A noncommutative Euclidean space can be obtained by deforming an algebra of functions on a commutative $\mathbb{R}^d$. This leads to the well known Moyal multiplication

$$f * g(x) = \int d^d u d^d v \, f(x + \theta u) g(x + v) e^{iuv}$$

To construct an algebra with this multiplication rule one needs to specify a class of functions $f(x)$ on $\mathbb{R}^d$. Here are 3 useful classes of functions each closed under Moyal multiplication:

- $\mathcal{S}(\mathbb{R}^d)$- Schwartz class functions. Leads to an algebra denoted $\mathcal{S}({\mathbb{R}^d}_\theta)$
- Smooth functions decreasing at infinity: $\mathbb{R}^d_\theta$.
- Smooth functions $f(x)$ satisfying

  $$|\partial_\alpha f(x)| \leq C_\alpha (1 + |x|^2)^{\frac{1}{2}(m - \rho|\alpha|)}$$

  where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multiindex, $0 < \rho \leq 1$, $m \in \mathbb{R}$; C_α are positive constants.
  For $\rho = 1$ and negative m this class of functions gives rise to an algebra $\Gamma^m(\mathbb{R}^d_\theta)$

Unitized algebras obtained by a formal adjunction of unity will be marked by upper tilde:

$$\tilde{\Gamma}^m(\mathbb{R}^d_\theta), \;\; \tilde{\mathcal{S}}(\mathbb{R}^d_\theta), \;\; \tilde{\mathbb{R}}^d_\theta$$

Projective modules

One can prove that any projective module over a noncommutative Euclidean space $\tilde{\mathcal{A}}$ is isomorphic to a direct sum

$$E \cong \mathcal{F}^k \oplus (\tilde{\mathcal{A}}^1)^N$$

where $\tilde{\mathcal{A}}^1$ is a rank one free module and $\mathcal{F}$ is a so called Fock module. $\mathcal{F}$ is constructed on an irreducible representation of the Heisenberg algebra

$$[\hat{x}^j, \hat{x}^k] = i\theta^{jk} \cdot \mathbf{1}$$

However as the coordinates x^i do not belong to any class of functions that are of interest to us, one should define $\mathcal{F}$, in a more precise manner, via an action of integral operators on Schwartz class functions of $d/2$ variables that emerges in Weyl quantization.

Connections

$\mathbb{R}^d$ acts on algebras of functions $\tilde{\mathcal{A}}$. On a free module ∂_i - a reference connection. On a Fock module $\nabla_j^{(0)}$ - the operators of (right) multiplication by $i\theta^{-1}_{jk}\hat{x}^k$ provides a connection with the curvature

$$[\nabla_j^{(0)}, \nabla_k^{(0)}] = -i\theta^{-1}_{jk} \cdot \mathbf{1}$$

Yang-Mills field - connection compatible with the Hermitian structure. On a module $\mathcal{F}^k \oplus (\tilde{\mathcal{A}})^N$ a general form of the Yang-Mills field is

$$\nabla_j = \nabla_j^0 + i \begin{pmatrix} (A_j)_i^k & \langle (b_j)_i^\beta | \\ |(b_j)_\alpha^k\rangle & (\hat{D}_j)_\alpha^\beta \end{pmatrix} .$$

where

$$\nabla_j^0 = \begin{pmatrix} i\theta^{-1}_{jk}\hat{x}^k & 0 \\ 0 & \partial_j \end{pmatrix}$$

Solitons in Yang-Mills-Higgs system

Consider an action functional

$$S = \mathrm{Tr}(\frac{1}{4}F_{jk}F^{jk} + \frac{1}{2}\sum_{\alpha}[\nabla_j, \Phi^{\alpha}][\nabla^j, \Phi^{\alpha}] + V(\Phi))$$

where Φ^{α}, $\alpha = 1, \ldots, n$ are endomorphisms representing n scalar fields, $V(\Phi)$ is a potential that we will assume to be a polynomial. The equations of motion read

$$[\nabla^j, [\nabla_j, \nabla_k]] = -\sum_{\alpha}\Phi^{\alpha}[\nabla_k, \Phi_{\alpha}]\,,$$
$$[\nabla^j, [\nabla_j, \Phi_{\alpha}]] = \partial_{\alpha}V(\Phi)$$

Look for solutions s.t. $S < \infty$ (solitons). Consider a single scalar field Φ. Let $\{\phi_0, \ldots, \phi_p\}$ be the set of extrema of the function $V(\Phi)$. Also assume that $V(\phi_0) = 0$.

The solutions have a block-diagonal form with respect to the decomposition $E = \mathcal{F}^k \oplus (\tilde{A})^N$. The Yang-Mills fields are

$$\nabla_j = \begin{pmatrix} i\theta^{-1}_{jk}\hat{x}^k & 0 \\ 0 & \partial_j \end{pmatrix} + i\begin{pmatrix} D^j_i & 0 \\ 0 & \hat{D}^{\alpha}_{\beta} \end{pmatrix}$$

where

$$(D_i^j) = \begin{pmatrix} d_1 & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & d_k \end{pmatrix}$$

$$\widehat{D}_\alpha^\beta = \begin{pmatrix} a_1 \cdot \mathbf{1} & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & a_N \cdot \mathbf{1} \end{pmatrix}$$

and d_i, a_α are numbers. The scalar field reads

$$\Phi = \begin{pmatrix} \begin{pmatrix} \lambda_1 & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & \lambda_k \end{pmatrix} & 0 \\ 0 & \phi_0 \cdot \mathbf{1} \end{pmatrix}$$

where each λ_i is an element of the set of extrema $\{\phi_i\}$. The value of the soliton action on the solutions

$$S = \sum_1^k V(\lambda_i) + \frac{k}{4}(\theta^{-1})_{in} g^{ij} g^{nl} (\theta^{-1})_{jl} \,.$$

The moduli d_i, a_α are zero modes of the solitons.

Scalar Solitons

Consider a single scalar field Φ on a module $\mathcal{A}^N$ with action functional

$$S = \frac{1}{g^2}\int d^dx(\frac{1}{2}[\partial_i, \Phi][\partial^i, \Phi] + V(\Phi))$$

where the function $V(x)$ is assumed to be a polynomial, $\Phi \in End_{\mathcal{A}}\mathcal{A}^N$ is Hermitian and g is a coupling constant.

We assume that θ^{ij} has the standard form with equal eigenvalues $\theta_1 = \theta_2 = \ldots = \theta_{d/2} = \theta > 0$.

To study solutions in $\theta \to \infty$ limit perform rescalings

$$x^i \mapsto x^i\sqrt{\theta}$$

$$g^2 \mapsto g^2/\theta^{d/2}$$

The action functional now reads

$$S' = \frac{1}{g^2}\int d^dx(\frac{1}{2\theta}[\partial_i, \Phi][\partial^i, \Phi] + V(\Phi)) .$$

The operators corresponding to rescaled coordinates satisfy the canonical commutation relations

$$[\widehat{x}^j, \widehat{x}^{j+n}] = i\,, \quad j = 1, \ldots, n$$

and θ enters S' as a formal parameter. When $\theta \to \infty$ the approximate equations of motion take a very simple form

$$V'(\Phi) = 0$$

where V' stands for the derivative of $V(x)$.

Let $\phi_1, \ldots, \phi_p$ be a set of zeroes of $V'(x)$. Considering Φ as a pseudodifferential operator acting in Hilbert space $\mathcal{H}$ we imply that any eigenvalue of operator Φ should coincide with one of the numbers ϕ_i.

$\Rightarrow$ we can represent Φ as

$$\Phi = \sum_{i=1}^{p} \phi_i P_i$$

where P_i are orthogonal projectors on subspaces $\mathcal{H}_i$ such that $\mathcal{H} = \oplus_{i=1}^{p} \mathcal{H}_i$.

The energy of this solution in the limit $\theta \to \infty$ is given by

$$E = \frac{1}{g^2} \sum_{i=1}^{p} \mathrm{Tr} P_i \cdot V(\phi_i) = \frac{1}{g^2} \sum_{i=1}^{p} \mathrm{dim} \mathcal{H}_i V(\phi_i)$$

$E < \infty \Rightarrow$ the dimensions $\mathrm{dim}\mathcal{H}_i$ are infinite only when the corresponding $V(\phi_i)$ vanish.

$$\Rightarrow \exists x \ \text{ s.t. } \ V(x) = 0\,, \quad V'(x) = 0$$

In the case when $x = 0$ is the only such point an arbitrary (asymptotic) finite energy solution can be written as

$$\Phi = \sum_{i=1}^{p} \phi_i P_i$$

where $\mathrm{Tr} P_i < \infty$ for all i.

First order correction

Consider an expansion of the exact solution

$$\Phi = \Phi_0 + \frac{1}{\theta}\Phi_1 + \dots$$

The first order correction to the energy is then given by

$$K \equiv \frac{1}{2\theta g^2}\mathrm{Tr}[\partial_i, \Phi_0][\partial^i, \Phi_0] \, .$$

Thus we may seek an improved solution by minimizing the kinetic energy over all zero order solutions Φ_0. Consider the case when

$$\Phi_0 = \phi_1 P = \phi_1 \sum_{i=0}^{k} |e_i\rangle\langle e_i|$$

where $|e_i\rangle \in \mathcal{H}$ is a finite orthonormal system of vectors. Substituting it in the kinetic energy we obtain

$$K = \frac{{\phi_1}^2}{\theta g^2} \sum_{\alpha=1}^{d/2} \mathrm{Tr}(F_\alpha^\dagger F_\alpha + P)$$

where

$$F_\alpha = (1 - P) a_\alpha P$$

$$\boxed{a_\alpha = \frac{1}{\sqrt{2\theta}}(\hat{x}^\alpha + i\hat{x}^{n+\alpha})}$$

⇒ the kinetic energy satisfies a Bogomolnyi type bound

$$K \geq \frac{{\phi_1}^2 k d}{2\theta g^2}$$

saturated by the projectors P satisfying

$$(1 - P) a_\alpha P = 0$$

⇒ P projects on a subspace invariant under annihilation operators a_α.

Such subspaces are spanned by coherent states

$$|z\rangle = e^{z^\alpha a_\alpha^\dagger}|0\rangle$$

- eigenstates of a_α with eigenvalues $z^\alpha \in \mathbb{C}$.

The numbers z^α can be considered as complex coordinates on the moduli space of approximate solutions to the equation of motion. $\Rightarrow$ finite θ corrections reduce the moduli space to a finite-dimensional complex manifold.

The existence of rotationally invariant solitons at finite θ was proved rigorously by
B. Durhuus, T. Jonsson, R. Nest.

Let P_i denote the orthogonal projectors onto the eigenspaces of the number operator

$$N = \sum_{\alpha=1}^{d/2} a_\alpha^\dagger a_\alpha$$

of the $d/2$-dimensional harmonic oscillator.

Theorem: *for any projection P on $\mathcal{H} \cong L_2(\mathbb{R}^{d/2})$, which is the sum of a finite number of projections P_i, there is a unique family Φ_θ of rotationally invariant solutions to the equation of motion*

$$[\partial_i, [\partial^i, \Phi]] = \theta V'(\Phi)$$

that depend smoothly on θ and satisfy $V''(\Phi_\theta) > 0$ and $\Phi_\theta \to \phi \cdot P$ in Hilbert-Schmidt operator norm as $\theta \to \infty$.

Here ϕ stands for the local minimum of $V(x)$. The theorem is proved in the assumption that V has exactly one local minimum and one local maximum.

Instantons on noncommutative $\mathbb{R}^d$

(N. Nekrasov, A. Schwarz, 1998)

We would like to solve the antiinstanton (self-duality) equation

$$F_{ij} = \frac{1}{2}\epsilon_{ijkl}F^{kl}$$

on a module $\mathcal{F}^k \oplus (\tilde{\mathcal{A}})^N$. The appropriate algebra is

$$\tilde{\mathcal{A}} = \tilde{\Gamma}_\theta \equiv \bigcup_{m<-1} \Gamma^m(\mathbb{R}^4_\theta)$$

Noncommutative ADHM (after Atiyah, Drinfeld, Hitchin, Manin) construction starts from a solution to some matrix equations that are further used to construct such a projector Π acting on a free module $(\tilde{\Gamma}_\theta)^N$ that the corresponding Levi-Civita connection $\Pi \cdot \partial_i \cdot \Pi$ is (anti)self-dual.

Let V and W be a pair of complex vector spaces of dimensions k and N respectively. Consider a Γ_θ-linear operator

$$D^\dagger : (V \oplus V \oplus W) \otimes \Gamma_\theta \to (V \oplus V) \otimes \Gamma_\theta$$

defined by the formula

$$D^\dagger = \begin{pmatrix} -B_2 + \hat{z}_2 & B_1 - \hat{z}_1 & I \\ B_1^\dagger - \widehat{\bar{z}}_1 & B_2^\dagger - \widehat{\bar{z}}_2 & J^\dagger \end{pmatrix}$$

where

$$B_1, B_2 : V \to V$$

$$I : W \to V$$

$$J : V \to W$$

are linear mappings, $\hat{z}_\alpha$, $\widehat{\bar{z}}_1$ are operators acting by star multiplication by the corresponding complex coordinates:

$$z_1 = \frac{1}{\sqrt{2}}(x^1 + ix^3)\,, \qquad z_2 = \frac{1}{\sqrt{2}}(x^2 + ix^4)$$

The requirement that $D^\dagger$ satisfies

$$D^\dagger D = \begin{pmatrix} \triangle & 0 \\ 0 & \triangle \end{pmatrix}$$

where $\triangle$ is some operator

$$\triangle : V \otimes \Gamma_\theta \to V \otimes \Gamma_\theta$$

is equivalent to the noncommutative ADHM equations

$$[B_1, B_2] + IJ = 0\,,$$

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta_r$$

where

$$\zeta_r = -(\theta_1 + \theta_2)$$

One can prove that for $\zeta_r \neq 0$ the operator $\triangle$ is invertible while in the commutative ADHM construction it is a separate assumption. This is related to a remarkable fact that *the moduli space of noncommutative instantons has no small instanton singularities.*

$\triangle$ is nondegenerate $\Rightarrow$ we can split the module

$$(V \oplus V \oplus W) \otimes \Gamma_\theta = \mathcal{E} \oplus \mathcal{E}'$$

into submodules

$$\mathcal{E} = \mathrm{Ker}(D^\dagger)$$

and

$$\mathcal{E}' = D(V \oplus V \oplus W) \otimes \Gamma_\theta \equiv \mathrm{Im}(D)$$

Each of these submodules can be identified with images of orthogonal projectors

$$\mathcal{E} = \Pi(V \oplus V \oplus W) \otimes \Gamma_\theta$$

$$\mathcal{E}' = (1 - \Pi)(V \oplus V \oplus W) \otimes \Gamma_\theta$$

where

$$\Pi = 1 - D \cdot (D^\dagger D)^{-1} \cdot D^\dagger .$$

The inverse of operator $D^\dagger D$ exists under the assumption that the operator $\triangle$ is invertible (true for $\theta^{ij} \neq 0$).

$\Rightarrow$ starting with a solution to the noncommutative ADHM equations we constructed an orthogonal projector Π. One can check that the Levi-Civita connection $\Pi \partial_i \Pi$ induced on the module $\mathcal{E}$ satisfies the self-duality equation.

String Field Theory

E. Witten (1986) ”Noncommutative geometry and string field theory”.

$\mathcal{H}$-Hilbert space of first quantized open string theory - Fock representation of infinite system of bosonic oscillators

$$[a_m^\mu, a_n^{\dagger\nu}] = \delta_{mn} g^{\mu\nu}\,, \quad \mu, \nu = 1, \ldots, 26$$

and a fermionic $b - c$ system

$$\{b_m, c_n\}_+ = \delta_{m,-n}$$

describing Faddeev-Popov ghosts arising from gauge fixing.

$\mathbb{Z}$-Grading by ghost number: $\#b - \#c$.
BRST operator

$$Q : \mathcal{H} \to \mathcal{H}$$

such that $Q^2 = 0$, gh$(Q) = 1$. The set of cogomologies of Q singles out physical states in $\mathcal{H}$.

Axioms of Witten’s SFT:

- $\mathcal{H}$ is endowed with an associative multiplication
$$*:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$$
that respects the ghost number grading, i.e.
$$\mathrm{gh}(A*B)=\mathrm{gh}(A)+\mathrm{gh}(B)\,.$$

- (super)trace $\int:\mathcal{H}\to\mathbb{C}$.
$$\int A*B=(-1)^{\epsilon(A)\epsilon(B)}\int B*A\,.$$

- BRST operator Q is a (super)derivation of the star algebra:
$$Q(A*B)=(QA)*B+(-1)^{\epsilon(A)}A*(QB)$$

-

$$\int QA=0$$

SFT action

$$S[A] = \frac{1}{g^2} \int (\frac{1}{2} A * Q(A) + \frac{1}{3} A * A * A) \, .$$

Equation of motion

$$QA + A * A = 0$$

Vacuum SFT (L. Rastelly, A. Sen, B. Zwiebach) has a simpler BRST operator. Star algebra projectors

$$P = P * P$$

are solutions of VSFT equation of motion.

A concrete realization of Witten's axioms can be given by specifying the so called vertices

$$\langle V_N| \in \mathcal{H}^{*\otimes N}$$

that satisfy

$$\langle V_N|\Psi_1\rangle \otimes |\Psi_2\rangle \otimes \ldots |\Psi_N\rangle = \int \Psi_1 * \Psi_2 * \ldots * \Psi_N \,.$$

These vertices are specified by infinite-dimensional matrices V^{rs}_{nm} so that

$$|V_N\rangle = e^{-\frac{1}{2} a^{\dagger(r)}_n V^{rs}_{nm} a^{\dagger(s)}_m} |0\rangle_1 \otimes \ldots \otimes |0\rangle_N \,.$$

Recently the spectrum of operators specified by matrices V_3 was found exactly. That allowed one to write Witten's star product in a form of infinite-dimensional Moyal product.

Such a Moyal product has the form of a functional integral on wave functionals $\Psi(x(k)) = \langle x(k)|\Psi\rangle$:

$$(\Psi_1 * \Psi_2)(x_3(k)) = \int \mathcal{D}[x_1(k)]\mathcal{D}[x_2(k)]$$
$$K(x_1(k), x_2(k), x_3(k))\Psi_1(x_1(k))\Psi_2(x_2(k))$$

with the kernel

$$K(x_1, x_2, x_3) = K \cdot \exp[\int_0^\infty d\kappa \frac{2i}{\theta(\kappa)} x^{(r)} \chi^{rs} x^{(s)}]$$

where

$$\theta(k) = 2\tanh\left(\frac{\pi k}{4}\right) .$$

(An important feature of $\theta(k)$ is the existence of a commuting mode at $k = 0$.)

Open problems:

- find a suitable class of wave functionals on which the continuous Moyal product is well defined
- find a suitable norm
- find projectors and K-theory for such a star algebra