Algebra of functions on n.c. torus

Linear generators $U_{\mathbf{n}}$ labelled by $\mathbf{n} = (n_1, \ldots, n_d)$, $n_i \in \mathbb{Z}$ satisfy

$$U_{\mathbf{n}} U_{\mathbf{m}} = e^{\pi i n_j \theta^{jk} m_k} U_{\mathbf{n}+\mathbf{m}}$$

Denote the algebra generated by $U_{\mathbf{n}}$'s as T^d_θ. Involution:

$$U^*_{\mathbf{n}} = U_{-\mathbf{n}} \, .$$

The elements $U_{\mathbf{n}}$ are assumed to be unitary with respect to this involution.

The elements $U_i = U_{\mathbf{e}^{\mathbf{i}}}$ where $\mathbf{e}^{\mathbf{i}} = (e^i_1, \ldots, e^i_d)$, $e^i_j = \delta^j_i$. satisfy the condition

$$U_j U_k = e^{2\pi i \theta^{jk}} U_k U_j \, .$$

These elements are algebraic generators as they satisfy

$$U_{\mathbf{n}} = {U_1}^{n_1} {U_2}^{n_2} \ldots {U_d}^{n_d} e^{-\pi i \sum_{j<k} n_j n_k \theta^{jk}} \, .$$

Consider linear combinations $\sum_{\mathbf{n}} C(\mathbf{n}) U_{\mathbf{n}}$ where the coefficients $C(\mathbf{n})$ tend to 0 faster than any power of $\|\mathbf{n}\|$. The algebraic operations can be extended to the set of such linear combinations. One can say that this set constitutes an algebra of smooth functions on a noncommutative torus. From now on we will use the notation T^d_θ for this algebra.

There is a canonical normalized trace on T^d_θ specified by the rule

$$\mathrm{Tr} U_{\mathbf{n}} = 0 \ \text{ if } \ \mathbf{n} \neq \mathbf{0} \, , \qquad \mathrm{Tr} U_{\mathbf{0}} \equiv \mathrm{Tr} 1 = 1 \, .$$

Projective modules over n.c. tori

A large class of examples of projective modules constitute the so called Heisenberg modules. They are constructed as follows.

Consider a group

$$G = \mathbb{R}^p \times \mathbb{Z}^q \times F$$

where F is a finite group. Then the dual group is

$$G^* \cong \mathbb{R}^p \times T^q \times F^*$$

Consider a linear space $\mathcal{S}(G)$ of functions on G decreasing at infinity faster than any power. Define operators

$$U_{(\gamma,\tilde{\gamma})} : \mathcal{S}(G) \to \mathcal{S}(G)$$

labelled by a pair $(\gamma, \tilde{\gamma}) \in G \times G^*$ acting as follows

$$(U_{(\gamma,\tilde{\gamma})} f)(x) = \tilde{\gamma}(x) f(x + \gamma)\,,$$

where $f \in \mathcal{S}(G)$.

If $(\gamma, \tilde{\gamma})$ run over a d-dimensional discrete subgroup $\Gamma \subset G \times G^*$, $\Gamma \cong \mathbb{Z}^d$ then the above formula defines a module over a d-dimensional noncommutative torus T^d_θ with

$$exp(2\pi i\theta_{ij}) = \tilde{\gamma}_i(\gamma_j)\tilde{\gamma}_j{}^{-1}(\gamma_i)$$

for a given basis $(\gamma_i, \tilde{\gamma}_i)$ of the lattice Γ. This module is projective if Γ is such that $G \times G^*/\Gamma$ is compact.

If the matrix θ_{ij} is irrational then any projective module over T^d_θ can be represented as a direct sum of Heisenberg modules (M. Rieffel).

If a Heisenberg module E cannot be represented as a direct sum of isomorphic modules $E = E' \oplus E' \oplus \ldots \oplus E'$ then it is called a basic module. Evidently by the result stated above if θ_{ij} is irrational any projective module can be represented as a direct sum of basic modules. Moreover the algebra $End_{T^d_\theta} E$ is isomorphic to a noncommutative torus $T^d_{\hat{\theta}}$ for some matrix $\hat{\theta}_{ij}$.

Connections

Define a commutative Lie algebra action

$$L_\theta : T_\theta^d \to T_\theta^d$$

by specifying a standard basis consisting of derivations δ_j, $j = 1, \ldots, d$ satisfying

$$\delta_j(U_{\mathbf{n}}) = 2\pi i n_j U_{\mathbf{n}} .$$

A connection on a module E over T_θ^d is a set of operators $\nabla_X : E \to E$, $X \in L_\theta$ depending linearly on X and satisfying

$$[\nabla_X, U_{\mathbf{n}}] = \delta_X(U_{\mathbf{n}}) \tag{1}$$

where $U_{\mathbf{n}}$ are operators $E \to E$ representing the corresponding generators of T_θ^d. In the standard basis this relation reads as

$$[\nabla_j, U_{\mathbf{n}}] = 2\pi i n_j U_{\mathbf{n}} . \tag{2}$$

On any Heisenberg module there exists a constant curvature connection.

K-theory. Chern character.

The K-groups of a noncommutative torus can be computed using the technique due to Pimsner and Voiculescu. The answer coincides with that for commutative tori:

$$K_0(T_\theta^d) \cong \mathbb{Z}^{2^{d-1}} \cong K_1(T_\theta^d)\,.$$

Define the *Chern character* $\mathrm{ch}(E)$ as

$$\mathrm{ch}(E) = \mathrm{Tr}\, exp\left(\frac{1}{2\pi i}F\right) \in \Lambda^{even}(L_\theta^*)$$

where $\Lambda^{even}(L_\theta^*)$ is the even part of the exterior algebra of L_θ^*.

This mapping gives rise to a map

$$\mathrm{ch} : K_0(T_\theta^d) \to \Lambda^{even}(L_\theta^*)\,.$$

Consider a subset $\Lambda^{even}(\mathbb{Z}^d) \subset \Lambda^{even}(L_\theta^*)$ that consists of polynomials in α^j (standard basis) having integer coefficients. The Chern character is injective and its range on $K_0(T_\theta^d)$ is given by the image of $\Lambda^{even}(\mathbb{Z}^d)$ under the action of the operator $exp\left(-\frac{1}{2}\frac{\partial}{\partial\alpha^j}\theta^{jk}\frac{\partial}{\partial\alpha^k}\right)$. (Elliott).

This fact implies that the K-group $K_0(T_\theta^d)$ can be identified with the additive group $\Lambda^{even}(\mathbb{Z}^d)$.

A K-theory class $\mu(E) \in \Lambda^{even}(\mathbb{Z}^d)$ of a module E can be computed from its Chern character by the formula

$$\boxed{\mu(E) = exp\left(\frac{1}{2}\frac{\partial}{\partial\alpha^j}\theta^{jk}\frac{\partial}{\partial\alpha^k}\right)\mathrm{ch}(E)\,.}$$

Morita equivalence

One says that two associative algebras A and $\hat{A}$ are Morita equivalent if the category of A-modules is equivalent to the category of $\hat{A}$-modules.

Given an $(\hat{A}, A)$-bimodule P we can define a correspondence $E \mapsto \hat{E}$ of modules by means of the formula

$$\hat{E} = P \otimes_A E\,.$$

This construction gives rise to a correspondence between A-linear and $\hat{A}$-linear maps. Namely, every A-linear map $\phi : E \to E'$ induces an $\hat{A}$-linear map

$$\hat{\phi} = 1 \otimes \phi : P \otimes_A E \to P \otimes_A E'$$

We say that P generates a Morita equivalence of algebras A and $\hat{A}$ if there exists an ($A, \hat{A}$)*-bimodule Q generating the inverse correspondence $\hat{E} \mapsto E$.* Then P is called an $(\hat{A}, A)$ Morita equivalence bimodule.

Examples

1. One set of examples of Morita equivalence is provided by automorphisms of algebra A. Consider an automorphism $\phi : A \to A$ one can consider a bimodule P that consists of elements of A itself with the right action defined by the multiplication by $a \in A$ from the right and the left multiplication defined as a multiplication by $\phi(a)$ from the left.

- Noncommutative tori parameterized by the matrices θ^{ij} and $\theta^{ij} + N^{ij}$ that differ by an antisymmetric matrix N^{ij} with integer entries, are isomorphic. The resulting transformation shuffles the topological numbers according to the Elliott formula.

- If $g \in SL(d, \mathbb{Z})$ then the tori with θ^{ij} and $\hat{\theta}^{ij} = (g^t \theta g)^{ij}$ are isomorphic. The corresponding Morita equivalence results in a change of basis of the Grassmann algebra $\Lambda(L^*)$: $\alpha^i \to (g^t)^i_j \alpha^j$ that changes the topological numbers in the corresponding way.

2. Consider the following module over a two-dimensional n.c. torus T_θ ($\theta = \theta_{12}$):

$$(U_1 f)(x) = f(x+\theta)\,, \quad (U_1 f)(x) = f(x)e^{2\pi i x}$$

Its endomorphisms are generated by

$$(Z_1 f)(x) = f(x+1)\,, \quad (Z_2 f)(x) = f(x)e^{2\pi i x \theta^{-1}}$$

From this construction we conclude that T^2_θ is Morita equivalent to $T^2_{1/\theta}$.

3. In general we can take for P any basic T^d_θ-module. Then the algebra $End_{T^d_\theta} P$ is again a noncommutative torus $T^d_{\hat{\theta}}$. One can prove that the matrix $\hat{\theta}$ is related to θ by means of a fractional linear transformation

$$\hat{\theta} = (M\theta + N)(R\theta + S)^{-1}$$

where

$$\begin{pmatrix} M & N \\ R & S \end{pmatrix}$$

is a $2d \times 2d$ matrix belonging to the group $SO(d,d|\mathbb{Z})$.

Moreover given an element of $SO(d,d|\mathbb{Z})$ the torus T^d_θ is Morita equivalent to the torus $T^d_{\hat{\theta}}$ where $\hat{\theta}$ is related to θ by means of the above fractional transformation (provided the denominator is invertible).

The group $SO(d,d|\mathbb{Z})$ acting on θ is generated by three kinds transformations: the shifts $\theta \to \theta + N$ that embed in the $SO(d,d|\mathbb{Z})$ as matrices with vanishing blocks M, R, S, the $SL(d,\mathbb{Z})$ rotations $\theta \to A^t\theta A$ corresponding to $M = A^t$, $R = (A^t)^{-1}$, $N = S = 0$, a single transformation σ called a flip that inverts any given 2×2 block in matrix θ. More precisely it acts as follows. Without loss of generality we can assume that θ has a block form

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}.$$

where θ_{11} is a 2×2 nondegenerate matrix. Then a flip σ sends θ into

$$\sigma_2(\theta) = \begin{pmatrix} \theta_{11}^{-1} & -\theta_{11}^{-1}\theta_{12} \\ \theta_{21}\theta_{11}^{-1} & \theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12} \end{pmatrix}$$

For the particular case of a two-dimensional torus the flip simply inverts the whole matrix θ. One can check that $SL(d,\mathbb{Z})$ rotations, shifts and the flip σ generate the whole $SO(d,d|\mathbb{Z})$.

Gauge Morita Equivalence.

Gauge Morita equivalence (introduced by A. Schwarz) allows one to transport connections

$$\begin{array}{rcl} E - \mathcal{A} - \text{module} & \longrightarrow & \hat{E} - \hat{\mathcal{A}} - \text{module} \\ \nabla - \text{connection on } E & \longrightarrow & \hat{\nabla} - \text{connection on } \hat{E} \end{array}$$

$(T^d_{\hat{\theta}}, T^d_\theta)$ -Morita equivalence bimodule P establishes a gauge Morita equivalence if it is endowed with operators ∇^P_X, $X \in L$ satisfying

$$\begin{aligned} \nabla^P_X(ea) &= (\nabla^P_X e)a + e(\delta_X a)\,, \\ \nabla^P_X(\hat{a}e) &= \hat{a}(\nabla^P_X e) + (\hat{\delta}_X \hat{a})e\,, \\ [\nabla^P_X, \nabla^P_Y] &= 2\pi i \sigma_{XY} \cdot \mathbf{1}\,. \end{aligned}$$

δ_X and $\hat{\delta}_X$ are standard derivations on T_θ and $T_{\hat{\theta}}$:

$$\delta : L \to L_\theta\,, \qquad \hat{\delta} : L \to L_{\hat{\theta}}\,.$$

A basic module E equipped with a standard constant curvature connection ∇_i is an example of gauge Morita equivalence bimodule.
Given a connection ∇_X on E an operator

$$1 \otimes \nabla_X + \nabla^P_X \otimes 1$$

on $E \otimes_{\mathbb{C}} P$ descends to a connection $\hat{\nabla}_X$ on $\hat{E} = P \otimes_{T_\theta} E$. This mapping preserves gauge equivalence relation. The curvatures shift

$$F^{\hat{\nabla}}_{XY} = \hat{F}^{\nabla}_{XY} + 1\sigma_{XY}$$

Yang-Mills equation of motion is preserved. A stronger statement: modified Yang-Mills functional

$$S_{YM} = \frac{V}{4g^2}\mathrm{Tr}(F_{jk} + \Phi_{jk} \cdot 1)(F^{jk} + \Phi^{jk} \cdot 1)$$

is preserved provided that

- the curvature shift is compensated by $\Phi_{XY} \mapsto \Phi_{XY} - \sigma_{XY}$

- change in the normalization of the trace and in the volume $V = \det g_{ij}$ is compensated by the coupling constant change $g \mapsto \widehat{g}$

Transformation rules. Duality group.

The curvature shift implies

$$\boxed{\mathrm{ch}(\hat{E}) = \frac{\dim(\hat{E})}{\dim(E)} e^{\alpha^j \sigma_{jk} \alpha^k} \mathrm{ch}(E)\,.}$$

This relation induces by Elliott formula a relation between K-theory classes $\mu(E)$ and $\mu(\hat{E})$. Integrality puts a strong restriction on the form of the transformation $\theta \mapsto \widehat{\theta}$ induced by (gauge) Morita equivalence.

Some technical things.

The Grassmann algebra $\Lambda \equiv \Lambda(L^*)$ can be considered as a fermionic Fock space carrying two irreducible representations $\Lambda = \Lambda^{even} \oplus \Lambda^{odd}$ of the group $O(d,d|\mathbb{C})$.

We have operators a^k of multiplication by α^k and operators $b_k = \frac{\partial}{\partial \alpha^k}$ acting on Λ and satisfying canonical anticommutation relations

$$\{a^k, b_l\}_+ = \delta^k_l\,, \quad \{a^k, a^l\}_+ = 0\,, \quad \{b_k, b_l\}_+ = 0$$

that correspond to a Clifford algebra specified by a metric in $\mathbb{R}^{2d}$

$$\begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}.$$

This metric has a signature (d, d). The group $O(d, d|\mathbb{C})$ can be regarded as a group of automorphisms of the above Clifford algebra. A transformation W_g given by the formulas

$$W_g : a^k \mapsto \tilde{a}^k = M^k_l a^l + N^{kl} b_l\,,$$
$$W_g : b_k \mapsto \tilde{b}_k = R_{kl} a^l + S^l_k b_l$$

preserves the canonical anticommutation relations iff the matrix

$$g = \begin{pmatrix} M & N \\ R & S \end{pmatrix}$$

belongs to the group $O(d, d|\mathbb{C})$.

Define a projective action of $O(d,d|\mathbb{C})$ on $\Lambda(L^*)$ assigning to every $g \in O(d,d|\mathbb{C})$ an operator

$$V_g : \Lambda(L^*) \to \Lambda(L^*)$$

that satisfies

$$V_g a^k V_g^{-1} = W_{g^{-1}}(a^k)\,,$$

$$V_g b_k V_g^{-1} = W_{g^{-1}}(b_k)$$

Requiring that the operators V_g preserve an appropriate bilinear form on $\Lambda(L^*)$ one can obtain a spinor representation of $SO(d,d|\mathbb{C})$.

We also define an action $\theta \mapsto g\theta = \hat{\theta}$ of $O(d,d|\mathbb{C})$ on the space of antisymmetric matrices by the formula

$$\hat{\theta} = (M\theta + N)(R\theta + S)^{-1}\,.$$

Consider two two noncommutative tori T_θ and $T_{\hat{\theta}}$ related by a gauge Morita equivalence.

It follows from the curvature shift and Elliott formula that

$$\mu(\hat{E}) = V\mu(E)$$

where

$$V_g = V_1 V_2 V_3 V_4$$

with

$$\begin{aligned} V_1 f &= \exp(-\frac{1}{2} b_k \widehat{\theta}^{kj} b_j) f \\ V_2 f &= \exp(a^k \sigma_{kj} a^j) f \\ V_3 f &= \frac{\dim \widehat{E}}{\dim E} f(A^t \alpha) \\ V_4 f &= \exp(\frac{1}{2} b_k \theta^{kj} b_j) f \end{aligned}$$

where $f \in \Lambda$.

- the operator V_1 relates $\mu(\widehat{E})$ and $ch(\widehat{E})$
- the operator V_4 relates $\mu(E)$ and $ch(E)$
- the operator $V_2 V_3$ relates $ch(\widehat{E})$ and $ch(E)$.

In the last relation one has to take into account that we should identify L_θ and $L_{\widehat{\theta}}$ by means of some linear operator A.

The operators V_1, V_2, V_3, V_4, and their product are linear canonical transformations. We know that $\mu(\widehat{E})$ and $\mu(E)$ are integral elements of $\Lambda(L^*)$. Therefore, the operators V, V^{-1} transform integral elements of $\Lambda(L^*)$ into integral elements. $\Rightarrow$ the linear canonical transformation V corresponds to an element of $SO(d,d|\mathbb{Z})$ and we proved that $\mu(\widehat{E})$ and $\mu(E)$ are related by a linear canonical transformation corresponding to an element of $SO(d,d|\mathbb{Z})$.

From preservation of the bilinear form on $\Lambda(L^*)$ it follows that

$$\frac{\mathrm{dim}\widehat{E}}{\mathrm{dim}E} = |det(A)|^{-1/2}.$$

Going from V_i's to the corresponding elements of $SO(d,d|\mathbb{C})$ we obtain

$$\begin{pmatrix} 1 & \widehat{\theta} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\sigma & 1 \end{pmatrix}\begin{pmatrix} (A^t)^{-1} & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} (A^t)^{-1} - \widehat{\theta}\sigma(A^t)^{-1} & -(A^t)^{-1}\theta - \widehat{\theta}(\sigma(A^t)^{-1} - A) \\ -\sigma(A^t)^{-1} & \sigma(A^t)^{-1}\theta + A \end{pmatrix}.$$

From the last formula one readily obtains

$$\widehat{\theta} = (M\theta + N)(R\theta + S)^{-1}\,,$$

$$A = R\theta + S\,,$$

$$\sigma = -RA^{t} = -R(R\theta + S)^{t}\,.$$

$\Rightarrow$ We proved that whenever two tori T_θ and $T_{\widehat{\theta}}$ are gauge Morita equivalent the matrices θ and $\widehat{\theta}$ are connected by a fractional transformation corresponding to a subgroup $SO(d,d|\mathbb{Z}) \subset SO(d,d|\mathbb{C})$- the group of automorphisms of the Clifford algebra $\Lambda(L^*)$. Conversely one can also prove (M. Rieffel and A. Schwarz) that if θ and $\widehat{\theta}$ are connected by a fractional transformation corresponding to some element of $SO(d,d|\mathbb{Z})$ then the tori T_θ and $T_{\widehat{\theta}}$ are (gauge) Morita equivalent.

Summarizing our results we see that *gauge Morita equivalence of d-dimensional noncommutative tori is governed by the group $SO(d,d|\mathbb{Z})$. This group acts on matrices θ by means of fractional transformations and on topological numbers of modules by means of a spinor representation.*

It follows from the above formulas that the curvature tensor, metric tensor, background field Φ_{ij} and the volume transform according to

$$\begin{aligned} F^{\widehat{\nabla}}_{ij} &= A^k_i F^{\nabla}_{kl} A^l_j + \sigma_{ij}\,, & \widehat{g}_{ij} &= A^k_i g_{kl} A^l_j\,, \\ \widehat{\Phi}_{ij} &= A^k_i \Phi_{kl} A^l_j - \sigma_{ij}\,, & \widehat{V} &= V|\mathrm{det}\,A| \end{aligned}$$

where σ_{ij} and A^i_j were given above.

With this transformation rules the (modified) SYM action is invariant provided that the coupling constant changes as

$$\widehat{g}^2 = g^2 \cdot |\mathrm{det}\,A|^{1/2}$$

An application:moduli spaces of constant curvature connections.

A constant curvature connection is a connection ∇_j that satisfies

$$[\nabla_j, \nabla_k] = 2\pi i f_{jk} \mathbf{1} \tag{3}$$

Lemma *For even d one can find an $SO(d,d|\mathbb{Z})$ transformation such that the transformed curvature tensor $\widehat{f}_{ij}$ is nondegenerate.*

Gauge Morita equivalence maps constant curvature connections into constant curvature ones respecting the gauge equivalence relation. $\Rightarrow$ if E admits a constant curvature connection, so does $\hat{E}$ and moreover, the corresponding moduli spaces are isomorphic.

By Stone-von Neumann theorem there is a unique irreducible representation $\mathcal{F}$ of this algebra. Suppose that a representation space E can be decomposed into a direct sum of a finite number of irreducible components:

$$E \cong \mathcal{F}^N \cong \mathcal{F} \otimes \mathbb{C}^N$$

.

Fix the representation $\mathcal{F}$ as follows. First bring the matrix f_{ij} to a canonical block-diagonal form

$$(f_{ij}) = \begin{pmatrix} f_1\epsilon & 0 & \dots & 0 \\ 0 & f_2\epsilon & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & f_g\epsilon \end{pmatrix}$$

where

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a 2×2 matrix and f_i are positive numbers.

Define a representation space as $L_2(\mathbb{R}^g)$ and the operators ∇_i as

$$\nabla_j = \sqrt{f_{(j+1)/2}}\partial_j \, , j - \text{odd} \, ,$$

$$\nabla_j = 2\pi i \sqrt{f_{j/2}} x_{j-1} \, , j - \text{even}$$

where ∂_j, x_k, $j, k = 1, \dots, g$ are derivative and multiplication by x^k operators acting on smooth functions $f(x) \in L_2(\mathbb{R}^g)$.

An arbitrary representation of the torus generators U_i, $i = 1, \dots, d$ has the form $U_i = U_i^{st} \cdot u_i$ where U_i^{st} is some standard representation satisfying

$$[\nabla_j, U_k^{st}] = 2\pi i \delta_{jk} U_k^{st}$$

and u_i is an $N \times N$ unitary matrix.

One can take U_i^{st} to be

$$U_k^{st} = e^{-(f^{-1})^{kl} \nabla_l} .$$

$$U_j^{st} U_k^{st} = e^{-2\pi i (f^{-1})^{jk}} U_k^{st} U_j^{st} .$$

Since $U_i = U_i^{st} \cdot u_i$ must give a representation of a noncommutative torus so must do the operators u_i:

$$\boxed{u_i u_j = e^{2\pi i n^{ij}/N} u_j u_i}$$

where N is a positive integer and n^{ij} is an integer valued antisymmetric matrix.

$\Rightarrow$ the moduli space of constant curvature connections on a module with fixed (N, n^{ij}) (fixed $\mu(E)$) can be described as a space of inequivalent representations of the matrix algebra generated by u_i. An irreducible representation of this algebra E_Λ is labelled by $\Lambda \in \tilde{T}^d$- a d-dimensional torus that can be canonically identified once a basis u_i is fixed.

Consider a decomposition into irreps

$$\mathbb{C}^N = E_{\Lambda_1} \oplus \ldots \oplus E_{\Lambda_{N_0}}$$

It can be proven that $N_0 = g.c.d.(\mu(E))$. Taking a quotient over residual gauge transformations one obtains that *the moduli space of constant curvature connections on a module E is isomorphic to*

$$\boxed{\mathcal{M}_{c.c.c.} \cong (\tilde{T}^d)^{g.c.d.(\mu(E))}/S_{g.c.d.(\mu(E))}}$$

For $d = 2$ this result was proved by A. Connes and M. Rieffel.