

# GOING-DOWN FUNCTORS, THE KÜNNETH FORMULA, AND THE BAUM-CONNES CONJECTURE.

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ABSTRACT. We study the connection between the Baum-Connes conjecture for a locally compact group  $G$  with coefficient  $A$  and the Künneth formula for the K-theory of tensor products by the corresponding crossed product  $A \rtimes_r G$ . The main tool for this is obtained by an application of a general Going-Down procedure which allows to analyze certain functors connected to the topological K-theory of a group in terms of their restrictions to compact subgroups. We also discuss several other interesting applications of the Going-Down method, including a general extension result for the Baum-Connes conjecture.

## 0. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let  $A$  and  $B$  be two  $C^*$ -algebras and denote by  $A \otimes B$  their minimal tensor product. We say that  $A$  satisfies the *Künneth formula* if for all  $B$ , there is a canonical short exact sequence

$$(0.1) \quad 0 \rightarrow K_*(A) \otimes K_*(B) \xrightarrow{\alpha} K_*(A \otimes B) \xrightarrow{\beta} \mathrm{Tor}(K_*(A), K_*(B)) \rightarrow 0,$$

where  $K_*$  denotes  $\mathbb{Z}/2\mathbb{Z}$ -graded K-theory. We shall see below (extending earlier results of Schochet [32]) that  $A$  satisfies the Künneth formula if and only if  $\alpha : K_*(A) \otimes K_*(B) \rightarrow K_*(A \otimes B)$  is an isomorphism for every  $C^*$ -algebra  $B$  with free abelian K-group  $K_*(B)$ . We denote by  $\mathcal{N}$  the class of all  $A$  which satisfy this property. Note that  $\mathcal{N}$  contains all elements in the bootstrap class  $\mathcal{B}$  (see §4 below for the definition) and, in particular, it contains all type I algebras. However, there are  $C^*$ -algebras which do not satisfy the Künneth formula (see Remark 4.3).

The main objective of this paper is to study the connection between the Baum-Connes conjecture for a locally compact group  $G$  and the Künneth formula for the (reduced) crossed products  $A \rtimes_r G$ , where  $A$  is a  $G$ -algebra. We say that  $G$  *satisfies BC for A* if the Baum-Connes assembly map  $\mu_A : K_*^{\mathrm{top}}(G; A) \rightarrow K_*(A \rtimes_r G)$  is an isomorphism, where  $K_*^{\mathrm{top}}(G; A)$  denotes the topological K-theory of  $G$  with coefficient  $A$  (see [3] for the basic definitions). We say that  $G$  *satisfies BCC*, if  $G$  satisfies BC for all  $G$ -algebras  $A$ . By recent results of Higson, Kasparov and Tu (see [19, 36]), every  $\mathfrak{a}$ - $T$ -menable and, in particular, every amenable group satisfies BCC.

We define an equivariant analogue  $\mathcal{N}_G$  of the class  $\mathcal{N}$  consisting of all  $G$ -algebras  $A$  such that a canonically defined map

$$\alpha_G : K_*^{\mathrm{top}}(G; A) \otimes K_*(B) \rightarrow K_*^{\mathrm{top}}(G; A \otimes B)$$

is an isomorphism for all  $B$  with free abelian K-group  $K_*(B)$ . If  $A \in \mathcal{N}_G$  then  $B \mapsto K_*^{\mathrm{top}}(G; A \otimes B)$  is a generalized Künneth functor in the sense of §3 below, and we obtain a mixed Künneth formula

$$(0.2) \quad 0 \rightarrow K_*^{\mathrm{top}}(G; A) \otimes K_*(B) \xrightarrow{\alpha_G} K_*^{\mathrm{top}}(G; A \otimes B) \xrightarrow{\beta_G} \mathrm{Tor}(K_*^{\mathrm{top}}(G; A), K_*(B)) \rightarrow 0$$

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for topological K-theory. If  $G$  satisfies BC for all tensor products  $A \otimes B$ , with respect to the trivial action of  $G$  on  $B$ , then we show in Proposition 4.9 below that  $A$  is in  $\mathcal{N}_G$  if and only if  $A \rtimes_r G$  is in  $\mathcal{N}$ . We then show that the class  $\mathcal{N}_G$  is fairly large:

**Theorem 0.1** (see §4 below). *Let  $A$  be a  $G$ -algebra such that  $A \rtimes K \in \mathcal{N}$  for all compact subgroups  $K$  of  $G$ . Then  $A \in \mathcal{N}_G$ . In particular, the class  $\mathcal{N}_G$  contains all type I  $G$ -algebras.*

As a corollary of this and of Proposition 4.9 we get

**Corollary 0.2.** *Let  $A$  be a  $G$ -algebra satisfying the conditions of Theorem 0.1. Suppose further that  $G$  satisfies BC for  $A \otimes B$  for every  $C^*$ -algebra  $B$  with trivial  $G$ -action. Then  $A \rtimes_r G$  satisfies the Künneth formula.*

*In particular, if  $G$  satisfies BC for all coefficients  $B$  with trivial  $G$ -action, then the reduced group algebra  $C_r^*(G)$  satisfies the Künneth formula.*

If  $G$  is K-amenable, which is the case for all a- $T$ -menable groups by [19, 36], the full and reduced crossed products are KK-equivalent, and hence the above result also applies to the full crossed products  $A \rtimes G$ . In [21], Julg announces the remarkable result that the group  $\mathrm{Sp}(n, 1)$  and all of its closed subgroups satisfy BCC. The above corollary then shows that  $C_r^*(\Gamma)$  satisfies the Künneth formula for every lattice  $\Gamma$  in  $\mathrm{Sp}(n, 1)$ , although, as proven by Skandalis in [33, 4.1], the algebras  $C_r^*(\Gamma)$  are not K-nuclear.

For a locally compact group  $G$  let  $G_0$  denote the connected component of the identity.  $G$  is called *almost connected* if  $G/G_0$  is compact. In Proposition 5.1 below we show that the full and reduced group algebras of almost connected groups are in the bootstrap class  $\mathcal{B}$ , and hence in  $\mathcal{N}$ . Combining this with Corollary 0.2, we obtain

**Corollary 0.3.** *Let  $G$  be a locally compact group such that  $G/G_0$  satisfies BCC. Then the reduced group algebra  $C_r^*(G)$  satisfies the Künneth formula. If, in addition,  $G/G_0$  is K-amenable, then the full group algebra  $C^*(G)$  satisfies the Künneth formula, too.*

The most interesting applications of Theorem 0.1 are possibly in the converse direction. Assume that, for a given group  $G$ , we already know that  $C_r^*(G)$  satisfies the Künneth formula (eg. by showing that  $C_r^*(G)$  is in the bootstrap class). Assume further that  $G$  satisfies BC for  $\mathbb{C}$ . Then  $K_*(C_r^*(G)) \cong K_*^{\mathrm{top}}(G; \mathbb{C})$  and the combination of the short exact sequences (0.1) and (0.2) should imply that

$$K_*^{\mathrm{top}}(G; B) \cong K_*(C_r^*(G) \otimes B) = K_*(B \rtimes_r G)$$

for every  $C^*$ -algebra  $B$  with trivial  $G$ -action. By checking that the various identifications fit with the assembly maps, we get

**Corollary 0.4** (see Corollary 5.2). *Suppose that  $G$  satisfies BC for the trivial coefficient algebra  $\mathbb{C}$ . If  $C_r^*(G) \in \mathcal{N}$ , then  $G$  satisfies BC for all  $C^*$ -algebras  $B$  with trivial  $G$ -action.*

*In general, if  $G$  satisfies BC for  $\mathbb{C}$ , then  $G$  satisfies BC for all  $B \in \mathcal{N}$  with trivial  $G$ -action.*

Since all almost connected groups satisfy BC for  $\mathbb{C}$  by [10], the first part of the above corollary applies to all almost connected groups. We also know from [10] that (the  $k$ -rational points of) linear algebraic groups over local fields of characteristic 0 satisfy BC for  $\mathbb{C}$ . By [15, Theorem 5.4] and [13] all such groups have type I reduced group algebras  $C_r^*(G)$ , which therefore satisfy the Künneth formula. Hence we get

**Corollary 0.5.** *Assume that  $G$  is almost connected, or that  $G$  is a linear algebraic group over a local field of characteristic 0. Then  $G$  satisfies BC for every  $C^*$ -algebra  $B$  with trivial  $G$ -action.*

Using Corollary 0.4 and a general result on the Baum-Connes conjecture for group extensions (see Theorem 2.1), we shall prove the following permanence property for direct products of groups.

**Theorem 0.6** (see Corollary 2.10 and Theorem 5.3). *Let  $G_1, G_2$  be two locally compact groups. If  $G_1$  satisfies BC for  $\mathbb{C}$  and  $G_2$  satisfies BC for  $C_r^*(G_1)$  with trivial action, then  $G_1 \times G_2$  satisfies BC for the trivial coefficient  $\mathbb{C}$ . In particular, if  $G_1$  and  $G_2$  satisfy BC for  $\mathbb{C}$  and if one of the algebras  $C_r^*(G_i)$ ,  $i = 1, 2$ , is in  $\mathcal{N}$ , then  $G_1 \times G_2$  satisfies BC for  $\mathbb{C}$ .*

We apply this theorem to the adelic group  $G(\mathbb{A})$  corresponding to a linear algebraic group  $G \subseteq \mathrm{GL}_n(k)$  over a finite extension  $k$  of  $\mathbb{Q}$ . For every place (i.e. valuation)  $\nu$  of  $k$  let  $k_\nu$  denote the completion of  $k$  with respect to  $\nu$  and let  $G(k_\nu)$  denote the group of  $k_\nu$ -rational points of  $G$ . We write  $G(\mathcal{O}_\nu)$  for the compact group  $G(k_\nu) \cap \mathrm{GL}_n(\mathcal{O}_\nu)$ , where  $\mathcal{O}_\nu$  denotes the  $\nu$ -adic integers in  $k_\nu$ . If  $\{\nu_i : i \in \mathbb{N}\}$  is any ordering of the finite places of  $k$ , the adelic group  $G(\mathbb{A})$  of  $G$  can be obtained as the direct limit of the open subgroups

$$G_n = K_n \times G(k_{\nu_1}) \times \cdots \times G(k_{\nu_n}) \times G_\infty,$$

where  $G_\infty$  (the product of the evaluations of  $G$  at the infinite places of  $k$ ) is a real algebraic group with finitely many components, and  $K_n = \prod_{i > n} G(\mathcal{O}_{\nu_i})$  is a compact group. As observed before, every factor of  $G_n$  satisfies the assumptions of the theorem, so we see that each  $G_n$  satisfies BC for  $\mathbb{C}$ . Using [4, Theorem 2], this implies that  $G(\mathbb{A})$  satisfies BC for  $\mathbb{C}$ . Since the bootstrap class is stable under taking tensor products and direct limits (e.g. see [2, 22.3.5]),  $C_r^*(G(\mathbb{A})) = \lim_n C_r^*(G_n)$  is also in the bootstrap class. Thus, using Corollary 0.4, we get

**Theorem 0.7.** *Suppose that  $G$  is a linear algebraic group over a finite extension  $k$  of  $\mathbb{Q}$ . Then  $G(\mathbb{A})$  satisfies BC for every  $C^*$ -algebra  $B$  with trivial  $G(\mathbb{A})$ -action.*

We should mention that the above result has been obtained for reductive groups over arbitrary local fields by Baum, Millington and Plymen in [4]. The argument used in [4] for showing that validity of BC (with trivial coefficient) for  $G_n$  is inherited from the factors depends heavily on the reductiveness of  $G$  and cannot be used in the general situation.

The strategy for proving Theorems 0.1 and 0.6 follows a line which has been used quite extensively in [8, 11]: Recall that

$$K_*^{\mathrm{top}}(G; A) = \lim_{X \subseteq \mathcal{E}(G)} \mathrm{KK}_*^G(C_0(X), A),$$

where  $X$  runs through the  $G$ -compact subsets of a given universal proper  $G$ -space  $\mathcal{E}(G)$ . Hence we may view the topological K-theory of  $G$  as a limit-functor of the functor  $X \mapsto \mathrm{KK}_*^G(C_0(X), A)$  on the category of proper  $G$ -spaces  $X$ . In the situation of Theorem 0.1 there are actually two such functors in the game, the functor  $X \mapsto \mathrm{KK}_*^G(C_0(X), A \otimes B)$  and the functor  $X \mapsto \mathrm{KK}_*^G(C_0(X), A) \otimes K_*(B)$ , and the problem is to show that they coincide in the limit. In [7, 8, 11], a principle was developed and used which, under certain conditions

on similar functors on the proper  $G$ -spaces, allowed to perform a two-step reduction of the problem:

Step 1: Reduce the problem to the case of almost connected groups (based on a certain compression isomorphism and a Mayer-Vietoris argument — similar ideas have been used earlier very successfully for discrete groups in [17, 18, 28]).

Step 2: Reduce the problem to compact groups (based on a reduction isomorphism due to Kasparov).

It now became clear to us that the same principle might lead to many other useful results about topological K-theory of locally compact groups, so we decided to give a general outline of the method which culminates in Theorem 1.4 below. With this theorem at hand, the proof of Theorem 0.1 will be reduced to the checking of a small number of key facts. Indeed, as we shall demonstrate in §1 and §2 below, the general technique allows to give relatively short proofs of many other interesting results: As one further application (see Theorem 1.8 below) we shall prove that, for every locally compact group  $G$  which has an amenable action on some compact space  $Y$ , the Baum-Connes assembly map  $\mu_A : K_*^{\text{top}}(G; A) \rightarrow K_*(A \rtimes_r G)$  is split-injective for any coefficient  $A$  — thus extending an important result of Higson’s ([18]) from discrete groups to arbitrary (second-countable) locally compact groups. In another application (see Theorem 1.10 below), the Going-Down technique is used to weaken the usual assumptions required for the  $\gamma$ -element approach to the Baum-Connes conjecture. In §2 we use the Going-Down technique to generalize all existing permanence results on the Baum-Connes conjecture for group extensions, and to streamline the previous results and their proofs (see Theorem 2.1 below). Together with the results on the Künneth formula as explained above, the proof of Theorem 0.6 will then be an easy consequence of the results of §2.

In §3 we develop some general nonsense on “abstract Künneth functors” which we shall apply in §4 to the usual Künneth theorem for  $C^*$ -algebras and to the  $G$ -equivariant Künneth theorem. The results of §3 are abstractions of some of the results obtained by Schochet in [32] in a way which allows to apply them conveniently to various situations. The proof of Theorem 0.1 is given in §4 and all applications discussed above, including Theorem 0.6 and its proof, are given in §5. In this paper, all  $C^*$ -algebras are assumed to be separable and all locally compact groups are second countable, although in some instances, these assumptions might not be necessary.

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## 1. GOING-DOWN FUNCTORS

In this section we develop a general framework, which allows to prove several useful results on the topological K-theory of groups. One particular application of this framework will be the proof of Theorem 0.1. But the framework can also be used for the proof of several other important results, like continuity of topological K-theory with respect to inductive limits of coefficients, or half-exactness with respect to short exact sequences of coefficients. Further applications are given below.

Let  $G$  be a second countable locally compact group. We denote by  $\mathcal{A}(G)$  the category of proper separable commutative  $G$ - $C^*$ -algebras, i.e., algebras of the form  $C_0(Y)$ , where  $Y$  is a second countable proper locally compact  $G$ -space. We denote by  $\mathcal{S}(G)$  the set of all subgroups of  $G$  which are either  $G$ , or an open almost connected subgroup of  $G$ , or a maximal compact subgroup of an open almost connected subgroup of  $G$ .

**Definition 1.1.** A *Going-Down functor* for  $G$  is a collection of functors  $\mathcal{F} = (\mathcal{F}_H)_{H \in \mathcal{S}(G)}$ , where  $\mathcal{F}_H$  is a contravariant additive functor from  $\mathcal{A}(H)$  to the category  $\mathbf{Ab}$  of abelian groups such that the following axioms are satisfied:

**Cohomology Axioms:** For every group  $H$  in  $\mathcal{S}(G)$ ,

- (i) the functor  $\mathcal{F}_H$  is homotopy invariant;
- (ii) the functor  $\mathcal{F}_H$  is half-exact, i.e., for every short exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

in  $\mathcal{A}(H)$ , the sequence

$$\mathcal{F}_H(A/J) \longrightarrow \mathcal{F}_H(A) \longrightarrow \mathcal{F}_H(J)$$

is exact in the middle.

**Induction Axiom:** For every open almost connected subgroup  $H$  of  $G$ , there is a natural equivalence  $I_H^G$  between the functors  $\mathcal{F}_H$  and  $\mathcal{F}_G \circ \text{Ind}_H^G$ , where  $\text{Ind}_H^G : \mathcal{A}(H) \rightarrow \mathcal{A}(G); A \mapsto \text{Ind}_H^G(A)$  denotes induction of  $C^*$ -algebras from  $H$ -algebras to  $G$ -algebras.

**Restriction Axiom:** For every open almost connected subgroup  $H$  of  $G$  and any maximal compact subgroup  $K$  of  $H$ , there is a natural equivalence  $R_K^H$  between  $\mathcal{F}_H$  and  $\mathcal{F}_K \circ \text{res}_K^H$ , where  $\text{res}_K^H : \mathcal{A}(H) \rightarrow \mathcal{A}(K)$  denotes the restriction functor, that is: if  $A \in \mathcal{A}(H)$  we simply restrict the action of  $H$  on  $A$  to  $K$ .

If  $\mathcal{F}$  is a Going-Down functor, we define for each  $n \in \mathbb{N}$  the higher Going-Down functor  $\mathcal{F}^n = (\mathcal{F}_H^n)_{H \in \mathcal{S}(G)}$  by  $\mathcal{F}_H^n(A) := \mathcal{F}_H(A \otimes C_0(\mathbb{R}^n))$ , where  $H$  acts trivially on  $\mathbb{R}^n$ .

Using a standard homotopy argument (e.g., see [2, §21]) we obtain:

**Lemma 1.2.** *For every short exact sequence  $0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$  in  $\mathcal{A}(H)$  there are natural maps  $\partial_n : \mathcal{F}_H^n(J) \longrightarrow \mathcal{F}_H^{n+1}(A/J)$  providing a long exact sequence*

$$\cdots \longrightarrow \mathcal{F}_H^n(A/J) \longrightarrow \mathcal{F}_H^n(A) \longrightarrow \mathcal{F}_H^n(J) \xrightarrow{\partial_n} \mathcal{F}_H^{n+1}(A/J) \longrightarrow \cdots$$

If  $\mathcal{F}$  is a Going-Down functor for  $G$ , we define

$$\mathcal{F}^n(G) := \lim_{Z \subseteq \mathcal{E}(G)} \mathcal{F}^n(C_0(Z)),$$

where  $\mathcal{E}(G)$  denotes a *locally compact* universal example for the proper actions of  $G$ , as introduced in [25], and where  $Z$  runs through the  $G$ -compact subsets of  $\mathcal{E}(G)$ . We refer to [9] for a discussion on how this notion of  $\mathcal{E}(G)$  is related to the notion of universal proper  $G$ -spaces as introduced in [3]. Notice that although  $\mathcal{E}(G)$  is only defined up to equivariant homotopies, the homotopy invariance of  $\mathcal{F}$  ensures that  $\mathcal{F}(G)$  is independent of the special realization of  $\mathcal{E}(G)$ .

**Definition 1.3.** Let  $\mathcal{F}$  and  $\mathcal{G}$  be two Going-Down functors for  $G$ . A Going-Down transformation is a collection  $\Lambda = (\Lambda_H)_{H \in \mathcal{S}(G)}$  of natural transformations between  $\mathcal{F}_H$  and  $\mathcal{G}_H$  such that

- (1)  $\mathbb{I}_H^G \circ \Lambda_H = \Lambda_G \circ \mathbb{I}_H^G$ , for every open almost connected subgroup  $H$  of  $G$  ;
- (2)  $\mathbb{R}_K^H \circ \Lambda_H = \Lambda_K \circ \mathbb{R}_K^H$ , for every open almost connected subgroup  $H$  of  $G$  and any maximal compact subgroup  $K$  of  $H$ .

By naturality, a Going-Down transformation  $\Lambda$  between two Going-Down functors  $\mathcal{F}$  and  $\mathcal{G}$  induces morphisms  $\Lambda^n(G) : \mathcal{F}^n(G) \longrightarrow \mathcal{G}^n(G)$ .

**Theorem 1.4.** *Let  $\mathcal{F}$  and  $\mathcal{G}$  be two Going-Down functors and let  $\Lambda$  be a Going-Down transformation between  $\mathcal{F}$  and  $\mathcal{G}$  such that for any compact group  $K$  in  $\mathcal{S}(G)$  and any linear action of  $K$  on a finite dimensional real vector space  $V$ , the map  $\Lambda_K : \mathcal{F}_K(C_0(V)) \longrightarrow \mathcal{G}_K(C_0(V))$  is an isomorphism. Then, for all  $n$ , the map  $\Lambda^n(G) : \mathcal{F}^n(G) \longrightarrow \mathcal{G}^n(G)$  is an isomorphism.*

The proof requires some preparations. If  $G$  is a locally compact group, we denote by  $G_0$  the connected component of the identity. The quotient group  $G/G_0$  is totally disconnected and we can realize  $\mathcal{E}(G/G_0)$  as a locally finite simplicial complex with type preserving  $G$ -action (see [7]). Note that under the action of  $G$ , the stabilizer of any simplex of  $\mathcal{E}(G/G_0)$  is an almost connected group. Let  $\mathcal{E}(G)$  be any universal proper  $G$ -space. Then, equipped with the diagonal action,  $\mathcal{E}(G) \times \mathcal{E}(G/G_0)$  is a universal proper  $G$ -space, too (see [8]). Thus, if  $\mathcal{F}$  is any Going-Down functor for  $G$ , we can write:

$$\mathcal{F}^n(G) = \lim_{L, W} \mathcal{F}_G^n(C_0(G \cdot (L \times W)))$$

where  $L$  runs through the compact subsets of  $\mathcal{E}(G)$  and  $W$  runs through the finite sub-complexes of  $\mathcal{E}(G/G_0)$ . We have to introduce further notation:

**Notations 1.5.** For any given finite sub-complex  $W \subseteq \mathcal{E}(G/G_0)$  we denote by  $\dim W$  the maximal dimension of the simplices in  $W$ , and we let  $\overset{\circ}{W}$  be the union of the interiors of the simplices of  $W$  which have dimension  $\dim W$ . Set  $|W| = W \setminus \overset{\circ}{W}$ . We define for any  $n \in \mathbb{N}$

$$\mathcal{F}^n(G)[W] := \lim_L \mathcal{F}_G^n(C_0(G \cdot (L \times W))) \quad \text{and}$$

$$\mathcal{F}^n(G)[\overset{\circ}{W}] := \lim_L \mathcal{F}_G^n(C_0(G \cdot (L \times \overset{\circ}{W}))),$$

where  $L$  runs through the compact subsets of  $\mathcal{E}(G)$ .

Notice that with the above notation, we can compute  $\mathcal{F}^n(G)$  as

$$(1.1) \quad \mathcal{F}^n(G) = \lim_W \mathcal{F}^n(G)[W],$$

where  $W$  runs through the finite sub-complexes of  $\mathcal{E}(G/G_0)$ .

A Going-Down transformation  $\Lambda$  between two Going-Down functors  $\mathcal{F}$  and  $\mathcal{G}$  induces morphisms  $\Lambda_G^n[W] : \mathcal{F}^n(G)[W] \longrightarrow \mathcal{G}^n(G)[W]$  and  $\Lambda_G^n[\overset{\circ}{W}] : \mathcal{F}^n(G)[\overset{\circ}{W}] \longrightarrow \mathcal{G}^n(G)[\overset{\circ}{W}]$ . Using equation (1.1), Theorem 1.4 will follow from

**Lemma 1.6.** *Assume that for every compact group  $K \in \mathcal{S}(G)$ , and any linear action of  $K$  on a finite dimension real vector space  $V$ , the map  $\Lambda_K : \mathcal{F}_K(C_0(V)) \longrightarrow \mathcal{G}_K(C_0(V))$  is an isomorphism. Then  $\Lambda_G^n[W] : \mathcal{F}^n(G)[W] \longrightarrow \mathcal{G}^n(G)[W]$  is an isomorphism for all  $n \in \mathbb{N}$ .*

*Proof.* To start with, let us note first that the assumption on  $\Lambda_K : \mathcal{F}_K(C_0(V)) \longrightarrow \mathcal{G}_K(C_0(V))$  clearly implies that  $\Lambda_K^n : \mathcal{F}_K^n(C_0(V)) \longrightarrow \mathcal{G}_K^n(C_0(V))$  is an isomorphism for all  $n$ . The proof for this lemma is done by induction on the dimension of  $W$  following ideas which have been developed in [7, 8, 11].

Since it is simpler and gives a clear picture of the technique of the proof, we start with the case when  $W$  is a single vertex of  $\mathcal{E}(G/G_0)$ . Let  $H$  be the stabilizer of  $W$  under the action of  $G$ . It is an open almost connected subgroup of  $G$ . The  $G$ -space  $G \cdot (L \times W)$  is  $G$ -homeomorphic to the induced space  $G \times_H (H \cdot (L \times W))$ , and since  $W = \{\text{pt}\}$  is a single vertex, we have  $H \cdot (L \times W) \cong H \cdot L$ . Hence we can use the Induction Axiom and part (1) of Definition 1.3 to get the commutative diagram

$$\begin{array}{ccc} \mathcal{F}_H^n(C_0(H \cdot L)) & \xrightarrow{\Lambda_H^n} & \mathcal{G}_H^n(C_0(H \cdot L)) \\ \text{I}_H^G \downarrow \cong & & \text{I}_H^G \downarrow \cong \\ \mathcal{F}_G^n(G \cdot (L \times W)) & \xrightarrow{\Lambda_G^n} & \mathcal{G}_G^n(G \cdot (L \times W)). \end{array}$$

Let  $K$  denote a maximal compact subgroup of  $H$ . Since  $H$  is a subgroup of  $G$ ,  $\mathcal{E}(G)$  is also universal for  $H$  and hence, it is homotopic to  $H/K$  as an  $H$ -space. Therefore, taking the inductive limit on the compact subsets  $L$  of  $\mathcal{E}(G)$ , we get

$$\lim_L \mathcal{F}_H^n(C_0(H \cdot L)) = \mathcal{F}_H^n(C_0(H/K)) \quad \text{and} \quad \lim_L \mathcal{G}_H^n(C_0(H \cdot L)) = \mathcal{G}_H^n(C_0(H/K)).$$

Thus, in the limit we obtain a commutative diagram

$$(1.2) \quad \begin{array}{ccc} \mathcal{F}_H^n(C_0(H/K)) & \xrightarrow{\Lambda_H^n} & \mathcal{G}_H^n(C_0(H/K)) \\ \cong \downarrow & & \cong \downarrow \\ \mathcal{F}^n(G)[W] & \xrightarrow{\Lambda_G^n[W]} & \mathcal{G}_G^n[W]. \end{array}$$

The Restriction Axiom and part (2) of Definition 1.3 give the commutative diagram

$$\begin{array}{ccc} \mathcal{F}_H^n(C_0(H/K)) & \xrightarrow{\Lambda_H^n} & \mathcal{G}_H^n(C_0(H/K)) \\ \text{R}_K^H \downarrow \cong & & \text{R}_K^H \downarrow \cong \\ \mathcal{F}_K^n(C_0(H/K)) & \xrightarrow{\Lambda_K^n} & \mathcal{G}_K^n(C_0(H/K)). \end{array}$$

But  $H/K$  is  $K$ -homeomorphic to a finite dimensional real vector space with linear  $K$ -action, so the bottom arrow is an isomorphism by our assumptions. Combining the above diagrams, we see that  $\Lambda_G^n[W] : \mathcal{F}^n(G)[W] \longrightarrow \mathcal{G}^n(G)[W]$  is an isomorphism when  $W$  is a single vertex of  $\mathcal{E}(G/G_0)$ .

When  $W$  is a single simplex of dimension greater than 0, we denote by  $H$  the stabilizer of  $\overset{\circ}{W}$ . We remark that the  $G$ -space  $G \cdot (L \times \overset{\circ}{W})$  is  $G$ -homeomorphic to the induced space  $G \times_H (H \cdot (L \times \overset{\circ}{W}))$  and we have  $H \cdot (L \times \overset{\circ}{W}) = H \cdot L \times \overset{\circ}{W}$ . Thus, proceeding as before, we

obtain a commutative diagram with bijective columns:

$$\begin{array}{ccc} \mathcal{F}_H^n(C_0(H \cdot L \times \overset{\circ}{W})) & \xrightarrow{\Lambda_H^n} & \mathcal{G}_H^n(C_0(H \cdot L \times \overset{\circ}{W})) \\ \mathcal{I}_H^G \downarrow \cong & & \mathcal{I}_H^G \downarrow \cong \\ \mathcal{F}_G^n(G \cdot (L \times \overset{\circ}{W})) & \xrightarrow{\Lambda_G^n} & \mathcal{G}_G^n(G \cdot (L \times \overset{\circ}{W})) \end{array}$$

Since  $\overset{\circ}{W}$  is homeomorphic to  $\mathbb{R}^{\dim(W)}$ , we can replace the upper corners of the above diagram by the groups  $\mathcal{F}_H^{n+\dim(W)}(C_0(H \cdot L))$  and  $\mathcal{G}_H^{n+\dim(W)}(C_0(H \cdot L))$ , respectively. Hence, using the same arguments as in the single vertex case, taking limits over the compact subsets  $L$  of  $\mathcal{E}(G)$ , we obtain the commutative diagram

$$(1.3) \quad \begin{array}{ccc} \mathcal{F}_H^{n+\dim(W)}(C_0(H/K)) & \xrightarrow{\Lambda_K^{n+\dim(W)}} & \mathcal{G}_H^{n+\dim(W)}(C_0(H/K)) \\ \cong \downarrow & & \cong \downarrow \\ \mathcal{F}^n(G)[\overset{\circ}{W}] & \xrightarrow{\Lambda_G^n[\overset{\circ}{W}]} & \mathcal{G}^n(G)[\overset{\circ}{W}] \end{array}$$

Proceeding further as in the case of a single vertex, we can use the Restriction Axiom to see that the top arrow is an isomorphism. Hence  $\Lambda_G^n[\overset{\circ}{W}] : \mathcal{F}^n(G)[\overset{\circ}{W}] \rightarrow \mathcal{G}^n(G)[\overset{\circ}{W}]$  is an isomorphism, too.

If  $W$  is a finite union of simplices, then  $\overset{\circ}{W}$  is a disjoint finite union of interiors of single simplices, and the isomorphism in this case follows from the additivity of the Going-Down functor.

We can now perform the induction: If  $W$  is a finite union of simplices with  $\dim W = k > 0$ , then  $|W|$  is a finite union of simplices with  $\dim |W| = k - 1$ . For a fixed compact set  $L \subseteq \mathcal{E}(G)$ , consider the exact sequence

$$0 \rightarrow C_0(G \cdot (L \times \overset{\circ}{W})) \rightarrow C_0(G \cdot (L \times W)) \rightarrow C_0(G \cdot (L \times |W|)) \rightarrow 0.$$

According to Lemma 1.2, it gives rise to two long exact sequences associated with  $\mathcal{F}$  and  $\mathcal{G}$ , respectively. By naturality of the boundary maps, these long exact sequences are linked in the following commutative diagram:

$$\begin{array}{ccccccccc} \dots \mathcal{F}^{n+2}(G)[\overset{\circ}{W}] & \longrightarrow & \mathcal{F}^{n+1}(G)[|W|] & \longrightarrow & \mathcal{F}^{n+1}(G)[W] & \longrightarrow & \mathcal{F}^{n+1}(G)[\overset{\circ}{W}] & \longrightarrow & \mathcal{F}^n(G)[|W|] \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots \mathcal{G}^{n+2}(G)[\overset{\circ}{W}] & \longrightarrow & \mathcal{G}^{n+1}(G)[|W|] & \longrightarrow & \mathcal{G}^{n+1}(G)[W] & \longrightarrow & \mathcal{G}^{n+1}(G)[\overset{\circ}{W}] & \longrightarrow & \mathcal{G}^n(G)[|W|] \dots \end{array}$$

We know that the map  $\mathcal{F}^n(G)[W] \rightarrow \mathcal{G}^n(G)[W]$  is an isomorphism for all  $n \in \mathbb{N}$  when  $\dim(W) = 0$ . Assume it is also a bijection when  $\dim(W) = k - 1$  and consider the above diagram. All vertical arrows except the middle one are known to be bijections, hence the result follows from the Five Lemma.  $\square$

**Remark 1.7.** It might be useful to remark, that in some situations it is convenient to consider Going-Down functors which only go half-way, i.e., in which we consider the set  $\mathcal{S}_{ac}(G)$  consisting only of  $G$  and the almost connected open subgroups  $H$  of  $G$ , and in which the

functors  $\mathcal{F} = (\mathcal{F}_H)_{H \in \mathcal{S}_{ac}(G)}$  of Definition 1.1 are only assumed to satisfy the Cohomology Axioms and the Induction Axiom. Then, if  $\Lambda = (\Lambda_H)_{H \in \mathcal{S}_{ac}(G)}$  is a family of natural transformations between two such functors  $\mathcal{F}$  and  $\mathcal{G}$  satisfying condition (1) of Definition 1.3 and such that  $\Lambda_H^n : \mathcal{F}^n(H) \rightarrow \mathcal{G}^n(H)$  is an isomorphism for all open almost connected subgroups  $H$  of  $G$ , then it follows from diagrams (1.2) and (1.3) (and the universality of  $H/K$ ), that  $\Lambda_G^n : \mathcal{F}^n(G) \rightarrow \mathcal{G}^n(G)$  is an isomorphism, too.

**1.1. Injectivity of the assembly map for groups which are amenable at infinity.** As indicated before, the main examples of Going-Down functors appear in connection with the topological K-theory of locally compact groups and the Baum-Connes conjecture. Indeed, if  $G$  is a second countable locally compact group and  $B$  is a fixed  $G$ -algebra define

$$\mathcal{F}_H^*(C_0(Z)) := \mathrm{KK}_*^H(C_0(Z), B), \quad \text{for } H \in \mathcal{S}(G) \text{ and } C_0(Z) \in \mathcal{A}(H).$$

Then  $\mathcal{F}^* := (\mathcal{F}_H^*)_{H \in \mathcal{S}(G)}$  is a  $\mathbb{Z}/2\mathbb{Z}$ -graded Going-Down functor in the sense of Definition 1.1. It follows from [24, Corollary of Proposition 6.2] that  $\mathcal{F}^*$  satisfies the Cohomology Axioms, [7, Proposition 5.14] implies that  $\mathcal{F}^*$  satisfies the Induction Axiom, and the Restriction Axiom follows from [23, Theorem 5.8]. By definition, the topological K-theory  $\mathrm{K}_*^{\mathrm{top}}(G; B)$  of  $G$  with coefficient  $B$  coincides with  $\mathcal{F}^*(G)$ .

Recall that a locally compact group  $G$  is said to be *amenable at infinity* if there exists a topologically amenable action (in the sense of [1]) of  $G$  on some compact Hausdorff space  $X$ . The following theorem extends a result of Higson, who gave the proof in case where  $G$  is discrete:

**Theorem 1.8** (cf [18, Theorem 1.1]). *Let  $G$  be a second countable locally compact group such that  $G$  is amenable at infinity and let  $A$  be a  $G$ -algebra. Then the Baum-Connes assembly map  $\mu_A : \mathrm{K}_*^{\mathrm{top}}(G; A) \rightarrow \mathrm{K}_*(A \rtimes_r G)$  is split-injective.*

According to [18, Lemma 3.5], we can assume that the compact set  $X$  on which  $G$  acts amenably is second countable. Moreover, up to replacing  $X$  by its space of Borel probability measures [18, Lemma 3.6], we can assume that  $X$  is a convex space equipped with an affine action of  $G$ . Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{K}_*^{\mathrm{top}}(G; A) & \xrightarrow{\mu_A} & \mathrm{K}_*(A \rtimes_r G) \\ i_* \downarrow & & \downarrow i_* \\ \mathrm{K}_*^{\mathrm{top}}(G; A \otimes C(X)) & \xrightarrow{\mu_{A \otimes C(X)}} & \mathrm{K}_*((A \otimes C(X)) \rtimes_r G), \end{array}$$

where the vertical arrows are induced by the inclusion  $i : \mathbb{C} \rightarrow C(X)$  and where the horizontal arrows are the Baum-Connes assembly maps. Since the action of  $G$  on  $X$  is amenable, it follows from [36] (but see also [11, Corollary 0.4]) that the bottom horizontal arrow is an isomorphism. Since  $X$  is convex and the action of  $G$  on  $X$  is affine, the space  $X$  is  $K$ -equivariantly contractible for any compact subgroup  $K$  of  $G$ . An easy diagram chase then shows the split-injectivity of the assembly map  $\mu_A$ , provided that we have the following extension of [18, Proposition 3.7] to arbitrary second countable locally compact groups.

**Proposition 1.9.** *Let  $G$  be a second countable locally compact group and let  $X$  be a second countable compact  $G$ -space such that  $X$  is  $K$ -contractible for every compact group  $K$  in  $\mathcal{S}(G)$ . Then the map  $i_* : \mathrm{K}_*^{\mathrm{top}}(G; A) \rightarrow \mathrm{K}_*^{\mathrm{top}}(G; A \otimes C(X))$  is an isomorphism.*

*Proof.* We define the additive contravariant functors  $\mathcal{F}_H^* : \mathcal{A}(H) \rightarrow \mathbf{Ab}$  and  $\mathcal{G}_H^* : \mathcal{A}(H) \rightarrow \mathbf{Ab}$  by

$$\mathcal{F}_H^*(C_0(Y)) = \mathrm{KK}_*^H(C_0(Y), A) \quad \text{and} \quad \mathcal{G}_H^*(C_0(Y)) = \mathrm{KK}_*^H(C_0(Y), A \otimes C(X))$$

for  $H \in \mathcal{S}(G)$  and any proper  $H$ -space  $Y$ . As observed above,  $\mathcal{F}^* = (\mathcal{F}_H^*)_{H \in \mathcal{S}(G)}$  and  $\mathcal{G}^* = (\mathcal{G}_H^*)_{H \in \mathcal{S}(G)}$  are Going-Down functors and the inclusion  $i : \mathbb{C} \rightarrow C(X)$  induces a natural transformation between  $\mathcal{F}^*$  and  $\mathcal{G}^*$ . Since induction and restriction are natural, these transformations satisfy the compatibility conditions of Definition 1.3. Since  $X$  is  $K$ -contractible for every compact group  $K \in \mathcal{S}(G)$ , the morphism  $i_* : \mathrm{KK}_*^K(C_0(V), A) \rightarrow \mathrm{KK}_*^K(C_0(V), A \otimes C(X))$  is an isomorphism for every linear action of  $K$  on  $V$ . Thus we can apply Theorem 1.4 to get the result.  $\square$

**1.2. A weak version of the  $\gamma$ -element.** Let  $G$  be a locally compact group and let  $D$  be a  $G$ -algebra. Let  $\gamma = \alpha \otimes_D \beta \in \mathrm{KK}_0^G(\mathbb{C}, \mathbb{C})$  for some fixed elements  $\alpha \in \mathrm{KK}_0^G(\mathbb{C}, D)$  and  $\beta \in \mathrm{KK}_0^G(D, \mathbb{C})$ . If  $G$  is discrete,  $D$  is a proper  $G$ -algebra (i.e.,  $D$  is an  $X \rtimes G$ -algebra for some proper  $G$ -space  $X$ ), and if the restriction  $\mathrm{res}_K^G(\gamma) \in \mathrm{KK}_0^K(\mathbb{C}, \mathbb{C})$  is equal to  $1_{\mathbb{C}}$  for every finite subgroup  $K$  of  $G$ , then Guentner, Higson and Trout showed in [17, Theorem 14.2] that the Baum-Connes assembly map for  $G$  is split injective, for any coefficient  $A$ .

Using our general Going-Down technique (together with [11, Corollary 0.4]) we can extend their arguments to obtain

**Theorem 1.10.** *Let  $G$  be a second countable locally compact group acting amenably on the second countable locally compact space  $X$ . Let  $D$  be an  $X \rtimes G$ -algebra and let  $\gamma = \alpha \otimes_D \beta \in \mathrm{KK}_0^G(\mathbb{C}, \mathbb{C})$  for some elements  $\alpha \in \mathrm{KK}_0^G(\mathbb{C}, D)$  and  $\beta \in \mathrm{KK}_0^G(D, \mathbb{C})$ , such that  $\gamma$  restricts to  $1_{\mathbb{C}} \in \mathrm{KK}_0^K(\mathbb{C}, \mathbb{C})$  for any compact group  $K$  in  $\mathcal{S}(G)$ . Then, for any  $G$ -algebra  $A$ , the Baum-Connes assembly map  $\mu_A : \mathrm{K}_*^{\mathrm{top}}(G; A) \rightarrow \mathrm{K}_*(A \rtimes_r G)$  is split-injective.*

*Proof.* Consider again the additive contravariant functor  $\mathcal{F}_H^* : \mathcal{A}(H) \rightarrow \mathbf{Ab}$  given by

$$\mathcal{F}_H^*(C_0(Y)) = \mathrm{KK}_*^H(C_0(Y), A)$$

for  $H \in \mathcal{S}(G)$  and any proper  $H$ -space  $Y$ . As observed before,  $\mathcal{F}^* = (\mathcal{F}_H^*)_{H \in \mathcal{S}(G)}$  is a Going-Down functor, and the collection  $(\Phi_H)_{H \in \mathcal{S}(G)}$  of all morphisms  $\Phi_H : \mathrm{KK}_*^H(C_0(Y), A) \rightarrow \mathrm{KK}_*^H(C_0(Y), A)$ ,  $\Phi_H(x) := \gamma \otimes_{\mathbb{C}} x$  is a natural transformation of  $\mathcal{F}^*$  in the sense of Definition 1.3. The condition on  $\mathrm{res}_K^G(\gamma)$  implies that  $\Phi_K$  is the identity of  $\mathcal{F}_K^*(C_0(V))$  for any compact  $K \in \mathcal{S}(G)$  and any finite dimensional real linear  $K$ -space  $V$ . Hence Theorem 1.4 implies that the composition of the vertical left arrows of the following commutative diagram is a bijection:

$$\begin{array}{ccccc} \mathrm{K}_*^{\mathrm{top}}(G; A) & \xrightarrow{\otimes_{\mathbb{C}} \alpha} & \mathrm{K}_*^{\mathrm{top}}(G; A \otimes D) & \xrightarrow{\otimes_D \beta} & \mathrm{K}_*^{\mathrm{top}}(G; A) \\ \mu_A \downarrow & & \mu_{A \otimes D} \downarrow & & \downarrow \mu_A \\ \mathrm{K}_*(A \rtimes_r G) & \xrightarrow{\otimes_{J_G(\sigma_A(\alpha))}} & \mathrm{K}_*((A \otimes D) \rtimes_r G) & \xrightarrow{\otimes_{J_G(\sigma_A(\beta))}} & \mathrm{K}_*(A \rtimes_r G) \end{array}$$

(where  $\sigma_A$  denotes the exterior tensor product with  $A$  and  $J_G$  denotes Kasparov's descent homomorphism). Since  $A \otimes D$  is an  $X \rtimes G$ -algebra and the action of  $G$  on  $X$  is topologically amenable, it follows from [11, Corollary 0.4] that the middle horizontal arrow is an isomorphism. An easy diagram chase then gives the result.  $\square$

2. THE BAUM-CONNES CONJECTURE FOR EXTENSIONS

In this section we use our general Going-Down technique to improve the results obtained in [7, 8, 28] on the Baum-Connes conjecture for group extensions. We prove

**Theorem 2.1.** *Let  $N$  be a closed normal subgroup of the (second countable) locally compact group  $G$  and let  $B$  be a separable  $G$ -algebra. Assume that the following condition (A) holds:*

(A) *If  $N \subseteq L$  is a closed subgroup of  $G$  and  $L/N$  is compact, then  $L$  satisfies BC for  $B$ .*

*Then  $G$  satisfies BC for  $B$  if and only if  $G/N$  satisfies BC for  $B \rtimes_r N$ .*

**Remark 2.2.** (1) Notice that, in general, the action of  $G/N$  on  $B \rtimes_r N$  cannot be realized as an ordinary action. Instead, we have to use Green's twisted action of the pair  $(G, N)$  on  $B \rtimes_r N$  such that  $B \rtimes_r G \cong (B \rtimes_r N) \rtimes_r (G, N)$  (see [7] for the details). However, by the stabilization trick of [14] (see [26] for the case of reduced crossed products) we know that the twisted action of  $(G, N)$  on  $B \rtimes_r N$  is Morita equivalent to the ordinary action of  $G/N$  on  $C_0(G/N, B) \rtimes_r G$ , where  $G$  acts by left translation on  $G/N$  and diagonally on  $C_0(G/N, B) \cong C_0(G/N) \otimes B$  and the action of  $G/N$  on  $C_0(G/N, B) \rtimes_r G$  is induced by the right-translation action of  $G/N$  on  $C_0(G/N, B)$ . Since the validity of the Baum-Connes conjecture is invariant under passing to equivariantly Morita equivalent coefficients (eg., see [7, Proposition 5.6]), the following statements are equivalent:

- (i)  $G/N$  satisfies BC for  $C_0(G/N, B) \rtimes_r G$ .
- (ii)  $G/N$  satisfies BC for  $B \rtimes_r N$ .

(2) The above theorem has been obtained for discrete groups in [28], and for more general locally compact groups in [7, Theorem 7.1] and [8, Corollary 3.4 and Corollary 3.10] under certain additional assumptions. The proof we give here follows some ideas presented in [28].

(3) At this place we should use the opportunity to point out an error in the statements of [8, Theorem 3.7 and Corollary 3.10]: If, as in those statements,  $\dot{K}$  denotes the maximal compact subgroup of the almost connected group  $G/N$ , then the assumption that  $K = q^{-1}(\dot{K})$  satisfies BC for  $B$  has to be replaced by the assumption that  $K$  satisfies BC for  $C_0(G/K) \otimes B$  with respect to the diagonal action, since this is used in the proof. Using similar arguments as given for Lemma 2.8 below one can show that this assumption can also be replaced by condition (A) of Theorem 2.1.

We divide the proof of the theorem into several steps. For the first step assume that  $G \times G'$  is a direct product of two groups acting on some  $C^*$ -algebra  $A$ . Let

$$\mu_{part, G'} : K_*^{\text{top}}(G \times G'; A) \rightarrow K_*^{\text{top}}(G'; A \rtimes_r G)$$

denote the partial assembly map of [6, 3.9] which can be constructed as follows: Let  $\mathcal{E}(G)$  and  $\mathcal{E}(G')$  denote universal proper  $G$ - resp.  $G'$ -spaces. Then  $\mathcal{E}(G) \times \mathcal{E}(G')$  is a universal proper  $G \times G'$ -space. The topological K-theory of  $G \times G'$  with coefficient  $A$  can be computed as

$$K_*^{\text{top}}(G \times G'; A) = \lim_{X, Y} \text{KK}_*^{G \times G'}(C_0(X \times Y), A),$$

where  $X$  and  $Y$  run through the  $G$ - resp.  $G'$ -compact subsets of  $\mathcal{E}(G)$  resp.  $\mathcal{E}(G')$ . If we fix a proper  $G'$ -space  $Y$  (not necessarily  $G'$ -compact), we define a map

$$\mu_{X, Y} : \text{KK}_*^{G \times G'}(C_0(X \times Y), A) \rightarrow \text{KK}^{G'}(C_0(Y), A \rtimes_r G),$$

similar to the usual assembly map, via the composition

(2.1)

$$\mathrm{KK}_*^{G \times G'}(C_0(X \times Y), A) \xrightarrow{J_G} \mathrm{KK}^{G'}(C_0(X) \rtimes G \otimes C_0(Y), A \rtimes_r G) \xrightarrow{[p_X] \otimes} \mathrm{KK}^{G'}(C_0(Y), A \rtimes_r G),$$

where  $J_G$  denotes Kasparov's descent homomorphism for  $G$  (which is equivariant for  $G'$ ) and  $[p_X] \in \mathrm{K}_*(C_0(X) \rtimes G)$  is the canonical class constructed via a cut-off function for  $X$  (see [6] for more details). If  $Y$  is  $G'$ -compact, then there is a canonical map from  $\mathrm{KK}_*^{G'}(C_0(Y), A \rtimes_r G)$  to  $\mathrm{K}_*^{\mathrm{top}}(G', A \rtimes_r G)$ . Composing this with  $\mu_{X,Y}$  and taking limits over  $X$  and  $Y$  gives the partial assembly map  $\mu_{\mathrm{part}, G'}$ .

Specializing to  $G' = G/N$  and  $A = C_0(G/N, B)$  with  $N$  and  $B$  as in Theorem 2.1, we get

**Proposition 2.3.** *Assume that  $N$  is a closed normal subgroup of  $G$  and that  $B$  is a  $G$ -algebra such that the partial assembly map*

$$\mu_{\mathrm{part}, G/N} : \mathrm{K}_*^{\mathrm{top}}(G \times G/N; C_0(G/N, B)) \rightarrow \mathrm{K}_*^{\mathrm{top}}(G/N; C_0(G/N, B) \rtimes_r G)$$

*is an isomorphism. Then  $G$  satisfies BC for  $B$  if and only if  $G/N$  satisfies BC for  $B \rtimes_r N$ .*

*Proof.* We first note that the  $G \times G/N$ -algebra  $C_0(G/N, B)$  is isomorphic to the induced algebra  $\mathrm{Ind}_{\Delta(G)}^{G \times G/N} B$  (we refer to [8, §2] for the notation), where  $\Delta(G) = \{(s, sN) : s \in G\} \cong G$  denotes the diagonal in  $G \times G/N$ , and the action of  $\Delta(G)$  on  $B$  is given via the canonical identification  $\Delta(G) \cong G$ . Consider the diagram:

$$\begin{array}{ccccc} \mathrm{K}_*^{\mathrm{top}}(G; B) & \xrightarrow[\cong]{\mathrm{Ind}_G^{G \times G/N}} & \mathrm{K}_*^{\mathrm{top}}(G \times G/N; C_0(G/N, B)) & \xrightarrow[\cong]{\mu_{\mathrm{part}, G/N}} & \mathrm{K}_*^{\mathrm{top}}(G/N; C_0(G/N, B) \rtimes_r G) \\ \mu_B \downarrow & & \mu_{C_0(G/N, B)} \downarrow & & \downarrow \mu_{C_0(G/N, B) \rtimes_r G} \\ \mathrm{K}_*(B \rtimes_r G) & \xrightarrow[\cong]{} & \mathrm{K}_*(C_0(G/N, B) \rtimes_r (G \times G/N)) & \xrightarrow[\cong]{} & \mathrm{K}_*((C_0(G/N, B) \rtimes_r G) \rtimes_r G/N), \end{array}$$

where the first isomorphism in the upper row follows from [8, Theorem 2.2], the second follows from our assumption, the first isomorphism of the lower row follows from Green's imprimitivity theorem, and the second follows from taking iterated crossed products. The first square commutes by [8, Proposition 2.3] and the second square commutes by [6, Proposition 3.10]. Hence, the result follows from this diagram and part (1) of Remark 2.2.  $\square$

So all we have to do is to show that condition (A) of Theorem 2.1 implies bijectivity of the partial assembly map of Proposition 2.3. For this we want to use our Going-Down technique. As a first step we prove an ‘‘equivariant’’ version of [23, Theorem 5.8].

**Lemma 2.4.** *Let  $G$  and  $H$  be almost connected locally compact groups and let  $A$  be a separable  $G \times H$ -algebra. Let  $C$  denote the maximal compact subgroup of  $G$ . Then for any proper  $G \times H$ -space  $Z$ , the restriction map  $\mathrm{res}_{C \times H}^{G \times H} : \mathrm{KK}_*^{G \times H}(C_0(Z), A) \rightarrow \mathrm{KK}_*^{C \times H}(C_0(Z), A)$  is an isomorphism.*

*Proof.* Let  $K$  denote the maximal compact subgroup of  $H$ . Then  $G \times H$  and  $C \times H$  are both almost connected groups with maximal compact subgroup  $C \times K$ . But [23, Theorem 5.8] implies that the second map of  $\mathrm{KK}_*^{G \times H}(C_0(Z), A) \xrightarrow{\mathrm{res}} \mathrm{KK}_*^{G \times K}(C_0(Z), A) \xrightarrow{\mathrm{res}} \mathrm{KK}_*^{C \times K}(C_0(Z), A)$  is an isomorphism, and the same is true for the composition, which is identical to  $\mathrm{res}_{C \times K}^{G \times H}$ . Thus the first map of the composition has to be an isomorphism, too.  $\square$

**Lemma 2.5.** *Let  $G$  and  $H$  be locally compact groups with  $H$  almost connected and let  $A$  be a separable  $G \times H$ -algebra. Let  $K$  denote the maximal compact subgroup of  $H$ . For a fixed proper  $H$ -space  $Y$  and for any proper  $G$ -space  $X$ , let*

$$\text{res}_{G \times K}^{G \times H} : \text{KK}_*^{G \times H}(C_0(X \times Y), A) \rightarrow \text{KK}_*^{G \times K}(C_0(X \times Y), A)$$

denote the restriction map, and let

$$\text{R}_K^H[Y] : \lim_X \text{KK}_*^{G \times H}(C_0(X \times Y), A) \rightarrow \lim_X \text{KK}_*^{G \times K}(C_0(X \times Y), A)$$

denote the resulting map on the limits taken over all  $G$ -compact subsets  $X$  of  $\mathcal{E}(G)$ . Then  $\text{R}_K^H[Y]$  is an isomorphism.

*Proof.* Let  $L$  be an element of  $\mathcal{S}(G)$  and let  $X$  be a proper  $L$ -space. We define

$$\mathcal{F}_L^*(C_0(X)) := \text{KK}_*^{L \times H}(C_0(X \times Y), A) \quad \text{and} \quad \mathcal{G}_L^*(C_0(X)) := \text{KK}_*^{L \times K}(C_0(X \times Y), A).$$

It follows from [24, Corollary of Proposition 6.2], [8, Proposition 5.14], [23, Theorem 5.8] and Lemma 2.4 that  $\mathcal{F}^* = (\mathcal{F}_L^*)_{L \in \mathcal{S}(G)}$  and  $\mathcal{G}^* = (\mathcal{G}_L^*)_{L \in \mathcal{S}(G)}$  are Going-Down functors, and it is easily checked that the restriction maps

$$\text{res}_{L \times K}^{L \times H} : \text{KK}_*^{L \times H}(C_0(X \times Y), A) \rightarrow \text{KK}_*^{L \times K}(C_0(X \times Y), A)$$

provide a Going-Down transformation from  $\mathcal{F}^*$  to  $\mathcal{G}^*$ . Let  $C$  be a compact subgroup of  $G$  acting linearly on a real vector space  $V$ . Then  $C \times H$  is an almost connected group with maximal compact subgroup  $C \times K$  and  $\text{res}_{C \times K}^{C \times H} : \text{KK}_*^{C \times H}(C_0(V \times Y), A) \rightarrow \text{KK}_*^{C \times K}(C_0(V \times Y), A)$  is an isomorphism by [23, Theorem 5.8]. Hence, the result follows from Theorem 1.4.  $\square$

**Proposition 2.6.** *Suppose that  $G \times G'$  is a direct product of locally compact groups and that  $A$  is a  $G \times G'$ -algebra. Assume that for any compact subgroup  $K$  of  $G'$  and for any finite dimensional real linear  $K$ -space  $V$  the group  $G \times K$  satisfies the Baum-Connes conjecture with coefficient  $A \otimes C_0(V)$  (with trivial action of  $G$  on  $V$ ). Then the partial assembly map  $\mu_{\text{part}, G'} : \text{K}_*^{\text{top}}(G \times G'; A) \rightarrow \text{K}_*^{\text{top}}(G'; A \rtimes_r G)$  is an isomorphism.*

*Proof.* If  $H \in \mathcal{S}(G')$  and  $C_0(Y) \in \mathcal{A}(H)$  we define

$$\mathcal{F}_H^*(C_0(Y)) := \lim_X \text{KK}_*^{G \times H}(C_0(X \times Y), A) \quad \text{and} \quad \mathcal{G}_H^*(C_0(Y)) := \text{KK}_*^H(C_0(Y), A \rtimes_r G),$$

where  $X$  runs through the  $G$ -compact subsets of  $\mathcal{E}(G)$ . The same references as used in the previous lemma show that  $\mathcal{F}^* = (\mathcal{F}_L^*)_{L \in \mathcal{S}(G)}$  and  $\mathcal{G}^* = (\mathcal{G}_L^*)_{L \in \mathcal{S}(G)}$  are functors which satisfy the cohomology and the induction axioms of Definition 1.1. It is clear that the functor  $\mathcal{G}^*$  satisfies the restriction axiom and the restriction axiom for  $\mathcal{F}^*$  follows from Lemma 2.5.

For any proper  $H$ -space  $Y$ , consider the map

$$\mu_{\text{part}, H}(Y) : \mathcal{F}_H^*(C_0(Y)) \rightarrow \mathcal{G}_H^*(C_0(Y)),$$

which is given by taking limits over  $X$  of the partial assembly maps

$$\mu_{X, Y} : \text{KK}_*^{G \times H}(C_0(X \times Y), A) \rightarrow \text{KK}_*^G(C_0(Y), A \rtimes_r G)$$

as defined in (2.1). One can check that  $(\mu_{\text{part}, H})_{H \in \mathcal{S}(G')}$  is a natural transformation from  $\mathcal{F}^*$  to  $\mathcal{G}^*$  which satisfies the conditions of Definition 1.3 (the necessary computations have been performed in a much more general setting in [8, §5] — the proof in the present case of direct

products is much easier, as can be seen in the case of discrete groups in [28]). So by Theorem 1.4 it suffices to show that

$$\mu_{part,K} : \mathcal{F}_K^*(C_0(V)) \rightarrow \mathcal{G}_K^*(C_0(V)),$$

is an isomorphism for all compact subgroups  $K$  of  $G'$  and for all finite dimensional real linear  $K$ -spaces  $V$ . But for any  $G$ -compact subset  $X$  of  $\mathcal{E}(G)$  we can use Kasparov's Bott-periodicity theorem [22, Theorem 7, §5] together with [7, 7.7 and proof of 7.6] to obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{KK}_*^{G \times K}(C_0(X \times V), A) & \xrightarrow{\cong} & \mathrm{KK}_*^{G \times K}(C_0(X), A \otimes C_0(V)) \\ \mu_{X,V} \downarrow & & \downarrow \mu_{X,pt} \\ \mathrm{KK}_*^K(C_0(V), A \rtimes_r G) & \xrightarrow[\cong]{} & \mathrm{KK}_*^K(\mathbb{C}, (A \rtimes_r G) \otimes C_0(V)). \end{array}$$

The Green-Julg theorem implies  $\mathrm{KK}_*^K(\mathbb{C}, (A \rtimes_r G) \otimes C_0(V)) \cong K_*((A \otimes C_0(V)) \rtimes_r (G \times K))$ . Using this identification and taking the limit over  $X$ , the right vertical arrow becomes the Baum-Connes assembly map for the  $G \times K$ -algebra  $A \otimes C_0(V)$ , which is an isomorphism by assumption. Hence, we also get an isomorphism for the left vertical arrow after passing to the limit over  $X$ .  $\square$

It remains to show that condition (A) in Theorem 2.1 implies the assumptions of Proposition 2.6 in the special case where  $G' = G/N$  and  $A = C_0(G/N, B)$ . We need

**Lemma 2.7.** *Let  $A$  be a  $G$ -algebra for some locally compact group  $G$ , and let  $0 \rightarrow J \rightarrow B \rightarrow B/J \rightarrow 0$  be a short exact sequence of nuclear  $K$ -algebras for some exact locally compact group  $K$ . Then, if  $G \times K$  satisfies the Baum-Connes conjecture for two of the algebras in the short exact sequence*

$$0 \rightarrow A \otimes J \rightarrow A \otimes B \rightarrow A \otimes B/J \rightarrow 0,$$

*then  $G \times K$  also satisfies the conjecture for the third algebra.*

*Proof.* The result follows from the arguments given in [8, §4] by observing that the sequence

$$0 \rightarrow (A \otimes J) \rtimes_r (G \times K) \rightarrow (A \otimes B) \rtimes_r (G \times K) \rightarrow (A \otimes B/J) \rtimes_r (G \times K) \rightarrow 0$$

is exact. Indeed, exactness of this sequence follows from the canonical isomorphisms

$$(A \otimes D) \rtimes_r (G \times K) \cong ((A \otimes D) \rtimes_r G) \rtimes_r K \cong ((A \rtimes_r G) \otimes D) \rtimes_r K,$$

which hold for any  $K$ -algebra  $D$ , the exactness of  $K$ , and the exactness of the sequence

$$0 \rightarrow (A \rtimes_r G) \otimes J \rightarrow (A \rtimes_r G) \otimes B \rightarrow (A \rtimes_r G) \otimes B/J \rightarrow 0,$$

which follows from the nuclearity of  $B$ ,  $J$  and  $B/J$ , and exactness of the maximal tensor product.  $\square$

**Lemma 2.8.** *Let  $G$  be a locally compact group and let  $K$  be a compact group acting linearly on the finite dimensional real vector space  $V$ . Assume further that  $A$  is a  $G \times K$ -algebra such that for every closed subgroup  $K'$  of  $K$  the group  $G \times K'$  satisfies BC for  $A$ . Then  $G \times K$  satisfies BC for  $A \otimes C_0(V)$ .*

*Proof.* We first recall some basic facts about linear actions of compact groups. If  $V$  is as above, there is a stratification

$$\emptyset = U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots \subseteq U_l = V$$

of  $K$ -invariant open subsets  $U_i$  of  $V$  such that the difference sets  $M_i := U_i \setminus U_{i-1}$  are the different orbit types for the action of  $K$  on  $V$  (i.e., the stabilizers for the action of  $K$  on  $M_i$  for a fixed  $i \in \{1, \dots, l\}$  are all conjugated to a fixed subgroup  $K_i$  of  $K$ ). By an iterated use of Lemma 2.7 we see that  $G \times K$  satisfies BC for  $A \otimes C_0(V)$  if, for all  $i \in \{1, \dots, l\}$ , it satisfies BC for  $A \otimes C_0(M_i)$ . So let  $M$  denote one of the sets  $M_i$  and let  $K' \subseteq K$  denote a representative for the stabilizers of  $M$ . Then the following are known (eg., see [5, p. 309, Corollary 2.5]):

- (i) The set  $M$  and the quotient  $K \backslash M$  are manifolds.
- (ii) For each  $x \in M$  there exists an open  $K$ -invariant neighborhood  $U$  of  $x$  in  $M$  such that  $U$  is  $K$ -homeomorphic to  $q(U) \times K/K'$ , where  $q : M \rightarrow K \backslash M$  denotes the quotient map.

Using these facts we can find a triangulation of  $K \backslash M$  such that each simplex  $\sigma$  trivializes the  $K/K'$ -fibre bundle  $M$ , i.e.  $q^{-1}(\sigma)$  is  $K$ -homeomorphic to  $\sigma \times K/K'$ . For any integer  $k$ , let  $M^k$  denote the inverse image under  $q$  of the  $k$ -skeleton of our triangulation. We show by induction that, for all  $k$ ,  $G \times K$  satisfies BC for  $A \otimes C_0(M^k)$ . Since  $M$  is finite dimensional, this will complete the proof. If  $k = 0$ , it follows from the continuity of K-theory and topological K-theory (e.g., see [8, §7]) that it is enough to show that  $G \times K$  satisfies BC for  $A \otimes C_0(q^{-1}(\{x\}))$  for all  $x$  in  $K \backslash M$ . Since  $q^{-1}(\{x\}) \cong K/K'$  as a  $K$ -space, we have identifications  $A \otimes C_0(q^{-1}(\{x\})) \cong A \otimes C_0(K/K') \cong \text{Ind}_{G \times K'}^{G \times K} A$  as  $G \times K$ -algebra. Since, by assumption,  $G \times K'$  satisfies BC for  $A$ , it follows then from [8, Theorem 2.5] that  $G \times K$  satisfies the BC for  $A \otimes C_0(q^{-1}(\{x\}))$  and thus for  $A \otimes C_0(M^0)$ .

For the step  $k \rightarrow k + 1$  assume that  $G \times K$  satisfies BC for  $A \otimes C_0(M^k)$ . Then Lemma 2.7 implies that  $G \times K$  satisfies BC for  $A \otimes C_0(M^{k+1})$  if and only if  $G \times K$  satisfies BC for  $A \otimes C_0(M^{k+1} \setminus M^k)$ . The set  $M^{k+1} \setminus M^k$  is the inverse image under  $q$  of the disjoint union of the interiors of all  $k + 1$ -dimensional simplices. Using continuity of K-theory and topological K-theory again, it is enough to prove that  $G \times K$  satisfies BC for  $A \otimes C_0(q^{-1}(\overset{\circ}{\sigma}))$  for every  $k + 1$ -dimensional simplex  $\sigma$ . Since  $\overset{\circ}{\sigma}$  trivialises the  $K/K'$ -fibre bundle  $M$ , we have  $A \otimes C_0(q^{-1}(\overset{\circ}{\sigma})) \cong A \otimes C_0(\overset{\circ}{\sigma} \times K/K') \cong \text{Ind}_{G \times K'}^{G \times K} (A \otimes C_0(\overset{\circ}{\sigma}))$ . Since  $\overset{\circ}{\sigma}$  is homeomorphic to an open ball in  $\mathbb{R}^{n+1}$ , it follows from Bott-periodicity that  $G \times K'$  satisfies BC for  $A \otimes C_0(\overset{\circ}{\sigma})$ . Hence, [8, Theorem 2.5] implies that  $G \times K$  satisfies BC for  $A \otimes C_0(q^{-1}(\overset{\circ}{\sigma}))$ , and thus for  $A \otimes C_0(M^{k+1} \setminus M^k)$ .  $\square$

The following corollary is now a direct consequence of Proposition 2.6 and Lemma 2.8.

**Corollary 2.9.** *Suppose that  $A$  is a  $G \times G'$ -algebra such that for all compact subgroups  $K'$  of  $G'$  the group  $G \times K'$  satisfies the Baum-Connes conjecture for  $A$ . Then the partial assembly map  $\mu_{part, G'} : K_*^{\text{top}}(G \times G'; A) \rightarrow K_*^{\text{top}}(G'; A \rtimes_r G)$  is an isomorphism.*

We are now ready to complete the

*Proof of Theorem 2.1.* By Proposition 2.3 and Corollary 2.9 it suffices to show that condition (A) of the theorem implies that  $G \times \dot{L}$  satisfies BC for  $A = C_0(G/N, B)$  for all compact

subgroups  $\dot{L}$  of  $G/N$ . Let  $L \subseteq G$  denote the inverse image of  $\dot{L}$  in  $G$ . Then, as in the proof of Proposition 2.3 for the case  $L = G/N$ , we see that  $C_0(G/N, B) \cong \text{Ind}_{\Delta(L)}^{G \times \dot{L}} B$ , with respect to the obvious action of  $\Delta(L) = \{(l, lN) : l \in L\} \cong L$  on  $B$ . By condition (A),  $L$  satisfies BC for  $B$ , and then  $G \times \dot{L}$  satisfies BC for  $C_0(G/N, B)$  by [8, Theorem 2.5].  $\square$

For direct products of groups, we get a much better result:

**Corollary 2.10.** *Let  $G, G'$  be two locally compact groups and assume that  $G$  satisfies BC for  $\mathbb{C}$ . Then the partial assembly map  $\mu_{part, G'} : K_*^{\text{top}}(G \times G'; \mathbb{C}) \rightarrow K_*^{\text{top}}(G', C_r^*(G))$  is an isomorphism. As a consequence,  $G \times G'$  satisfies BC for  $\mathbb{C}$  if and only if  $G'$  satisfies BC for  $C_r^*(G)$ .*

*Proof.* To prove bijectivity of the partial assembly map, by Corollary 2.9, it suffices to show that  $G \times K'$  satisfies BC for  $\mathbb{C}$  whenever  $K'$  is a compact subgroup of  $G'$ . But by Theorem 2.1, applied to the normal subgroup  $K'$  of  $G \times K'$ , this is true whenever  $K \times K'$  satisfies BC for  $\mathbb{C}$  for any compact subgroup  $K \subseteq G$ , which follows from the Green-Julg theorem, and if  $G$  satisfies BC for  $C^*(K')$  with respect to the trivial action, which follows from the fact that  $G$  satisfies BC for any matrix algebra with trivial  $G$ -action and that  $C^*(K')$  is a countable direct sum of matrix algebras. The second assertion now follows from [6, Proposition 3.10].  $\square$

### 3. THE ABSTRACT KÜNNETH THEOREM

In this section we want to derive an abstract Künneth theorem which, when specialized to  $C^*$ -algebras, gives the Künneth theorem of Schochet ([32]), but which applies also to other functors, like the topological K-theory of a group with coefficients in certain  $C^*$ -algebras. Note that all arguments used below are already given in [32] and [2] in the special case of the Künneth theorem for  $C^*$ -algebras. However, the conclusions we draw here are more general. In what follows we shall denote by  $\mathcal{S}$  the category of separable  $C^*$ -algebras with  $*$ -homomorphisms as morphisms, and we write  $\mathcal{C}$  for a given sub-category of  $\mathcal{S}$  which satisfies the following basic requirements:

- (C1)  $\mathcal{C}$  contains all separable commutative  $C^*$ -algebras.
- (C2)  $\mathcal{C}$  is closed under stabilization and under suspension, i.e., if  $B \in \mathcal{C}$ , then  $\mathcal{K} \otimes B$  and  $SB := C_0(\mathbb{R}) \otimes B$  are in  $\mathcal{C}$ .
- (C3) If  $0 \rightarrow J \rightarrow B \xrightarrow{q} B/J \rightarrow 0$  is a semi-split short exact sequence in  $\mathcal{C}$  (i.e., there exists a completely positive contractive cross section for  $q : B \rightarrow B/J$ ) such that two of the algebras are in  $\mathcal{C}$ , then so is the third.

As for  $\mathcal{S}$ , the morphisms in  $\mathcal{C}$  are the  $*$ -homomorphisms. For instance, one could take the category of all separable nuclear  $C^*$ -algebras for  $\mathcal{C}$ , but we shall also apply the results of this section to other sub-categories of  $\mathcal{S}$  such as the category  $\mathcal{N}$  as defined in the introduction. Note that it follows directly from the above conditions that  $\mathcal{C}$  contains the cone  $CB := C_0((0, 1], B)$  and the algebra  $C([0, 1], B)$  for all  $B \in \mathcal{C}$ .

**Definition 3.1.** Let  $\mathcal{C}$  be as above. A *Künneth functor* on  $\mathcal{C}$  is an additive covariant  $\mathbb{Z}/2\mathbb{Z}$ -graded functor  $F_* : \mathcal{C} \rightarrow \mathbf{Ab}$  from  $\mathcal{C}$  into the category  $\mathbf{Ab}$  of abelian groups, which satisfies the following set of axioms:

- (K1)  $F_*$  is stable and homotopy-invariant.

- (K2) If  $0 \rightarrow J \rightarrow B \rightarrow B/J \rightarrow 0$  is a semi-split short exact sequence in  $\mathcal{C}$ , then the sequence  $F_*(J) \rightarrow F_*(B) \rightarrow F_*(B/J)$  is exact.
- (K3) (Suspension) For all  $B \in \mathcal{C}$  there are natural isomorphisms  $F_i(SB) \cong F_{i+1}(B)$ .
- (K4) There exists a natural zero-graded homomorphism  $\alpha : F_*(\mathbb{C}) \otimes K_*(B) \rightarrow F_*(B)$ , for every  $B \in \mathcal{C}$ , such that  $\alpha$  is an isomorphism whenever  $K_*(B)$  is free abelian.

By [22, Lemmas 4 and 5 in §7] (see also [34, Lemma 1.3] and [2, Lemma 19.5.3 and Theorem 21.4.3]), it follows from conditions (K1) – (K3) that every semi-split short exact sequence  $0 \rightarrow J \rightarrow B \rightarrow B/J \rightarrow 0$  in  $\mathcal{C}$  gives rise to a six-term exact sequence

$$(3.1) \quad \begin{array}{ccccc} F_0(J) & \longrightarrow & F_0(B) & \longrightarrow & F_0(B/J) \\ \partial \uparrow & & & & \downarrow \partial \\ F_1(B/J) & \longleftarrow & F_1(B) & \longleftarrow & F_1(J). \end{array}$$

Note that the proof of that result only uses constructions which by (C1) – (C3) stay inside our category  $\mathcal{C}$ , so we can use the result here.

Given any Künneth functor  $F_*$ , we shall show that the constructions of [32] give natural homomorphisms  $\beta : F_*(B) \rightarrow \text{Tor}(F_*(\mathbb{C}), K_*(B))$  of grade one such that for all  $B \in \mathcal{C}$  the sequence

$$(3.2) \quad 0 \longrightarrow F_*(\mathbb{C}) \otimes K_*(B) \xrightarrow{\alpha} F_*(B) \xrightarrow{\beta} \text{Tor}(F_*(\mathbb{C}), K_*(B)) \longrightarrow 0$$

is exact. The main ingredient for the construction of  $\beta$  and the proof of the exactness of (3.2) is Schochet's construction of the geometrical resolution for  $B$  [32, Lemma 3.1]: the construction given there for a unital separable  $C^*$ -algebra  $B$  can be extended to any separable algebra as explained in [2, Proposition 23.5.1] to provide a separable commutative  $C^*$ -algebra  $R$  and a morphism  $\mu : R \rightarrow B \otimes \mathcal{K}$  such that  $K_*(R)$  is free abelian and the induced map  $\mu_* : K_*(R) \rightarrow K_*(B \otimes \mathcal{K}) \cong K_*(B)$  is surjective. If we denote by  $C_\mu$  the mapping cone of  $\mu$ , i.e.

$$C_\mu = \{(h, x) \in C(B \otimes \mathcal{K}) \oplus R : h(1) = \mu(x)\},$$

where  $C(B \otimes \mathcal{K}) := C_0([0, 1], B \otimes \mathcal{K})$  denotes the cone over  $B \otimes \mathcal{K}$ , then the semi-split short exact sequence

$$(3.3) \quad 0 \rightarrow S(B \otimes \mathcal{K}) \rightarrow C_\mu \rightarrow R \rightarrow 0$$

is called a *geometrical resolution for  $B$* . Since  $R$  is commutative, it follows from (C1) – (C3) together with (3.3) that  $R$  and  $C_\mu$  are in  $\mathcal{C}$  if  $B \in \mathcal{C}$ .

By (3.1), (K1) and (K3) we obtain a six-term exact sequence

$$(3.4) \quad \begin{array}{ccccc} F_1(B) & \longrightarrow & F_0(C_\mu) & \longrightarrow & F_0(R) \\ \partial \uparrow & & & & \downarrow \partial \\ F_1(R) & \longleftarrow & F_1(C_\mu) & \longleftarrow & F_0(B) \end{array}$$

for every Künneth functor  $F$ . Since the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & S(B \otimes \mathcal{K}) & \longrightarrow & C_\mu & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \mu & & \\ 0 & \longrightarrow & S(B \otimes \mathcal{K}) & \longrightarrow & C(B \otimes \mathcal{K}) & \longrightarrow & B \otimes \mathcal{K} & \longrightarrow & 0 \end{array}$$

is commutative, the naturality of  $F$  and the construction of the boundary  $\partial$  in [2, Theorem 21.4.3] implies that the diagram

$$\begin{array}{ccccccc} F_i(C_\mu) & \longrightarrow & F_i(R) & \xrightarrow{\partial} & F_{i-1}(S(B \otimes \mathcal{K})) & \longrightarrow & F_{i-1}(C_\mu) \\ \downarrow & & \downarrow^{\mu_*} & & \downarrow^{\cong} & & \downarrow \\ F_i(C(B \otimes \mathcal{K})) & \longrightarrow & F_i(B \otimes \mathcal{K}) & \longrightarrow & F_{i-1}(S(B \otimes \mathcal{K})) & \longrightarrow & F_{i-1}(C(B \otimes \mathcal{K})) \end{array}$$

commutes. Since  $F_*$  is homotopy invariant and  $C(B \otimes \mathcal{K})$  is contractible, we have  $F_i(C(B \otimes \mathcal{K})) = \{0\}$ , and hence the diagram implies:

**Lemma 3.2.** *The boundary map  $\partial : F_i(R) \rightarrow F_i(B) (\cong F_{i-1}(S(B \otimes \mathcal{K})))$  of (3.4) coincides with the map  $\mu_* : F_i(R) \rightarrow F_i(B)$  induced by the  $*$ -homomorphism  $\mu : R \rightarrow B \otimes \mathcal{K}$ .*

Applying this lemma to the K-theory functor (which clearly is a Künneth functor) and using the fact that  $\mu_* : K_*(R) \rightarrow K_*(B)$  is surjective, (3.4) provides a free resolution

$$(3.5) \quad 0 \rightarrow K_*(C_\mu) \rightarrow K_*(R) \rightarrow K_*(B) \rightarrow 0$$

of  $K_*(B)$ . Hence, if  $F$  is a Künneth functor, it follows from the definition of the Tor-group, that the geometrical resolution for  $B$  provides an exact sequence

$$0 \rightarrow \mathrm{Tor}(F_*(\mathbb{C}), K_*(B)) \rightarrow F_*(\mathbb{C}) \otimes K_*(C_\mu) \rightarrow F_*(\mathbb{C}) \otimes K_*(R).$$

By (K4) we have natural isomorphisms  $F_*(C_\mu) \cong F_*(\mathbb{C}) \otimes K_*(C_\mu)$  and  $F_*(R) \cong F_*(\mathbb{C}) \otimes K_*(R)$  and the inclusion  $\mathrm{Tor}(F_*(\mathbb{C}), K_*(B)) \hookrightarrow F_*(\mathbb{C}) \otimes K_*(C_\mu)$  induces an isomorphism

$$\mathrm{Tor}(F_*(\mathbb{C}), K_*(B)) \cong [\ker : F_*(C_\mu) \rightarrow F_*(R)].$$

Applying  $F_*$  to the geometric resolution (3.3) for  $B$ , we obtain a natural grade-one map

$$(3.6) \quad \beta : F_*(B) \rightarrow [\ker : F_*(C_\mu) \rightarrow F_*(R)] \cong \mathrm{Tor}(F_*(\mathbb{C}), K_*(B)),$$

which by (K2) is always surjective.

**Theorem 3.3.** *Assume that  $F_* : \mathcal{C} \rightarrow \mathbf{Ab}$  is a Künneth functor. Then the Künneth sequence (3.2) is exact.*

*Proof.* We already observed that  $\beta : F_*(B) \rightarrow \mathrm{Tor}(F_*(\mathbb{C}), K_*(B))$  is surjective. To prove injectivity of  $\alpha$ , consider the commutative diagram

$$(3.7) \quad \begin{array}{ccccccc} F_*(\mathbb{C}) \otimes K_*(C_\mu) & \longrightarrow & F_*(\mathbb{C}) \otimes K_*(R) & \xrightarrow{\mathrm{id} \otimes \mu_*} & F_*(\mathbb{C}) \otimes K_*(B) & \longrightarrow & 0 \\ \cong \downarrow & & \cong \downarrow & & \downarrow \alpha & & \\ F_*(C_\mu) & \longrightarrow & F_*(R) & \xrightarrow{\mu_*} & F_*(B) & & \end{array}$$

It follows from (3.4) and Lemma 3.2 that the horizontal rows are exact, and injectivity of  $\alpha$  follows from a simple diagram chase.

To see that the sequence (3.2) is exact in the middle term, recall that we identified  $\mathrm{Tor}(F_*(\mathbb{C}), K_*(B))$  with a subgroup of  $F_*(C_\mu)$ . It follows from (3.6) that the composition  $\beta \circ \alpha$  is then given by the composition  $F_*(\mathbb{C}) \otimes K_*(B) \rightarrow F_*(B) \rightarrow F_*(C_\mu)$ . To check that  $\ker \beta = \mathrm{Im} \alpha$ , we observe from (3.4) that  $\ker \beta = \mu_*(F_*(R)) \subseteq F_*(B)$ . Hence, commutativity and exactness of the upper horizontal row of the diagram (3.7) implies that  $\ker \beta = \mu_*(F_*(R)) = \mathrm{Im} \alpha$ .  $\square$

4. THE CLASSES  $\mathcal{N}$  AND  $\mathcal{N}_G$ 

If  $A$  and  $B$  are  $C^*$ -algebras, then the map  $\alpha : K_*(A) \otimes K_*(B) \rightarrow K_*(A \otimes B)$ , (which is natural in  $A$  and  $B$ ) can most easily be described by using the external Kasparov product

$$K_*(A) \otimes K_*(B) = \text{KK}_*(\mathbb{C}, A) \otimes \text{KK}_*(\mathbb{C}, B) \xrightarrow{\otimes_{\mathbb{C}}} \text{KK}_*(\mathbb{C}, A \otimes B).$$

Recall that the Kasparov-product over  $\mathbb{C}$  is given via the composition

$$\text{KK}_*(\mathbb{C}, A) \otimes \text{KK}_*(\mathbb{C}, B) \xrightarrow{\text{id} \otimes \sigma_A} \text{KK}_*(\mathbb{C}, A) \otimes \text{KK}_*(A, A \otimes B) \xrightarrow{\otimes_A} \text{KK}_*(\mathbb{C}, A \otimes B),$$

where  $\sigma_A : \text{KK}_*(\mathbb{C}, B) \rightarrow \text{KK}_*(A, A \otimes B)$  is Kasparov's external tensor product in KK-theory and  $\otimes_A$  denotes the usual Kasparov product over  $A$ . Let  $\mathcal{N}$  denote the class of all separable  $C^*$ -algebras  $A$  such that  $\alpha : K_*(A) \otimes K_*(B) \rightarrow K_*(A \otimes B)$  is an isomorphism for all separable  $C^*$ -algebras  $B$  such that  $K_*(B)$  is free abelian. It is clear that, if  $A \in \mathcal{N}$ , then  $F_*(B) := K_*(A \otimes B)$  is a functor on the full category  $\mathcal{S}$  of separable  $C^*$ -algebras which satisfies conditions (K1), (K3) and (K4) of Definition 3.1 and condition (K2) follows from the following easy lemma:

**Lemma 4.1.** *If  $0 \rightarrow J \rightarrow B \rightarrow B/J \rightarrow 0$  is a semi-split short exact sequence of  $C^*$ -algebras, the same is true for  $0 \rightarrow A \otimes J \rightarrow A \otimes B \rightarrow A \otimes B/J \rightarrow 0$ , for every  $C^*$ -algebra  $A$ .*

*Proof.* If  $\sigma : B/J \rightarrow B$  denotes the completely positive contractive cross-section for the quotient map  $q : B \rightarrow B/J$ , just check that  $(\text{id}_{A \otimes B} - \text{id}_A \otimes \sigma \circ q)(A \otimes B) = A \otimes J$ .  $\square$

Thus, the abstract Künneth theorem (Theorem 3.3) implies:

**Proposition 4.2** (cf [32] and [2, 23.1.3]). *If  $A \in \mathcal{N}$  then, for all  $C^*$ -algebras  $B$ , the Künneth sequence*

$$0 \longrightarrow K_*(A) \otimes K_*(B) \xrightarrow{\alpha} K_*(A \otimes B) \xrightarrow{\beta} \text{Tor}(K_*(A), K_*(B)) \longrightarrow 0$$

*is exact.*

**Remark 4.3.** (1) Georges Skandalis pointed out to us that any algebra  $A$  which is in  $\mathcal{N}$  is automatically K-exact, i.e., the functor  $B \rightarrow K_*(A \otimes B)$  is half-exact. To see this, recall from [20, p. 335–336] that for any sequence of  $C^*$ -algebras  $s : D \rightarrow E \rightarrow F$  (with zero composition), there is a double mapping cone  $C(s)$  such that  $K_*(C(s)) = 0$  implies that the sequence  $K_*(D) \rightarrow K_*(E) \rightarrow K_*(F)$  is exact. Since for any  $*$ -homomorphism  $\phi : E \rightarrow D$ , the mapping cone  $C_{\text{id}_A \otimes \phi}$  is equal to  $A \otimes C_\phi$  it follows that  $C(A \otimes s) = A \otimes C(s)$ .

Now, if  $A$  is not K-exact, there exists a short exact sequence of  $C^*$ -algebras  $s : 0 \rightarrow J \rightarrow B \rightarrow B/J \rightarrow 0$  such that  $K_*(A \otimes J) \rightarrow K_*(A \otimes B) \rightarrow K_*(A \otimes B/J)$  fails to be exact. Then  $K_*(C(s)) = 0$  (see [20]) and  $K_*(A \otimes C(s)) \neq 0$  which clearly shows that  $A \notin \mathcal{N}$ .

An example of a non-separable  $C^*$ -algebra  $A$  which is not K-exact is given by  $A = \prod_{i=1}^{\infty} M_n(\mathbb{C})$  (e.g., [29, Theorem A1]). By writing  $A$  as an inductive limit of separable subalgebras  $A_i$  one also gets separable  $C^*$ -algebras which are not K-exact, and hence do not satisfy the Künneth formula. It is still an open question whether all nuclear  $C^*$ -algebras are in  $\mathcal{N}$ .

(2) Of course, the result of Proposition 4.2 is symmetric in  $A$  and  $B$ , so a similar statement can be given if  $B \in \mathcal{N}$  and  $A$  is arbitrary. In this reversed situation the construction of the homomorphism  $\beta$  should use a geometrical resolution of the  $C^*$ -algebra  $A$ . However, it

follows from Lemma 4.4 below that the class  $\mathcal{N}$  satisfies conditions (C1) – (C3), and hence the functor  $F_* : \mathcal{N} \rightarrow \mathbf{Ab}$ ;  $F_*(B) := K_*(A \otimes B)$  is a Künneth functor on  $\mathcal{N}$  for every fixed  $C^*$ -algebra  $A$ . Hence we also get exactness of the Künneth sequence by using the geometrical resolution for  $B$ . This observation is used in the proof of Proposition 4.10 below.

Recall from [2, Definition 22.3.4] that the bootstrap class  $\mathcal{B}$  is the smallest class of separable nuclear  $C^*$ -algebras such that  $\mathcal{B}$  contains  $\mathbb{C}$ ,  $\mathcal{B}$  is closed under countable inductive limits and under KK-equivalences, and if  $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$  is a short exact sequence of  $C^*$ -algebras such that two of the algebras are in  $\mathcal{B}$ , then so is the third. It is shown in [2, 22.3.5] that  $\mathcal{B}$  is quite large and that it contains all type I algebras. We now state similar properties of the class  $\mathcal{N}$  — some of them have already been stated in [2, 23.4].

**Lemma 4.4.** *The class  $\mathcal{N}$  contains the bootstrap class  $\mathcal{B}$  and enjoys the following properties:*

- (i) *If  $A \in \mathcal{N}$  and  $B$  is KK-dominated by  $A$  (see [2, 23.10.6]), then  $B \in \mathcal{N}$ ; in particular,  $\mathcal{N}$  is stable under KK-equivalence and hence it is stable under Morita equivalence.*
- (ii) *If  $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$  is a semi-split short exact sequence of  $C^*$ -algebras such that two of them are in  $\mathcal{N}$ , then so is the third.*
- (iii) *If  $A, B \in \mathcal{N}$ , then  $A \otimes B \in \mathcal{N}$ .*
- (iv) *If  $A = \lim A_i$ , such that all structure maps are injective and all  $A_i \in \mathcal{N}$ , then  $A \in \mathcal{N}$ .*

*Proof.* First of all  $\mathcal{N}$  contains the bootstrap class  $\mathcal{B}$  by [2, 23.4]. To see (i) let  $D$  be any  $C^*$ -algebra with free K-theory group. Let  $x \in \text{KK}_0(A, B)$  and  $y \in \text{KK}_0(B, A)$  such that  $y \otimes x = 1 \in \text{KK}_0(B, B)$  and consider the commutative diagram:

$$\begin{array}{ccccc} K_*(B) \otimes K_*(D) & \xrightarrow{\otimes y} & K_*(A) \otimes K_*(D) & \xrightarrow{\otimes x} & K_*(B) \otimes K_*(D) \\ \alpha \downarrow & & \downarrow \alpha & & \downarrow \alpha \\ K_*(B \otimes D) & \xrightarrow{\otimes \sigma_D(y)} & K_*(A \otimes D) & \xrightarrow{\otimes \sigma_D(x)} & K_*(B \otimes D). \end{array}$$

Since the compositions of the horizontal arrows are the identities and the middle vertical arrow is an isomorphism, it follows from an easy diagram chase that the other vertical arrows are isomorphisms, too.

Statement (ii) is an easy consequence of Lemma 4.1 and the Five Lemma. For (iii) recall from §3 that the geometric resolution for  $K_*(B)$ :

$$(4.1) \quad 0 \rightarrow K_*(C_\mu) \rightarrow K_*(R) \rightarrow K_*(B) \rightarrow 0$$

provides a realization of  $\text{Tor}(K_*(A), K_*(B))$  as  $\ker [K_*(A) \otimes K_*(C_\mu) \rightarrow K_*(A) \otimes K_*(R)]$ , where we use the notations of (3.3). Tensoring (4.1) with the free abelian group  $K_*(D)$ , we obtain a free resolution  $0 \rightarrow K_*(C_\mu) \otimes K_*(D) \rightarrow K_*(R) \otimes K_*(D) \rightarrow K_*(B) \otimes K_*(D) \rightarrow 0$  of  $K_*(B) \otimes K_*(D)$ . Since  $K_*(D)$  is free abelian, we then obtain canonical identifications

$$\begin{aligned} \text{Tor}(K_*(A), K_*(B) \otimes K_*(D)) &\cong \ker [K_*(A) \otimes K_*(C_\mu) \otimes K_*(D) \rightarrow K_*(A) \otimes K_*(R) \otimes K_*(D)] \\ &\cong \left( \ker [K_*(A) \otimes K_*(C_\mu) \rightarrow K_*(A) \otimes K_*(R)] \right) \otimes K_*(D) \\ &\cong \text{Tor}(K_*(A), K_*(B)) \otimes K_*(D). \end{aligned}$$

Now consider the following diagram:

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 K_*(A) \otimes K_*(B) \otimes K_*(D) & \xrightarrow{\text{id} \otimes \alpha_{B,D}} & K_*(A) \otimes K_*(B \otimes D) \\
 \downarrow & & \downarrow \\
 K_*(A \otimes B) \otimes K_*(D) & \xrightarrow{\alpha_{A \otimes B, D}} & K_*(A \otimes B \otimes D) \\
 \downarrow & & \downarrow \\
 \text{Tor}(K_*(A), K_*(B) \otimes K_*(D)) & \xrightarrow{\text{Tor}(\text{id}, \alpha_{B,D})} & \text{Tor}(K_*(A), K_*(B \otimes D)) \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

The first vertical column is the Künneth sequence for  $(A, B)$  tensored with the free abelian group  $K_*(D)$ , and hence is exact by Proposition 4.2 since  $A \in \mathcal{N}$ . The second vertical column is the Künneth sequence for the pair  $(A, B \otimes D)$ , and hence it is exact, too. The top and the bottom horizontal rows are isomorphisms because  $B \in \mathcal{N}$ . Hence the middle row is an isomorphism, too, which proves that  $A \otimes B$  belongs to  $\mathcal{N}$ .

Item (iv) follows from the continuity of K-theory.  $\square$

We are now going to apply the abstract Künneth theorem to the topological K-theory of a locally compact group  $G$  with appropriate coefficients. If  $A$  is a separable  $G$ -algebra and if  $B$  is any separable  $C^*$ -algebra equipped with the trivial  $G$ -action, then the trivial homomorphism  $G \rightarrow \{e\}$  induces a “inflation map”  $\text{inf} : K_*(B) = \text{KK}_*(\mathbb{C}, B) \rightarrow \text{KK}_*^G(\mathbb{C}, B)$ . Thus, if  $X$  is a locally compact proper  $G$ -space, we obtain a natural map

$$(4.2) \quad \alpha_X : \text{KK}_*^G(C_0(X), A) \otimes K_*(B) \rightarrow \text{KK}_*^G(C_0(X), A \otimes B),$$

which is defined by the composition

$$\text{KK}_*^G(C_0(X), A) \otimes K_*(B) \xrightarrow{\text{id} \otimes \text{inf}} \text{KK}_*^G(C_0(X), A) \otimes \text{KK}_*^G(\mathbb{C}, B) \xrightarrow{\otimes_{\mathbb{C}}} \text{KK}_*^G(C_0(X), A \otimes B),$$

where the second map is given by taking Kasparov-product over  $\mathbb{C}$  (compare with the definition of  $\alpha$  as given above). It is straightforward to check that the  $\alpha_{\bullet}$  are compatible with inclusion of  $G$ -compact spaces. Consequently, if  $A$  is a  $G$ -algebra and  $B$  is a  $C^*$ -algebra, then we have a well defined morphism  $\alpha_G : K_*^{\text{top}}(G; A) \otimes K_*(B) \rightarrow K_*^{\text{top}}(G; A \otimes B)$ .

**Definition 4.5.** We denote by  $\mathcal{N}_G$  the class of all  $G$ -algebras  $A$  such that  $\alpha_G$  is an isomorphism for all  $B$  with  $K_*(B)$  free abelian.

**Proposition 4.6.** *Let  $A$  be a  $G$ -algebra. Then  $A \in \mathcal{N}_G$  if and only if  $A$  satisfies the  $G$ -equivariant Künneth formula, i.e., for every  $C^*$ -algebra  $B$ , there exists a canonical homomorphism  $\beta_G : K_*^{\text{top}}(G; A \otimes B) \rightarrow \text{Tor}(K_*^{\text{top}}(G; A), K_*(B))$  such that the sequence*

$$(4.3) \quad 0 \longrightarrow K_*^{\text{top}}(G; A) \otimes K_*(B) \xrightarrow{\alpha_G} K_*^{\text{top}}(G; A \otimes B) \xrightarrow{\beta_G} \text{Tor}(K_*^{\text{top}}(G; A), K_*(B)) \longrightarrow 0$$

is exact.

*Proof.* It follows from the general properties of topological K-theory that the functor  $F_* : \mathcal{S} \rightarrow \mathbf{Ab}$ ;  $F_*(B) = K_*^{\text{top}}(G; A \otimes B)$  is a generalized Künneth functor, where condition (K2) of Definition 3.1 follows from half-exactness of topological K-theory ([25, Appendix]) together with Lemma 4.1. Hence the proposition follows from Theorem 3.3.  $\square$

Exactly the same arguments as used in Lemma 4.4 for the class  $\mathcal{N}$ , give

**Lemma 4.7.** *The class  $\mathcal{N}_G$  enjoys the following properties:*

- (i) *If  $A \in \mathcal{N}_G$  and  $B$  is  $\text{KK}^G$ -dominated by  $A$  (see [2, 23.10.6]), then  $B \in \mathcal{N}_G$ ; in particular,  $\mathcal{N}_G$  is stable under  $\text{KK}^G$ -equivalence and hence it is stable under  $G$ -equivariant Morita equivalence.*
- (ii) *If  $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$  is a semi-split short exact sequence of  $G$ -algebras such that two of them are in  $\mathcal{N}_G$ , then so is the third.*
- (iii) *If  $A \in \mathcal{N}_G$  and  $B \in \mathcal{N}$ , then  $A \otimes B \in \mathcal{N}_G$  where  $B$  is equipped with trivial  $G$ -action.*
- (iv) *If  $A = \lim A_i$  is an equivariant limit of  $G$ -algebras, such that all structure maps are injective and all  $A_i \in \mathcal{N}_G$ , then  $A \in \mathcal{N}_G$ .*

**Remark 4.8.** Note that the completely positive cross section in (ii) is not required to be  $G$ -equivariant. Indeed, the semi-split assumption in (ii) is only used to ensure that for all  $C^*$ -algebras  $B$ , the sequence  $0 \rightarrow I \otimes B \rightarrow A \otimes B \rightarrow A/I \otimes B \rightarrow 0$  is exact, which follows from Lemma 4.1. Statement (ii) then follows from half-exactness of topological K-theory.

A priori, it is not clear that for a given group  $G$  the class  $\mathcal{N}_G$  is not empty, but we shall see below that it is indeed fairly large. In fact it contains all type I  $G$ -algebras. Recall from [3] that for any  $G$ -algebra  $A$ , the Baum-Connes assembly map is given on the level of  $G$ -compact subsets  $X \subseteq \mathcal{E}(G)$  via the map  $\mu_{X,A} : \text{KK}_*^G(C_0(X), A) \rightarrow K_*(A \rtimes_r G)$ , which is defined as the composition

$$\text{KK}_*^G(C_0(X), A) \xrightarrow{J_G} \text{KK}_*(C_0(X) \rtimes_r G, A \rtimes_r G) \xrightarrow{[p_X] \otimes \cdot} K_*(A \rtimes_r G),$$

where the first map is Kasparov's descent morphism, and the second map is given by taking the Kasparov product with a certain fundamental class  $[p_X] \in K_0(C_0(X) \rtimes_r G)$ .

**Proposition 4.9.** *Let  $A$  be a  $G$ -algebra and let  $B$  be a  $C^*$ -algebra. Then the diagram*

$$(4.4) \quad \begin{array}{ccc} K_*^{\text{top}}(G; A) \otimes K_*(B) & \xrightarrow{\mu_A \otimes \text{id}} & K_*(A \rtimes_r G) \otimes K_*(B) \\ \alpha_G \downarrow & & \downarrow \alpha \\ K_*^{\text{top}}(G; A \otimes B) & \xrightarrow{\mu_{A \otimes B}} & K_*((A \otimes B) \rtimes_r G) \end{array}$$

*commutes. In particular, if  $G$  satisfies BC with coefficients in  $A \otimes B$ , for all  $C^*$ -algebras  $B$  equipped with the trivial  $G$ -action, then  $A \in \mathcal{N}_G$  if and only if  $A \rtimes_r G \in \mathcal{N}$ .*

*Proof.* Since  $J_G(\sigma_A(x)) = \sigma_{A \rtimes_r G}(x)$ , for all  $x \in K_*(B)$ , where we view  $\sigma_A$  as a map from  $K_*(B)$  to  $\text{KK}_*^G(A, A \otimes B)$ , the commutativity of (4.4) is easily checked on the level of each

$G$ -compact set  $X \subseteq \mathcal{E}(G)$ : Starting with  $y \in \text{KK}_*^G(C_0(X), A)$  and  $x \in \text{K}_*(B)$ , we compute

$$\begin{aligned} \mu_{X, A \otimes B} \circ \alpha_X(y \otimes x) &= [p_X] \otimes_{C_0(X) \rtimes G} J_G(y \otimes \sigma_A(x)) \\ &= [p_X] \otimes_{C_0(X) \rtimes G} (J_G(y) \otimes_{A \rtimes_r G} J_G(\sigma_A(x))) \\ &= ([p_X] \otimes_{C_0(X) \rtimes G} J_G(y)) \otimes_{A \rtimes_r G} \sigma_{A \rtimes_r G}(x) \\ &= \alpha(\mu_{X, A}(y) \otimes x). \end{aligned}$$

The final statement is now obvious.  $\square$

From Propositions 4.6 and 4.9 we deduce:

**Proposition 4.10.** *Suppose that  $A$  is a  $G$ -algebra such that  $G$  satisfies BC for  $A$ . Then the following are true:*

- (i) *If  $A \in \mathcal{N}_G$  and  $A \rtimes_r G \in \mathcal{N}$ , then  $G$  satisfies BC for  $A \otimes B$  for all separable  $C^*$ -algebras  $B$  equipped with the trivial  $G$ -action.*
- (ii) *If  $A \in \mathcal{N}_G$ , then  $G$  satisfies BC for  $A \otimes B$  for all  $B \in \mathcal{N}$  with the trivial  $G$ -action.*

*Proof.* In both situations we consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{K}_*^{\text{top}}(G; A) \otimes \text{K}_*(B) & \xrightarrow{\alpha_G} & \text{K}_*^{\text{top}}(G; A \otimes B) & \xrightarrow{\beta_G} & \text{Tor}(\text{K}_*^{\text{top}}(G; A) \otimes \text{K}_*(B)) & \longrightarrow & 0 \\ & & \mu_A \otimes \text{id} \downarrow & & \mu_{A \otimes B} \downarrow & & \downarrow \text{Tor}(\mu_A, \text{id}) & & \\ 0 & \longrightarrow & \text{K}_*(A \rtimes_r G) \otimes \text{K}_*(B) & \xrightarrow{\alpha} & \text{K}_*((A \rtimes_r G) \otimes B) & \xrightarrow{\beta} & \text{Tor}(\text{K}_*(A \rtimes_r G), \text{K}_*(B)) & \longrightarrow & 0. \end{array}$$

In situation (i), both horizontal rows are exact by Propositions 4.2 and 4.6. Since  $G$  satisfies BC for  $A$ , the left and right vertical arrows are isomorphisms. Hence the Five Lemma implies that the middle vertical arrow is an isomorphism, too.

In situation (ii) the upper horizontal sequence is exact by Proposition 4.6 and the lower horizontal sequence is exact by Proposition 4.2, where by Remark 4.3, in both sequences the maps  $\beta_G$  resp.  $\beta$  can be constructed via a common geometric resolution of  $B$ . This implies that the diagram commutes also in this situation, and the proof then follows again from the Five Lemma.  $\square$

In view of the above results, it is quite useful to find easy conditions which imply that a given  $G$ -algebra is in the class  $\mathcal{N}_G$ . Such conditions are given in Theorem 0.1 as stated in the introduction:

**Theorem 0.1** *Let  $A$  be a  $G$ -algebra such that  $A \rtimes K \in \mathcal{N}$  for all compact subgroups  $K$  of  $G$ . Then  $A \in \mathcal{N}_G$ . In particular, the class  $\mathcal{N}_G$  contains all type I  $G$ -algebras.*

Notice first, that the last assertion follows from the first since all type I  $C^*$ -algebras are in the bootstrap class and since crossed products of type I algebras by compact groups are type I (which follows easily from [35, Theorem 6.1]).

For the proof of the theorem we use the Going-Down technique of §1. Let  $B$  be a fixed  $C^*$ -algebra with free abelian  $\text{K}$ -group  $\text{K}_*(B)$ . Using the notation of §1, for each  $H \in \mathcal{S}(G)$ , we define the additive contravariant functors  $\mathcal{F}_H : \mathcal{A}(H) \rightarrow \mathbf{Ab}$  and  $\mathcal{G}_H : \mathcal{A}(H) \rightarrow \mathbf{Ab}$  by

$$\mathcal{F}_H(C_0(X)) := \text{KK}_*^H(C_0(X), A) \otimes \text{K}_*(B) \quad \text{and} \quad \mathcal{G}_H(C_0(X)) := \text{KK}_*^H(C_0(X), A \otimes B),$$

where  $X$  is any proper  $H$ -space. As noticed before, it follows from [24, Corollary of Proposition 6.2] that these functors are half exact, and [7, Proposition 5.14] implies that they satisfy the Induction Axiom. The Restriction Axiom follows from [23, Theorem 5.8]. For all  $H \in \mathcal{S}(G)$  and  $C_0(X) \in \mathcal{A}(H)$  the same construction as used for the map  $\alpha_X$  in (4.2) determines natural transformations  $\alpha_H : \mathcal{F}_H \rightarrow \mathcal{G}_H$ . Since induction preserves right Kasparov products with elements in  $K_*(B)$ , and restriction preserves (left and right) Kasparov products, these transformations satisfy the compatibility conditions of Definition 1.3. Hence, by Theorem 1.4, the proof of Theorem 0.1 reduces to check that

$$\alpha_K : \mathrm{KK}_*^K(C_0(V), A) \otimes K_*(B) \rightarrow \mathrm{KK}_*^K(C_0(V), A \otimes B)$$

is an isomorphism for every compact group  $K \in \mathcal{S}(G)$  acting linearly on the real vector space  $V$ . For this we first prove:

**Lemma 4.11.** *Assume that  $K$  is a compact group acting linearly on a real vector space  $V$  and that  $A$  is a  $K$ -algebra such that  $(A \otimes C_0(V)) \rtimes K$  is in  $\mathcal{N}$ . Then*

$$\alpha_K : \mathrm{KK}_*^K(C_0(V), A) \otimes K_*(B) \rightarrow \mathrm{KK}_*^K(C_0(V), A \otimes B)$$

*is an isomorphism for all  $B$  with free abelian  $K$ -group  $K_*(B)$ .*

*Proof.* By [22, §5, Theorem 7] and [7, Lemma 7.7], we know that for every  $K$ -algebra  $D$ , tensoring by  $C_0(V)$  induces an isomorphism  $\sigma_{C_0(V)} : \mathrm{KK}_*^K(C_0(V), D) \rightarrow \mathrm{KK}_*^K(C_0(V) \otimes C_0(V), C_0(V) \otimes D)$  and that  $C_0(V) \otimes C_0(V)$  is  $\mathrm{KK}^K$ -equivalent to  $\mathbb{C}$ . Let  $\mathcal{M} \in \mathrm{KK}_0^K(\mathbb{C}, C_0(V) \otimes C_0(V))$  denote this  $\mathrm{KK}$ -equivalence. Consider the diagram

$$\begin{array}{ccc} \mathrm{KK}_*^K(C_0(V), A) \otimes K_*(B) & \xrightarrow{\alpha_K} & \mathrm{KK}_*^K(C_0(V), A \otimes B) \\ (\mathcal{M} \otimes \cdot) \otimes \mathrm{id} \downarrow & & \downarrow \mathcal{M} \otimes \cdot \\ \mathrm{KK}_*^K(\mathbb{C}, C_0(V) \otimes A) \otimes K_*(B) & \xrightarrow{\alpha_K} & \mathrm{KK}_*^K(\mathbb{C}, C_0(V) \otimes A \otimes B) \\ \mu_{C_0(V) \otimes A} \otimes \mathrm{id} \downarrow & & \downarrow \mu_{C_0(V) \otimes A \otimes B} \\ K_*((A \otimes C_0(V)) \rtimes K) \otimes K_*(B) & \xrightarrow{\alpha} & K_*(((A \otimes C_0(V)) \rtimes K) \otimes B). \end{array}$$

All vertical rows of this diagram are bijections, since compact groups satisfy the Baum-Connes conjecture. Commutativity of the upper square follows from associativity of the Kasparov product, and commutativity of the lower square follows from Proposition 4.9. Since  $\alpha$  is an isomorphism by assumption, all arrows are isomorphisms.  $\square$

The next lemma shows that the assumptions of Theorem 0.1 imply the assumptions of the previous lemma, and hence it completes the proof of the theorem:

**Lemma 4.12.** *Let  $K$  be a compact group acting linearly on a finite dimensional real vector space  $V$ . Suppose further that  $A$  is a  $K$ -algebra such that  $A \rtimes L \in \mathcal{N}$  for every closed subgroup  $L$  of  $K$ . Then  $(A \otimes C_0(V)) \rtimes K \in \mathcal{N}$ .*

*Proof.* First note that whenever we have a short exact sequence  $0 \rightarrow C_0(U) \rightarrow C_0(X) \rightarrow C_0(X \setminus U) \rightarrow 0$  of abelian  $K$ -algebras, then the corresponding short exact sequence

$$0 \rightarrow (A \otimes C_0(U)) \rtimes K \rightarrow (A \otimes C_0(X)) \rtimes K \rightarrow (A \otimes C_0(X \setminus U)) \rtimes K \rightarrow 0$$

is semi-split. This can be shown by averaging a cross-section from  $A \otimes C_0(X \setminus U)$  to  $A \otimes C_0(X)$  over the compact group  $K$  and using [31, Theorem 3.5].

We then proceed as in Lemma 2.8, exploiting the fact that the class  $\mathcal{N}$  is stable under semi-split extensions and under direct limits with injective structure maps (see Lemma 4.4). Indeed, if  $M_i \subseteq K$ ,  $i \in \{1, \dots, l\}$  are the orbit types as in the proof of Lemma 2.8, then stability under semi-split extensions implies that we only need to show that  $(A \otimes C_0(M_i)) \rtimes K$  is in  $\mathcal{N}$  for all  $i \in \{1, \dots, l\}$ . Now let  $M$  be one of the sets  $M_i$ . Proceeding precisely as in the proof of Lemma 2.8, this reduces to show that  $(A \otimes C_0(U \times K/K')) \rtimes K \in \mathcal{N}$ , where  $K'$  is a closed subgroup of  $K$  and  $U$  is a trivial  $K'$ -space. But then  $(A \otimes C_0(U \times K/K')) \rtimes K$  is Morita equivalent to  $(A \otimes C_0(U)) \rtimes K' \cong (A \rtimes K') \otimes C_0(U)$ , which is in  $\mathcal{N}$  by Lemma 4.4. By the same lemma, the class  $\mathcal{N}$  is stable under Morita equivalence and the result follows.  $\square$

## 5. APPLICATIONS

We are now going to discuss some interesting consequences of Theorem 0.1. First of all, Corollary 0.2 of the introduction is a direct consequence of Theorem 0.1 and Proposition 4.9. For the proof of Corollary 0.3 we need the following strengthening of Connes's result that almost connected groups have nuclear group algebras (see [12, Corollary 7]):

**Proposition 5.1.** *Let  $G$  be a (second countable) almost connected group and let  $\alpha : G \rightarrow \text{Aut}(\mathcal{K}(H))$  be an action of  $G$  on the compact operators on some separable Hilbert space  $H$ . Then the full and reduced crossed products  $\mathcal{K}(H) \rtimes G$  and  $\mathcal{K}(H) \rtimes_r G$  are in the bootstrap class. In particular, the full and reduced group algebras  $C^*(G)$  and  $C_r^*(G)$  of  $G$  are in the bootstrap class.*

Before we give the proof of this proposition, we want to draw the consequences of this result. Recall that  $G_0$  denotes the component of the identity of the locally compact group  $G$ . We start with the proof of

**Corollary 0.3** *Let  $G$  be a locally compact group such that  $G/G_0$  satisfies BCC. Then the reduced group algebra  $C_r^*(G)$  satisfies the Künneth formula. If, in addition,  $G/G_0$  is K-amenable (which is always the case if  $G/G_0$  is  $\alpha$ -T-amenable by [36, 19]), then also the full group algebra  $C^*(G)$  satisfies the Künneth formula.*

*Proof.* By the stabilization trick (see [14, Theorem 1] and [26, §1]) there exist  $C^*$ -algebras  $B$  and  $B_r$  and actions of  $G/G_0$  on  $B$  and  $B_r$  such that for all subgroups  $H$  of  $G$  containing  $G_0$  the full (resp. reduced) group algebra  $C_{(r)}^*(H)$  is Morita equivalent to the full (resp. reduced) crossed product  $B_{(r)} \rtimes_{(r)} (H/G_0)$ . In particular,  $B$  is Morita equivalent to  $C^*(G_0)$  and  $B_r$  is Morita equivalent to  $C_r^*(G_0)$ . Now, if  $K$  is a subgroup of  $G$  containing  $G_0$  such that  $K/G_0$  is compact, then  $K$  is almost connected, and it follows from Proposition 5.1 that  $C_{(r)}^*(K)$  and hence  $B_{(r)} \rtimes (K/G_0)$  (by Morita equivalence) is in the bootstrap class. It follows then from Corollary 0.2 that the reduced crossed products  $B_{(r)} \rtimes_r (G/G_0)$  satisfy the Künneth formula. Hence  $C_r^*(G) \sim_M B_r \rtimes_r (G/G_0)$  satisfies the Künneth formula.

Assume now that  $G/G_0$  is K-amenable. Then  $C^*(G) \sim_M B \rtimes (G/G_0)$  is KK-equivalent to  $B \rtimes_r (G/G_0)$ . Hence  $C^*(G)$  also satisfies the Künneth formula.  $\square$

Next, we apply our results to the situation of Corollary 0.4 and Corollary 0.5 of the introduction. We are grateful to Bachir Bekka for showing us the reference to Gootman's and Kallman's paper [15], which is used below.

**Corollary 5.2.** *Assume that  $G$  satisfies BC for  $\mathbb{C}$ . Then the following are true:*

- (i) *If  $C_r^*(G) \in \mathcal{N}$ , then  $G$  satisfies BC for  $B$  for all separable  $C^*$ -algebras  $B$  equipped with the trivial  $G$ -action.*
- (ii)  *$G$  satisfies BC for all  $B \in \mathcal{N}$  equipped with the trivial  $G$ -action.*

*In particular, if  $G$  is almost connected, or if  $G$  is a linear algebraic group over a local field of characteristic zero, then  $G$  satisfies BC for all separable  $C^*$ -algebras  $B$  with trivial  $G$ -action.*

*Proof.* By Theorem 0.1 we have  $\mathbb{C} \in \mathcal{N}_G$ . Thus the first part of the corollary follows from part (i) of Proposition 4.10 and the second part follows from part (ii) of Proposition 4.10.

For the final statement note first that all almost connected groups and all linear algebraic groups over local fields with characteristic zero satisfy BC for  $\mathbb{C}$  by [10]. Proposition 5.1 shows that  $C_r^*(G)$  satisfies the Künneth formula if  $G$  is almost connected, and by [15, Theorem 5.4],  $C_r^*(G)$  is type I, and hence satisfies the Künneth formula, if  $G$  is linear algebraic over a local field of characteristic zero.  $\square$

Theorem 0.6, as stated in the introduction, is now a consequence of Corollary 2.10 and:

**Theorem 5.3.** *Let  $G_1, G_2$  be two locally compact groups.*

- (i) *If  $G_1$  and  $G_2$  satisfy BC for  $\mathbb{C}$  and if  $C_r^*(G_1)$  or  $C_r^*(G_2)$  belong to  $\mathcal{N}$ , then  $G_1 \times G_2$  satisfies BC for  $\mathbb{C}$ .*
- (ii) *Let  $G_1, G_2, \dots, G_n$  be locally compact groups such that all  $G_k$  satisfy BC for  $\mathbb{C}$  and such that all  $C_r^*(G_k)$  belong to  $\mathcal{N}$ , for  $k = 1, \dots, n$ . Then  $G := G_1 \times G_2 \times \dots \times G_n$  satisfies BC for all  $C^*$ -algebras  $B$  equipped with the trivial  $G$ -action.*

*Proof.* The first statement follows directly from Corollaries 2.10 and 5.2. For the second statement note that (i) implies that  $G = G_1 \times \dots \times G_n$  satisfies BC for  $\mathbb{C}$ . Since the class  $\mathcal{N}$  is closed under taking tensor products by Lemma 4.4, it follows that  $C_r^*(G) = C_r^*(G_1) \otimes \dots \otimes C_r^*(G_n)$  is in  $\mathcal{N}$ . Thus, the result follows from Corollary 5.2.  $\square$

We are now going to give the proof of Proposition 5.1. The proof uses quite heavily the structure theory for almost connected groups. We start with the following lemma, which uses some ideas of Tu:

**Lemma 5.4** (cf. [36, §10.2]). *Assume that  $G$  is an amenable almost connected group acting on a type I  $C^*$ -algebra  $A$ . Then the crossed product  $A \rtimes G$  is in the bootstrap class.*

*Proof.* Note first that  $A \rtimes G$  is nuclear, since  $A$  is type I (hence nuclear) and  $G$  is amenable. By results of Kasparov [23], there is a  $\text{KK}^G$ -equivalence between  $\mathbb{C}$  and the algebra  $C_0(T^*V)$ , where  $V = G/K$  is the quotient of  $G$  by its maximal compact subgroup and  $T^*V$  denotes the cotangent bundle of  $V$ . Tensoring with  $A$  and using Kasparov's descent morphism implies that  $A \rtimes G$  is  $\text{KK}$ -equivalent to  $(A \otimes C_0(T^*V)) \rtimes G$ . But since  $T^*V$  is a proper  $G$ -space,  $A \otimes C_0(T^*V)$  is a proper type I algebra. Hence it follows from [36, Proposition 10.3] that  $(A \otimes C_0(T^*V)) \rtimes G$  is type I, and hence in the bootstrap class. By  $\text{KK}$ -equivalence,  $A \rtimes G$  is in the bootstrap class, too.  $\square$

In the next step, we give the proof for almost connected Lie-groups.

**Lemma 5.5.** *Assume that  $G$  is an almost connected Lie group acting on  $\mathcal{K}(H)$ . Then the full (resp. reduced) crossed product  $\mathcal{K}(H) \rtimes_{(r)} G$  is in the bootstrap class.*

*Proof.* Assume without loss of generality that  $H$  is infinite dimensional and write  $\mathcal{K} = \mathcal{K}(H)$ . Let  $\omega \in H^2(G, \mathbb{T})$  be the Mackey-obstruction for the action of  $G$  on  $\mathcal{K}$  and let  $\tilde{G} = G \times_{\omega} \mathbb{T}$  denote the corresponding central extension of  $G$  by  $\mathbb{T}$  (see [10, §2] for the notation). Let  $G_0$  denote the connected component of  $G$  and let  $N$  denote the nil-radical of  $G$ , i.e.,  $N$  is the maximal connected closed nilpotent normal subgroup of  $G$ . Then  $\tilde{N} = N \times_{\omega} \mathbb{T}$  is the nil-radical of  $\tilde{G}$ . Let  $K = [G_0, G_0]N$ . Then  $\tilde{K} = K \times_{\omega} \mathbb{T}$  coincides with  $[\tilde{G}_0, \tilde{G}_0]\tilde{N}$ . By [30, §1.4, Lemma 5], the full (resp. reduced) group algebra  $C_{(r)}^*(\tilde{K})$  of  $\tilde{K}$  is type I. By the stabilization trick we can write  $C_{(r)}^*(\tilde{K}) \otimes \mathcal{K} \cong C_0(\mathbb{Z}, \mathcal{K}) \rtimes K$ , where the action of  $K$  on the summand  $\mathcal{K}$  of  $C_0(\mathbb{Z}, \mathcal{K})$  corresponding to  $1 \in \mathbb{Z}$  is Morita equivalent to the original action of  $K$  on  $\mathcal{K}$  (see [10, Lemma 2.4]). Hence  $\mathcal{K} \rtimes_{(r)} K$  can be realized as a quotient of the type I algebra  $C_{(r)}^*(\tilde{K}) \otimes \mathcal{K}$ , which implies that  $\mathcal{K} \rtimes_{(r)} K$  is a type I algebra itself. Now, using the stabilization trick again (assuming without loss of generality that  $\mathcal{K} \rtimes K$  is stable), we can write the full (resp. reduced) crossed product  $\mathcal{K} \rtimes_{(r)} G$  as an iterated crossed product  $(\mathcal{K} \rtimes_{(r)} K) \rtimes G/K$ . Since  $G/K$  is an extension of a finite group by a connected abelian group,  $G/K$  is amenable and almost connected, and the result then follows from Lemma 5.4.  $\square$

*Proof of Proposition 5.1.* Let  $G$  be an arbitrary almost connected group acting on  $\mathcal{K} = \mathcal{K}(H)$ . We assume without loss of generality that  $H$  is infinite dimensional. By the structure theorem for such groups (see [27]), there exists a compact normal subgroup  $C$  of  $G$  such that  $G/C$  is an almost connected Lie group. Using the stabilization trick, we can write the full (resp. reduced) crossed products  $\mathcal{K} \rtimes_{(r)} G$  as  $(\mathcal{K} \rtimes C) \rtimes_{(r)} G/C$  with respect to a suitable action of  $G/C$  on the stable  $C^*$ -algebra  $\widehat{\mathcal{K} \rtimes C}$ . Since  $C$  is compact,  $\mathcal{K} \rtimes C$  is isomorphic to a direct sum  $\bigoplus_{x \in X} \mathcal{K}_x$ , where  $X = \widehat{\mathcal{K} \rtimes C}$  is the discrete dual of  $\mathcal{K} \rtimes C$  (to see that  $X$  is discrete, just note that  $X$  is a subspace of the unitary dual of the compact group  $C \times_{\omega} \mathbb{T}$ , where  $\omega$  denotes the Mackey obstruction for the action of  $C$  on  $\mathcal{K}$ ). Alternatively, we can write  $\mathcal{K} \rtimes C$  as a direct sum  $\bigoplus_{G(x) \in G \setminus X} A_{G(x)}$ , where  $A_{G(x)} = \bigoplus_{y \in G(x)} \mathcal{K}_y$ , which then implies that  $\mathcal{K} \rtimes_{(r)} G \cong \bigoplus_{G(x) \in G \setminus X} (A_{G(x)} \rtimes_{(r)} G/C)$ . Since  $G(x)$  is discrete for all  $x \in X$ , the stabilizer  $(G/C)_x$  of  $x$  in  $G/C$  contains the connected component  $(G/C)_0$  of  $G/C$ , and hence  $(G/C)_x$  is an almost connected Lie group. Now Green's imprimitivity theorem implies that  $A_{G(x)} \rtimes_{(r)} G/C$  is Morita equivalent to  $\mathcal{K}_x \rtimes (G/C)_x$ , which is in the bootstrap class by Lemma 5.5. Since the bootstrap class is invariant under Morita equivalence and under taking countable direct sums, this finishes the proof.  $\square$

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