

## Matrix model of M-theory

1996. A conjecture by Banks, Fischler, Shenker and Susskind:

M-theory is described by a matrix supersymmetric quantum mechanics.

The Lagrangian of this theory depends on 10  $N \times N$  Hermitian matrices  $X^\mu, \mu = 0, \dots, 9$  and 16 Hermitian matrices  $\psi^\alpha, \alpha = 1, \dots, 16$  whose entries are of odd Grassmann parity:

$$L = \text{Tr} \left( \frac{1}{2} \nabla_t X^i \nabla_t X_i - \frac{1}{4} [X_i, X_j]^2 + \frac{1}{2} \psi \sigma^0 \nabla_t \psi + \frac{1}{2} \psi^\dagger \sigma^i [\psi, X_i] \right)$$

where  $\nabla_t = \partial_t + X_0$ , the indices  $i, j$  run from 1 to 9.

## Symmetries of M(atrix) Theory

This Lagrangian is invariant with respect to **gauge transformations**

$$\delta X_i = [X_i, g], \quad \delta X_0 = \partial_t g + [X_0, g], \quad \delta \psi^\alpha = [\psi^\alpha, g]$$

where  $g = g(t) \in U(N)$  and with respect to two kinds of **supersymmetry transformations**:

$$\delta_\epsilon X_i = \epsilon \sigma_i \psi$$

$$\delta_\epsilon \psi = \frac{1}{2} [X_i, X_j] \sigma^{ij} \epsilon + [\nabla_t, X_j] \sigma^{0j} \epsilon.$$

and transformations of the second kind

$$\tilde{\delta}_\epsilon X_i = \delta X_0 = 0, \quad \tilde{\delta}_\epsilon \psi = \epsilon.$$

The Lagrangian (1) is also invariant under  $SO(9)$  rotations and under translations

$$X_i \mapsto X_i + c_i \tag{1}$$

where  $c_i$  belong to the center of  $u(N)$ .

## Quantization of M(atrix) Model

To canonically quantize the model fix an orthonormal basis  $T_a \in u(N)$  so that  $f_{abc} = -2\text{Tr}T_a[T_b, T_c]$  are structure constants in this basis.

Canonical operators with respect to this basis:  $\pi_a^i$ ,  $x_i^a$ ,  $\hat{\psi}_a^\alpha$  satisfying the (anti)commutation relations

$$[x_j^a, \pi_b^k] = i\delta_j^k \delta_b^a, \quad \{\hat{\psi}_a^\alpha, \hat{\psi}_b^\beta\} = \delta^{\alpha\beta} \delta_{ab}.$$

The Hamiltonian is

$$\hat{H} = \frac{1}{2} \pi_a^i \pi_a^i + \frac{1}{4} f_{abc} x_i^b x_j^c f_{ade} x_i^d x_j^e - \frac{i}{2} f_{abc} x_a^j \hat{\psi}_\alpha^b \sigma_j^{\alpha\beta} \hat{\psi}_\beta^c.$$

**Spectrum:** One can prove that for each  $N > 1$  the spectrum of this model is continuous, starts from zero and at zero one has a normalizable eigenstate. (The presence of maximal number of supersymmetries is crucial!)

## Scattering in M(atrix) model quantum mechanics.

The expression

$$V = -\frac{1}{4}[X_i, X_j][X^i, X^j] = \frac{1}{4}f_{abc}X_i^b X_j^c f_{ade}X_i^d X_j^e$$

plays the role of potential energy.  $\min(V) = 0$ , the minimum is achieved when the matrices  $X_i$  all commute between themselves.

More generally consider a wave function concentrated on block-diagonal matrices

$$\begin{pmatrix} X_1^i & 0 & 0 & \dots \\ 0 & X_2^i & 0 & \dots \\ 0 & 0 & X_3^i & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where  $X_a^i$  are  $N_a \times N_a$  matrices and  $N_1 + N_2 + \dots + N_n = N$ . Define a relative distance between a pair of blocks according to

$$R_{ab} = \sqrt{\sum_i \left( \frac{\text{Tr} X_a^i}{N_a} - \frac{\text{Tr} X_b^i}{N_b} \right)^2}.$$

In the limit  $R_{ab} \rightarrow \infty$  contribution of the blocks  $X_a, X_b$  to the potential energy vanishes power-like (faster than the second power). Moreover this remains true after one takes into account interaction induced by off-diagonal blocks.

One can set up a scattering problem with the asymptotic wave function

$$\psi_{N_1}(X_1) \otimes \psi_{N_2}(X_2) \otimes \dots \otimes \psi_{N_n}(X_n)$$

a tensor product of  $n$  ground states  $\psi_{N_a}(X_a)$  for each block.

Thus the Hilbert space of the model contains asymptotic multiparticle Fock space to be identified by the **BFSS** conjecture with multigraviton states in  $R^{11}$ .

SUPERSYMMETRY



Classical (pseudo)Riemannian geometry at large distances.

no well-defined positions of gravitons at small separation



(primitive) Noncommutative Geometry

## Compactifications of M(atrix) Theory

Compactification of the "coordinates"  $X^i$  on a  $d$ -dimensional torus formally implies the following set of identifications

$$X^1 \sim X^1 + \mathbf{1} \cdot R_1,$$

$$X^2 \sim X^2 + \mathbf{1} \cdot R_2,$$

⋮

$$X^d \sim X^d + \mathbf{1} \cdot R_d$$

where  $\mathbf{1}$  is the identity matrix and  $R_a \leq 0$ - radii of compactification. (Shifts by constant matrices are symmetries of M(atrix) theory Lagrangian). The identification should be generically understood as equality up to a gauge transformation:

$$U_j X_k U_j^{-1} = X_k + \delta_{kj} 2\pi R_k \mathbf{1}, \quad j, k = 1, \dots, d,$$

$$U_j X_I U_j^{-1} = X_I, \quad I > d,$$

$$U_j \psi^\alpha U_j^{-1} = \psi^\alpha$$

where  $U_i$ ,  $i = 1, \dots, d$  - unitary matrices. By taking a trace of both sides of the above equations we immediately see that they cannot be satisfied by finite matrices unless all  $R_a$  are identically zero.

The above equations imply that  $U_j U_k U_j^{-1} U_k^{-1}$  commute with all  $X_i$ 's. Hence it is natural to assume that

$$U_j U_k = e^{2\pi i \theta^{jk}} U_k U_j$$

where  $\theta^{jk}$  is a constant  $d \times d$  matrix that WLOG can be chosen to be antisymmetric.

- $U_i$ 's generate an algebra of functions on a  $d$ -dimensional noncommutative torus  $T_\theta^d$ .
- A representation of these commutation relations in terms of operators in a Hilbert space specifies a module  $E$  over the algebra  $T_\theta^d$ .
- $X_j$ 's define a connection on  $E$ .
- $X_I$  for  $I > d$  and  $\psi^\alpha$  are endomorphisms of  $E$ .

This description of compactification of M(atric) theory on noncommutative tori was obtained in a seminal paper by [A. Connes](#), [M. Douglas](#) and [A. Schwarz](#) "Noncommutative Geometry and Matrix Theory: compactification on tori", JHEP 02 (1998) 003.

## Super Yang Mills Theory on projective modules

**Yang-Mills theory.** Let  $L$  be a Lie algebra acting on an associative algebra  $\mathcal{A}$  (by means of derivations). Consider a projective module  $E$  over  $\mathcal{A}$  and  $\nabla_X, X \in L$  a connection on  $E$  defined with respect to  $L$ . A curvature  $F_{XY} \in \Lambda^2 L \otimes \text{End}_{\mathcal{A}} E$ :

$$F_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

We assume that our module  $E$  is equipped with a Hermitian  $\mathcal{A}$ -valued inner product  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ . We call a *Yang-Mills field on  $E$*  a connection  $\nabla_X$  compatible with the Hermitian structure, that is a connection satisfying

$$\langle \nabla_X \xi, \eta \rangle_{\mathcal{A}} + \langle \xi, \nabla_X \eta \rangle_{\mathcal{A}} = \delta_X (\langle \xi, \eta \rangle_{\mathcal{A}}).$$

Let  $g_{ij}$  be a metric tensor on  $L$ .  $E$  - projective,  $\mathcal{A}$ -unital  $\Rightarrow \text{End}_{\mathcal{A}} E$  is equipped with a canonically normalized trace. We can write then a functional

$$S(\nabla_i) = \frac{1}{4g^2} \text{Tr} F_{ij} g^{ik} g^{il} F_{kl}$$

-Yang-Mills functional.

Its variation produces Yang-Mills equations of motion

$$[\nabla_i, F^{ij}] = 0$$

( $L$  is abelian here).

## Super-Yang Mills theory.

The (maximally supersymmetric) super-Yang-Mills theory functional depends on a connection  $\nabla_i$  and endomorphisms  $X_I$ ,  $I = d+1, \dots, 9$ ,  $\psi_\alpha$ ,  $\alpha = 1, \dots, 16$  of even and odd Grassmann parity respectively.

$$S(\nabla_i, X_I, \psi_\alpha) = \frac{1}{4g^2} \text{Tr}(F_{jk} F^{jk} + [\nabla_i, X_J][\nabla^i, X^J] + [X_I, X_J][X^I, X^J] - 2\psi^\alpha \sigma_{\alpha\beta}^j [\nabla_j, \psi^\beta] - 2\psi^\alpha \sigma_{\alpha\beta}^J [X_J, \psi^\beta])$$

where  $\sigma_{\alpha\beta}^A$  are Dirac Gamma matrices for  $SO(9,1)$  spinor representation.

Specializing to  $\mathcal{A} = T_\theta^d$  we take  $L$  - a commutative  $d$ -dimensional algebra of translations whose generators  $\delta_i$  act on the torus generators as

$$\delta_i U_j = 2\pi i \delta_{ij} U_j.$$

### Rank of the gauge group:

we say that  $N$  is the rank of the gauge group if the module  $E$  can be represented as a direct sum of  $N$  isomorphic modules  $E = E' \oplus \dots \oplus E'$ . More precisely we should take the largest number  $N$  with this property. It is easy to see then that the algebra of endomorphisms  $\text{End}_{T_\theta} E$  is isomorphic to the matrix algebra  $\text{Mat}_N(\text{End}_{T_\theta} E')$ .

This means that the group of gauge transformations is a subalgebra in this matrix algebra that consists of unitary endomorphisms.

### Supersymmetry transformations of SYM action:

denoted  $\delta_\epsilon$ ,  $\tilde{\delta}_\epsilon$  and defined as

$$\begin{aligned}\delta_\epsilon\psi &= \frac{1}{2}(\sigma^{jk}F_{jk}\epsilon + \sigma^{jI}[\nabla_j, X_I]\epsilon + \sigma^{IJ}[X_I, X_J]\epsilon), \\ \delta_\epsilon\nabla_j &= \epsilon\sigma_j\psi, \quad \delta_\epsilon X_J = \epsilon\sigma_J\psi, \\ \tilde{\delta}_\epsilon\psi &= \epsilon, \quad \tilde{\delta}_\epsilon\nabla_j = 0, \quad \tilde{\delta}_\epsilon X_J = 0.\end{aligned}$$

where  $\epsilon$  is a constant 16-component Majorana-Weyl spinor.

To check that the SYM action is invariant under the above transformations one needs to use the Fierz identities for 10-dimensional Gamma matrices and the identities

$$\text{Tr}[A, B] = 0, \quad \text{Tr}[\nabla_i, A] = 0$$

that hold for any endomorphisms  $A, B$ . The first of these identities is obvious while the last one can be easily proved by noting that it suffices to prove it for a any single connection  $\nabla_i^0$  and then explicitly checking that it holds for the Levi-Civita connection.

Solutions to the SYM equations of motions invariant under half of the supersymmetries are called 1/2 BPS solutions. One finds that such solutions satisfy

$$[\nabla_j, \nabla_k] = 2\pi i f_{jk} \mathbf{1},$$
$$\psi^\alpha = 0, \quad [\nabla_j, X_J] = 0, \quad [X_J, X_K] = 0$$

where  $\mathbf{1}$  is the identity operator and  $f_{jk}$  is a constant anti-symmetric matrix. Thus we see that, as far as gauge fields are concerned, 1/2 BPS configurations correspond to constant curvature connections.

## M(atrix) model compactified on orbifolds of n.c. tori

Let  $D \subset \mathbb{R}^d$  be a  $d$ -dimensional lattice embedded in  $\mathbb{R}^d$  and let  $G$  be a finite group acting on  $\mathbb{R}^d$  by linear transformations mapping the lattice  $D$  to itself. For an element  $g \in G$  we will denote the corresponding representation matrix  $R_i^j(g)$ .

One can write down a formal set of constraints describing compactification of M(atrix) theory on the orbifold  $T^d/G$ , where  $T^d = \mathbb{R}^d/D$ :

$$X_j + \delta_{ij} 2\pi \cdot \mathbf{1} = U_i^{-1} X_j U_i,$$

$$X_I = U_i^{-1} X_I U_i \quad \psi_\alpha = U_i^{-1} \psi_\alpha U_i,$$

$$R_i^j(g) X_j = W^{-1}(g) X_i W(g),$$

$$\Lambda_{\alpha\beta}(g) \psi_\beta = W^{-1}(g) \psi_\alpha W(g),$$

$$X_I = W^{-1}(g) X_I W(g).$$

Here  $i, j = 1, \dots, d$  are indices for directions along the torus,  $I = d + 1, \dots, 9$  is an index corresponding to the transverse directions,  $\alpha$  is a spinor index;  $\Lambda_{\alpha\beta}(g)$  is the matrix of spinor representation of  $G$  obeying  $\Lambda^\dagger(g) \Gamma^i \Lambda(g) = R_{ij}(g) \Gamma_j$ ;  $U_i, W(g)$  - unitary operators.

One can check that the quantities  $U_i U_j U_i^{-1} U_j^{-1}$ ,  $W(gh)W^{-1}(g)W^{-1}(h)$  and  $W^{-1}(g)U_{\mathbf{n}}W(g)U_{R^{-1}(g)\mathbf{n}}^{-1}$  commute with all  $X_i$ ,  $X_I$ , and  $\psi_\alpha$ . It is natural to set them to be proportional to the identity operator.

One can define an algebra of functions on a noncommutative orbifold as an algebra generated by the operators  $U_{\mathbf{n}}$  - linear generators of a noncommutative torus and  $W(g)$  satisfying

$$W^{-1}(g)U_{\mathbf{n}}W(g) = U_{R^{-1}(g)\mathbf{n}},$$

$$W(g)W(h) = W(gh).$$

This construction specifies a crossed product  $T_\theta \rtimes_R G$ . This algebra can be equipped with an involution  $*$  by setting  $U_{\mathbf{n}}^* = U_{-\mathbf{n}}$ ,  $W^*(g) = W(g)$ . A projective module over an orbifold can be considered as a projective module  $E$  over  $T_\theta$  equipped with operators  $W(g)$ ,  $g \in G$  satisfying the above commutation relations.  $X_i$  specifies a  $G$ - *equivariant connection* on  $E$ , i.e.  $X_j = i\nabla_j$  where  $\nabla_j$  is a  $T_\theta$ -connection satisfying

$$R_i^j(g)\nabla_j = W^{-1}(g)\nabla_i W(g).$$

The fields  $X_I$  are endomorphisms of  $E$ , commuting both with  $U_n$  and  $W(g)$  and the spinor fields  $\psi_\alpha$  can be called *equivariant spinors*.

Compactification of M(atrrix) Theory on

orbifolds of n.c. tori



Equivariant Yang-Mills Theory on projective modules

over crossed products of n.c. tori

Above we obtained SYM theory on projective modules over noncommutative tori following the procedure of compactification of M(atric) theory. The last one is a conjectured nonperturbative theory that unifies all known string theories. A different approach to noncommutative spaces within *per-turbative string theory* was put forward by N. Seiberg and E. Witten in their seminal 1999 paper "*String Theory and Noncommutative Geometry*".

The developments that followed that paper brought into consideration classical and quantum field theories on various noncommutative spaces including noncommutative Euclidean spaces  $\mathbb{R}_\theta^d$ , noncommutative tori  $T_\theta^d$  and fuzzy spheres. The construction of classical field theories proceeds via writing down a real numbers valued functional over the set of fields (variables) which can be:

- elements of the algebra itself (scalar fields)
- elements of projective module over the algebra (scalar fields in the fundamental representation of the gauge group)
- connections (Yang-Mills fields)
- endomorphisms (scalar fields in the adjoint representation)

The construction of the action functional uses algebraic operations, algebra-valued inner products on modules and traces defined on the algebra and the algebra of endomorphisms of a given projective module.

For example suppose we have a projective module  $E$  over an involutive unital algebra  $\mathcal{A}$ . One can construct a functional that depends on a number of elements  $\Psi_A \in E$  (sections) and a (Hermitian) connection  $\nabla_i$ :

$$S(\Psi_A, \nabla_i) = \frac{1}{g^2} \text{Tr} \left( \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \sum_A \langle \nabla_j \Psi_A, \nabla^j \Psi_A \rangle_{\mathcal{A}} + V(\langle \Psi_A, \Psi_B \rangle_{\mathcal{A}}) \right)$$

where  $V$  is a polynomial function specifying a potential for scalar fields  $\Psi_A$ .

Once the action functional is written its variation produces equations of motion. One is interested then in finding

- spaces of solutions to the equations of motion modulo gauge transformations
- As the moduli spaces break into connected components according to topological numbers of projective modules one needs to know K-theory ( D-brane charges)
- spaces of solutions that preserve a certain number of supersymmetries. For example constant curvature connections, instanton solutions