}$ for all closed subgroups $H \subset S^1$

In this context

- $\mathcal{C}$ is the full subcategory of finite S^1-spectra
- the functor $C \mapsto [-, C]$ takes a spectrum C to the presheaf that takes value $[\Sigma^n S \wedge S^1/H_+, C]_n^{S^1}$ on $\Sigma^n S \wedge S^1/H_+$.
- This gives the homotopy group Mackey functor

$$\underline{\pi}_*^{S^1}(C)(S^1/H_+) = [S \wedge S^1/H_+, C]_*^{S^1} = \pi_*^H(C)$$

The S^1-Equivariant Generating Hypothesis

Conjecture (S^1-Equivariant Generating Hypothesis)

The functor $\underline{\pi}_*^{S^1}(-) : \mathrm{Ho}S^1\mathrm{S} \to \underline{\pi}_*^{S^1}$-modules is faithful on restriction to finite S^1-spectra.

This conjecture posits that if $f : X \to Y$ is a map of finite S^1-spectra and

$$f_* : \pi_n^H(X) \to \pi_n^H(Y)$$

is the zero map for all closed subgroups $H \subset S^1$ and all integers $n \in \mathbb{Z}$, then f is nullhomotopic.

The Rational S^1-Equivariant Generating Hypothesis

To simplify our category, we restrict to rational S^1-spectra.

Theorem (B.)

The generating hypothesis does not hold in the category of rational S^1-equivariant spectra.

In other words, there exist finite rational S^1-spectra X and Y and a map $f : X \to Y$ such that

- $f_* : \pi_*^H(X) \to \pi_*^H(Y)$ is zero for all closed subgroups $H \subset S^1$
- f is not nullhomotopic

This is unexpected!

The generating hypothesis fails for rational S^1-spectra.

- Recall the generating hypothesis holds for the derived category of a von Neumann regular ring.
- The rational Burnside ring of compact Lie group is a von Neumann regular ring.
- So we might expect the generating hypothesis to hold for S^1-spectra, but it fails.
- On second glance, the analogy between rings and equivariant spectra isn't exactly right–the structure of $\mathrm{Ho}S^1\mathcal{S}$ is more complicated.

The Short Exact Sequence for Rational S^1-Spectra

Proof Ideas:

- Use Greenlees's description of the category of rational S^1-spectra.
- If X and Y are free rational S^1-equivariant spectra, we have an Adams short exact sequence:

$$0 \to \mathrm{Ext}_k(\Sigma\pi_*^{S^1}(X), \pi_*^{S^1}(Y)) \to [X, Y]_*^{S^1} \to \mathrm{Hom}_k(\pi_*^{S^1}(X), \pi_*^{S^1}(Y)) \to 0$$

  where $k = \mathbb{Q}[c]$, with c in degree -2 coming from the Euler class of the representation of S^1 on $\mathbb{C}$.
- We need a map that induces zero on $\pi_*^H(-)$ for all $H \subset S^1$, so it's not enough to find a case where the Ext term is nonzero.

Controlling $\pi_*^H(X)$

We need to get a handle on $\pi_*^H(X)$

Computational Lemma (B.)

If X is a free rational S^1-equivariant space and $\mathbb{X} = \Sigma_+^\infty X$, then

$$\pi_*^H(\mathbb{X}) = \begin{cases} \pi_*(\Sigma\mathbb{X}/S^1) & \text{if } H = S^1 \\ \pi_*(\mathbb{X}) & \text{if } H \neq S^1 \end{cases}$$

Proof: The tom Dieck splitting theorem.

Controlling $\pi_*^H(X)$

Let X and Y be free rational S^1-equivariant spaces.

- Set $\mathbb{X} = \Sigma_+^\infty X$ and $\mathbb{Y} = \Sigma_+^\infty Y$.
- Then $[\Sigma_+^\infty(S^1 \wedge X), \mathbb{Y}]_*^{S^1} = [\mathbb{X}, \mathbb{Y}]_*$.
- Let $\mathbb{W} = \Sigma_+^\infty(S^1 \wedge X)$
- Then, for $H \neq S^1$,

$$\begin{aligned} \mathrm{Hom}(\pi_*^H(\mathbb{X}), \pi_*^H(\mathbb{Y})) &= \mathrm{Hom}(\pi_*(\mathbb{X}), \pi_*(\mathbb{Y})) \\ &= [\mathbb{X}, \mathbb{Y}]_* \\ &= [\mathbb{W}, \mathbb{Y}]_*^{S^1}. \end{aligned}$$

A map of short exact sequences

Recall $\mathbb{W} = \Sigma^\infty_+(S^1 \wedge X)$. The projection $p : S^1 \wedge X \to X$ gives a map of short exact sequences

$$\begin{array}{ccccc}
\mathrm{Ext}_k(\Sigma\pi_*^{S^1}(\mathbb{X}), \pi_*^{S^1}(\mathbb{Y})) & \longrightarrow & [\mathbb{X}, \mathbb{Y}]_*^{S^1} & \longrightarrow & \mathrm{Hom}_k(\pi_*^{S^1}(\mathbb{X}), \pi_*^{S^1}(\mathbb{Y})) \\
\downarrow{\scriptstyle p_*} & & \downarrow & & \downarrow \\
\mathrm{Ext}_k(\Sigma\pi_*^{S^1}(\mathbb{W}), \pi_*^{S^1}(\mathbb{Y})) & \longrightarrow & [\mathbb{W}, \mathbb{Y}]_*^{S^1} & \longrightarrow & \mathrm{Hom}_k(\pi_*^{S^1}(\mathbb{W}), \pi_*^{S^1}(\mathbb{Y}))
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{Ext}_k(\Sigma\pi_*^{S^1}(\mathbb{X}), \pi_*^{S^1}(\mathbb{Y})) & \longrightarrow & [\mathbb{X}, \mathbb{Y}]_*^{S^1} & \longrightarrow & \mathrm{Hom}_k(\pi_*^{S^1}(\mathbb{X}), \pi_*^{S^1}(\mathbb{Y})) \\
\downarrow{\scriptstyle p_*} & & \downarrow & & \downarrow \\
\mathrm{Ext}_k(\Sigma\pi_*^{S^1}(\mathbb{W}), \pi_*^{S^1}(\mathbb{Y})) & \rightarrow & \mathrm{Hom}(\pi_*^{H}(\mathbb{X}), \pi_*^{H}(\mathbb{Y})) & \rightarrow & \mathrm{Hom}_k(\pi_*^{S^1}(\mathbb{W}), \pi_*^{S^1}(\mathbb{Y}))
\end{array}$$

Hence, to show that the generating hypothesis fails, we just need to find X and Y such that the map on Ext groups has a nontrivial kernel.

The Counterexample

Take X to be the unit sphere in the representation of S^1 on $\mathbb{C}^a$ and Y to be the unit sphere in the representation of S^1 on $\mathbb{C}^b$ for $a, b > 1$.
By the computational lemma,

$$\begin{aligned} \pi_*^{S^1}(\mathbb{X}) &= \pi_*(\Sigma\mathbb{X}/S^1) \\ &= \pi_*(\mathbb{C}P^{a-1}) \\ &= \Sigma^{2a-1}\mathbb{Q}[c]/(c^a) \end{aligned}$$

Similarly, $\pi_*^{S^1}(\mathbb{Y}) = \Sigma^{2b-1}\mathbb{Q}[c]/(c^b)$ and $\pi_*^{S^1}(\mathbb{W}) = \Sigma\mathbb{Q} \oplus \Sigma^{2a}\mathbb{Q}$.

The Counterexample

The problem reduces to calculating Ext groups over a graded polynomial ring and we get

$$\mathrm{Ext}_k(\Sigma\pi_*^{S^1}(\mathbb{X}), \pi_*^{S^1}(\mathbb{Y})) = \begin{cases} \Sigma^{2b-1}\mathbb{Q}[c]/(c^a) & \text{if } a \le b \\ \Sigma^{2b-1}\mathbb{Q}[c]/(c^b) & \text{if } a > b \end{cases}$$

$$\mathrm{Ext}_k(\Sigma\pi_*^{S^1}(\mathbb{W}), \pi_*^{S^1}(\mathbb{Y})) = \Sigma^{2b-1}\mathbb{Q} \oplus \Sigma^{2b-2a}\mathbb{Q}.$$

Hence there is no injective map

$$\mathrm{Ext}_k(\Sigma\pi_*^{S^1}(\mathbb{X}), \pi_*^{S^1}(\mathbb{Y})) \to \mathrm{Ext}_k(\Sigma\pi_*^{S^1}(\mathbb{W}), \pi_*^{S^1}(\mathbb{Y}))$$

In particular, p_* is not injective, and we've found our counterexample.

Further Directions

- Does the generating hypothesis fail rationally for other infinite compact Lie groups?
  - Note that the rational generating hypothesis trivially holds for finite groups by work of Greenlees and May.
- Does the generating hypothesis hold nonrationally?
- Investigate the generating hypothesis in other triangulated categories with multiple generators, such as the stable module category of a finite group.