A Coarse Baum-Connes Conjecture with Coefficients

(joint work with H. Emerson)

May 7, 2003

The Coarse Baum-Connes Conjecture

The conjecture states that for (X, d) a proper metric space the assembly map

$$KX_*(X) \to K_*(C^*X)$$

is an isomorphism.

The groups $KX_*(X)$ and $K_*(C^*X)$ are 'large scale' invariants of (X, d).

Known results

The conjecture holds for bounded geometry subsets of Hilbert space (Yu). This suggests an inheritance property.

Idea: Introduce a coarse Baum-Connes conjecture with coefficients for which inheritance can be made precise.

Background

For $(X, d), (X', d')$ proper metric spaces, a map $\phi\colon (X, d) \to (X', d')$ is *coarse* if

1. for all r there exists s such that ϕ(set of diameter r) has diameter at most s;

2. ϕ^{-1}(bounded set) is bounded

For a set S two maps $\phi, \psi\colon S \to (X, d)$ are *close* if there exists r such that $d(\phi(s), \psi(s)) < r$ for all $s \in S$.

A coarse map which has a coarse inverse modulo closeness is called a *coarse equivalence*.

The Roe Algebra

Let H_X be an ample representation of $C_0(X)$. Then for operators on H_X we have a notion of support. (cf. the support of an operator with integral kernel $k(x,y)$ on $L^2(X)$)

Note supports obey matrix multiplication rule: $(x,x') \in \operatorname{Supp} S, (x',x'') \in \operatorname{Supp} T$ for some x' $\Leftarrow (x,x'') \in \operatorname{Supp} ST$.

An operator T is controlled if there exists r such that $d(x,x') \leq r$ for $(x,x') \in \operatorname{Supp} T$.

Definition. The *Roe algebra* is $C^*X =$ completion of the algebra of operators T on H_X with

1. T controlled;

2. fT, Tf compact for all $f \in C_0(X)$.

The Coarse Baum-Connes Conjecture

For simplicity we'll state the uniformly contractible case.

Definition. The *controlled dual* is $D^*X =$ completion of the algebra of operators T on H_X with

1. T controlled;
2. $[f, T]$ compact for all $f \in C_0(X)$ (cf. definition of K-homology).

We may identify the K-homology of X as

$$K_*(X) = KK(C_0(X), \mathbb{C}) = K_{*+1}(D^*(X)/C^*(X))$$

The conjecture asserts that if X is uniformly contractible then the boundary map

$$\mu\colon K_{*+1}(D^*(X)/C^*(X)) \to K_*(C^*(X))$$

is an isomorphism.

Coefficients

Idea: Replace H_X by a Hilbert module carrying representation(s) of $C_0(X)$.

We might use $H_X \otimes B$ for B any C^*-algebra (constant coefficients), but a more powerful approach is to use a 'field of C^*-algebras over X.'

We will say B is an X-algebra if it is equipped with a homomorphism $C_0(X) \to M(B)$. More generally a Hilbert module $\mathcal{E}$ is an X-module if there is a map $C_0(X) \to \mathcal{B}(\mathcal{E})$.

We will define $C^*(X; \mathcal{E})$ as a subalgebra of $\mathcal{B}(H_X \otimes \mathcal{E})$. Note that $H_X \otimes \mathcal{E}$ carries two commuting representations of $C_0(X)$.

The Roe Algebra with Coefficients

A point in the support of an operator T on $H_X \otimes \mathcal{E}$ is a quadruple $((x,y),(x',y'))$ and we have the matrix multiplication rule:
$((x,y),(x',y')) \in \operatorname{Supp} S, ((x',y'),(x'',y'')) \in \operatorname{Supp} T$ for some $(x',y') \Leftarrow ((x,y),(x'',y'')) \in \operatorname{Supp} ST$.
Definition. An operator T on $H_X \otimes \mathcal{E}$ is controlled if there exists r such that $d(x,x'), d(x,y)$, $d(x',y') < r$ for $((x,y),(x',y')) \in \operatorname{Supp} T$.
Definition. The *Roe algebra* is $C^*(X,\mathcal{E}) =$ completion of the algebra of operators T on $H_X \otimes \mathcal{E}$ with

1. T controlled;

2. $(f \otimes 1)T, T(f \otimes 1)$ compact for all $f \in C_0(X)$.

Key examples

1. Define $\Delta_r = (x, y) \in X \times X$ with $d(x, y) < r$. If $\mathcal{E} = H_X$ then $T \in B(H_X \otimes H_X)$ is controlled if for some r, T is an operator on $H_{\Delta_r} = \chi_{\Delta_r} H_X \otimes H_X$ and is controlled in the old sense, i.e. $d((x, y), (x', y')) < r$ for $((x, y), (x', y')) \in \operatorname{Supp} T$. Hence $C^*(X; H_X) = \varinjlim_r C^*(\Delta_r)$ and this has the same K-theory as $C^*(X)$ since X is coarsely equivalent to Δ_r for all r.

2. If $\mathcal{E} = C_0(X)$ regarded as a Hilbert module over itself, then note that the support of an operator consists of semi-diagonal elements $((x, y), (x', y))$. Hence the multiplication is local in the second variable. The algebra $C^*(X; C_0(X))$ is the completion of the algebra of bounded continuous functions from X to $C^*(X)$ satisfying the control condition.

The Coarse Baum-Connes Conjecture with Coefficients

Again we'll restrict to the uniformly contractible case.

Definition. The *controlled dual* is $D^*(X;\mathcal{E}) =$ completion of the algebra of operators T on $H_X \otimes \mathcal{E}$ with

1. T controlled;
2. $[(f \otimes 1), T]$ compact for all $f \in C_0(X)$.

The 'conjecture' asserts that if X is uniformly contractible then the boundary map

$$\mu_{\mathcal{E}} \colon K_{*+1}(D^*(X;\mathcal{E})/C^*(X;\mathcal{E})) \to K_*(C^*(X;\mathcal{E}))$$

is an isomorphism.

Functoriality

If $\phi\colon X \to Y$ is a coarse and continuous map, define $\phi_*(\mathcal{E})$ to be $\mathcal{E}$ with $C_0(Y)$ representated by $f\xi = (f \circ \phi)\xi$. Then there is an induced map

$$\phi_*\colon K_*(C^*(X;\mathcal{E})) \to K_*(C^*(Y;\phi_*(\mathcal{E}))).$$

If $\sigma\colon \mathcal{E} \to \mathcal{E}'$ a map of X-modules such that $f\sigma(\xi) = \sigma(f\xi)$ then there is an induced map

$$\sigma_*\colon K_*(C^*(X;\mathcal{E})) \to K_*(C^*(X;\mathcal{E}')).$$

For example if $\mathcal{E} = C_0(X)$ and Y is a closed subspace of X then the restriction map $C_0(X) \to C_0(Y)$ induces a restriction map

$$K_*(C^*(X;C_0(X)) \to K_*(C^*(X;C_0(Y)).$$

Commutative coefficients

In the case of commutative coefficients we get a (cohomological) Mayer-Vietoris sequence in the coefficients.

We can show directly that the conjecture holds for $(X; C_0(Y))$ with Y a discrete subset of X. This makes use of the fact that an operator cannot propagate from (x, y) to (x', y') with $y' \neq y$, so the algebras involved are direct products.

From these two observations we obtain the following result:
Theorem. *If X is a uniformly contractible finite dimensional simplicial complex with spherical metric then $\mu_{C_0(X)}$ is an isomorphism.*

We have 'forgotten control' by introducing the commutative coefficients. Can we regain control?

Dirac/Dual Dirac Method

Let X be an even-dimensional simply connected spin^C manifold of non-positive curvature. Let $A = C^*(X; C_0(X)), B = C^*(X; L^2(S))$, where S is the spinor bundle. We can construct a Dirac element $[D]$ in the group $KK(A, B)$, and a dual-Dirac element $[\hat{D}] \in KK(C^*(X), A)$. These are defined as follows:

Let D be the Dirac operator on sections of $L^2(S)$, and define $F = 1 \otimes D(D^2 + 1)^{-1/2}$ on $H_X \otimes L^2(S)$. Now regard $C^*(X; L^2(S))$ as a Hilbert module over itself, and note that

1. F defines an adjointable operator on this module;
2. there is a multiplication by elements of $C^*(X; C_0(X))$;
3. F commutes with this multiplication modulo elements of $C^*(X; L^2(S))$.

Hence we have an element $[D] \in KK(A, B)$.

Dual Dirac

For the dual Dirac element the Hilbert module is $C^*(X;\Gamma(S))$ where $\Gamma(S)$ is the module of continuous sections of S vanishing at infinity. This is a Hilbert module over $C^*(X;C_0(X))$, and $C^*(X)$ acts on it by left multiplication.

For $z \in X$ define functions $C_z\colon X \to T_zX$ by $C_z(x) = \exp_z^{-1}(x) \in T_zX$. Define an (unbounded) operator C on $C^*(X;\Gamma(S))$ (regarded as functions from X to operators on $H_X \otimes T_zX$) by $C\xi(z) = C_z \cdot \xi(z)$. As each C_z is contractive we get estimates on the norm of commutators of C with $T \in C^*(X)$. Hence we have an element $[\hat{D}]$ in $KK_*(C^*(X), C^*(X;C_0(X)))$.

We define the γ element $\gamma = [\hat{D}] \otimes_{C^*(X;C_0(X))} [D]$, and showing this is 1 gives a proof that the (original) coarse Baum-Connes conjecture holds for X, by reducing it to a proven case of the coefficients conjecture.

Dirac/Dual Dirac for subspaces

The definition of $C^*(X;\mathcal{E})$ etc. can be generalized to define $C^*(X;\mathcal{E}_0\otimes\mathcal{E}_1\otimes\dots)$ where each $\mathcal{E}_i$ carries a representation of $C_0(X)$. In particular for $\mathcal{E}$ an X-module we can carry out the Dirac/Dual Dirac construction and produce:

$$[D_{\mathcal{E}}] \in KK(C^*(X;C_0(X)\otimes\mathcal{E}), C^*(X;L^2(S)\otimes\mathcal{E}))$$

and

$$[\hat{D}_{\mathcal{E}}] \in KK(C^*(X;\mathcal{E}), C^*(X;C_0(X)\otimes\mathcal{E}))$$

and a γ element $\gamma_{\mathcal{E}} = [\hat{D}] \otimes [D]$.

For Y a closed subpace of X let γ_Y denote the γ element for $\mathcal{E} = H_Y$ an ample representation of Y.

Proving that $\gamma_Y = 1$ for closed subspaces Y of X will prove the coarse Baum-Connes conjecture for subspaces of non-positively curved manifolds, by reducing it to the coefficients conjecture.