NONCOMMUTATIVE GEOMETRY OF THE DISCRETE HEISENBERG GROUP

.

Clay Mathematics Institute / Shanks school in Noncommutative geometry and applications

TOM HADFIELD

UNIVERSITY COLLEGE
CORK, IRELAND

t.hadfield@ucc.ie

Tuesday May 13th, 2003

Introduction

Recently, there has been considerable interest in the idea of a "noncommutative manifold". There are several different approaches to this problem.

Connes' operator-algebraic approach takes as its starting point the commutative algebra $C^\infty(M)$, where M is a compact Riemannian spin manifold, together with the Dirac operator coming from the spin connection, and then gives an axiomatic algebraic formulation which extends to noncommutative algebras.

The most accessible examples of "noncommutative differentiable manifolds" in this sense are provided by the noncommutative tori. Recently Connes' program has enjoyed considerable success.

A second approach, via quantum groups, has been much more examples driven, proceeding with case by case studies of the many interesting algebras arising from quantum groups. Rather than constructing spectral triples, this work has focussed on classifying the possible differential calculi over a given algebra.

This talk falls between these two approaches. We wanted to see how much of Connes' formalism could be extended to the setting of (reduced) group C*-algebras of discrete groups. This talk concerns one specific example - the group C*-algebra of the (three-dimensional) discrete Heisenberg group.

In Connes' picture, a noncommutative geometry over a C*-algebra A consists of a spectral triple (or K-cycle) over A equipped with additional structures. The bounded formulation of spectral triples are Fredholm modules, equivalence classes of which make up the Kasparov K-homology groups $KK^i(A, \mathbf{C})$. The main focus of this work is thus to calculate the K-homology groups and exhibit the generating Fredholm modules.

Also of interest for noncommutative geometry are the derivations and cyclic cohomology of appropriate dense subalgebras of the group C*-algebra.

Analytic K-homology

Let A be a C*-algebra.

The odd K-homology group $K^1(A) = KK^1(A, \mathbf{C})$ consists of equivalence classes of odd Fredholm modules.

An odd Fredholm module over A is a triple (H, π, F), with H a Hilbert space, a *-representation

$$\pi : A \to \mathbf{B}(H)$$

and $F \in \mathbf{B}(H)$ satisfying $F = F^*$, $F^2 = 1$ and

$$[F, \pi(a)] = F\pi(a) - \pi(a)F \in \mathbf{K}(H)$$

for all $a \in A$.

The even K-homology group $K^0(A) = KK^0(A, \mathbf{C})$ consists of equivalence classes of even Fredholm modules (H, π, F, γ).

An even Fredholm module is the same data as the odd case, plus a $\mathbf{Z}_2$-grading γ of the Hilbert space H,

$$\gamma \in \mathbf{B}(H), \quad \gamma = \gamma^*, \quad \gamma^2 = 1$$

that satisfies $[\gamma, \pi(a)] = 0, \quad F\gamma = -\gamma F$

The discrete Heisenberg group

The discrete Heisenberg group H_3 can be defined abstractly as the group generated by elements a and b such that the commutator $c = aba^{-1}b^{-1}$ is central.

$$H_3 = \{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{Z}\}.$$

It is a semidirect product $H_3 \cong \mathbf{Z}^2 \times_\alpha \mathbf{Z}$, with the action α of $\mathbf{Z}$ on $\mathbf{Z}^2$ being given by

$$\alpha^k(m, n) = (m, km + n)$$

H_3 is amenable and nilpotent.

The group C*-algebra $C^*(H_3)$

$C^*(H_3)$ is the universal C*-algebra generated by unitaries U, V and W satisfying

$$VU = WUV$$

with W central.

It is a crossed product $C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$, where the action α of $\mathbf{Z}$ on $C^*(U, W) \cong C(\mathbf{T}^2)$ is given by

$$\alpha(U) = VUV^* = WU, \quad \alpha(W) = VWV^* = W$$

For an irreducible representation

$$\pi : C^*(H_3) \to \mathbf{B}(H)$$

W central, hence $\pi(W) = \lambda I$, for some $\lambda \in \mathbf{C}$, $|\lambda| = 1$, $\lambda = \exp(2\pi i\theta)$.

For each $\theta \in \mathbf{R}$ we have a surjective *-homomorphism

$$q_\theta : C^*(H_3) \to A_\theta$$

$$U \mapsto U, \quad V \mapsto V, \quad W \mapsto \lambda I$$

Hence $C^*(H_3)$ is a continuous field of rotation algebras A_θ over the circle.

Classification of derivations

We classify derivations of a smooth subalgebra A^∞ of the group C*-algebra A.

$$A^\infty = \{\Sigma a_{p,q,r} U^p V^q W^r\}$$

where $\{a_{p,q,r}\} \in S(\mathbf{Z}^3)$, functions on $\mathbf{Z}^3$ vanishing at infinity faster than any polynomial in p, q and r.

A^∞ coincides with Jollisaint's rapid decay algebra ${H^\infty}_L(H_3)$ for the word length function on H_3. Hence H_3 has property (RD) and the algebra A^∞ is closed under holomorphic functional calculus.

Let $Der(A^\infty)$ be the vector space of all derivations

$$\partial : A^\infty \to A^\infty$$

"Derivation" means a linear map ∂ satisfying the Leibnitz rule

$$\partial(ab) = \partial(a)b + a\partial(b)$$

We define fiberwise derivations ∂_1, ∂_2 by

$$\partial_1(U) = U, \quad \partial_1(V) = 0, \quad \partial_1(W) = 0$$

$$\partial_2(U) = 0, \quad \partial_2(V) = V, \quad \partial_2(W) = 0.$$

and extend to A^∞ by linearity. ∂_1 and ∂_2 are unbounded derivations and hence not inner.

We would like a third derivation ∂_3, corresponding to "going round the circle", satisfying

$$\partial_3(W) = W, \quad \partial_3(U) =??, \quad \partial_3(V) =??$$

Then we could make an obvious Dirac operator

$$\partial = \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3$$

where σ_i are appropriate Pauli matrices.

However ..

Proposition: For any derivation $\partial : A^\infty \to A^\infty$, we have $\partial(W) = 0$.

Proposition: $\partial(W) = 0$ for any ∂.

Proof: We have $W = VUV^*U^*$. So

$$\partial(W) = \partial(V)UV^*U^* + V\partial(U)V^*U^* +$$
$$VU\partial(V^*)U^* + VUV^*\partial(U^*).$$

Since U and V are unitaries,

$$\partial(U^*) = -U^*\partial(U)U^*, \quad \partial(V^*) = -V^*\partial(V)V^*.$$

Suppose that

$$\partial(U) = \Sigma a_{p,q,r}U^pV^qW^r, \quad \partial(V) = \Sigma b_{p,q,r}U^pV^qW^r$$

where $\{a_{p,q,r}\}$, $\{b_{p,q,r}\} \in S(\mathbf{Z}^3)$. Then

$$\partial(W) = \Sigma_{p,q,r}[(b_{p,q+1,r-1} - b_{p,q+1,r+q-1}) +$$
$$(a_{p+1,q,r-p+q-1} - a_{p+1,q,r+q-1})]U^pV^qW^r.$$

Terms in W^r have coefficients

$$b_{0,1,r-1} - b_{0,1,r-1} + a_{1,0,r-1} - a_{1,0,r-1} = 0.$$

So all terms in W^r vanish. But $\partial(W)$ must be central, so we must have $\partial(W) = 0$. $\square$

Theorem: As a left $Z(A^\infty)$-module, $Der(A^\infty)$ is spanned by the derivations ∂_1 and ∂_2, together with the inner derivations ∂_x. Any derivation $\partial : A^\infty \to A^\infty$ can be written uniquely as

$$\partial = z_1\partial_1 + z_2\partial_2 + \partial_x$$

for some z_1, $z_2 \in Z(A^\infty)$, and $x \in A^\infty$.

Viewing $A = C^*(H_3)$ as a continuous field of C*-algebras over the circle, this tells us that derivations only act fiberwise - there is no derivation corresponding to "going round the circle". From the viewpoint of derivations, this group C*-algebra does not really look three-dimensional.

K-theory: (Anderson-Paschke) The K-groups of $C^*(H_3)$ are both $\mathbf{Z}^3$.

K-homology: The K-homology groups of $C^*(H_3)$ are

$$K^i(C^*(H_3)) \cong \mathbf{Z}^3, \quad (i = 0, 1)$$

Proof: Rosenberg and Schochet's Universal Coefficient Theorem:
For an algebra A, if the K-groups $K_i(A)$ are both free abelian, then so are the K-homology groups, and
$K^i(A) \cong K_i(A)$. $\square$

Want to identify the generating Fredholm modules.

1. Via the quotient maps $q_\theta : C^*(H_3) \longrightarrow A_\theta$

2. Via the Pimsner-Voiculescu six term cyclic sequence on K-homology for crossed products by $\mathbf{Z}$:

$$C^*(H_3) \cong C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$$

$C^*(H_3)$ is a continuous field of rotation algebras A_θ over the circle. For each $\theta \in \mathbf{R}$, we have a surjective *-homomorphism $q_\theta : C^*(H_3) \to A_\theta$,

$$U \mapsto U, \quad V \mapsto V, \quad W \mapsto \lambda I$$

where $\lambda = \exp(2\pi i\theta)$.

Pull back Fredholm modules over the A_θ to Fredholm modules over $C^*(H_3)$:

$${q_\theta}^* : K^i(A_\theta) \to K^i(C^*(H_3))$$

$$(H, \pi, F) \mapsto (H, \pi \circ q_\theta, F)$$

Sufficient just to do this for $\theta = 0$:

$$q_0 : C^*(H_3) \to A_0 = C(\mathbf{T}^2)$$

$$U \mapsto U, \quad V \mapsto V, \quad W \mapsto I$$

Canonical even Fredholm module

Given a C*-algebra A, and a *-homomorphism

$$\phi : A \to \mathbf{C}$$

Canonical even Fredholm module $\mathbf{z}_0 \in K^0(A)$:

$$\mathbf{z}_0 = (H_0 = \mathbf{C}^2, a \mapsto \begin{pmatrix} \phi(a) & 0 \\ 0 & 0 \end{pmatrix}, F_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}).$$

In general $\mathbf{z}_0$ may be a trivial element of K-homology.

Lemma: If A is unital, and ϕ nonzero, then

$$< ch_*(\mathbf{z}_0), [1] >= 1$$

Canonical odd Fredholm module

Given a unital C*-algebra A, a nonzero *-homomorphism

$$\phi : A \to \mathbf{C}$$

and an automorphism α of A.

The crossed product algebra $A \times_\alpha \mathbf{Z}$ is generated by $a \in A$, together with a unitary V implementing α :

$$VaV^* = \alpha(a)$$

Canonical odd Fredholm module

$$\mathbf{z}_V = (H = l^2(\mathbf{Z}), \pi_V, F) \in K^1(A \times_\alpha \mathbf{Z})$$

$$\pi_V(a)e_n = \phi(\alpha^n(a))e_n$$

$$\pi_V(V)e_n = e_{n+1}$$

π_V is the usual representation of $A \times_\alpha \mathbf{Z}$ induced from the representation ϕ of A.

$$Fe_n = sign(n)e_n = \begin{cases} e_n & : n \geq 0 \\ -e_n & : n < 0 \end{cases}$$

Lemma: $< ch_*(\mathbf{z}_V), [V] >= 1$.

The commutative C*-algebra $C(\mathbf{T}^2)$

Universal C*-algebra generated by commuting unitaries U and V.

$K_0(C(\mathbf{T}^2)) \cong \mathbf{Z}^2$, generated by [1] and $[Bott]$, the Bott projection $Bott \in M_2(C(\mathbf{T}^2))$.

$K_1(C(\mathbf{T}^2)) \cong \mathbf{Z}^2$, generated by $[U]$ and $[V]$.

K-homology:

The even K-homology $K^0(C(\mathbf{T}^2)) \cong \mathbf{Z}^2$ is generated by Fredholm modules $\mathbf{z}_0$ and **Dirac**.

The odd K-homology $K^1(C(\mathbf{T}^2)) \cong \mathbf{Z}^2$ is generated by Fredholm modules $\mathbf{z}_U$ and $\mathbf{z}_V$.

Even K-homology of $C(\mathbf{T}^2)$

$\mathbf{z}_0$ is the canonical even Fredholm module corresponding to the unital *-homomorphism

$$\phi : C(\mathbf{T}^2) \to \mathbf{C}, \quad U, V \mapsto 1$$

$$\mathbf{z}_0 = (H = \mathbf{C}^2, \pi = \phi \oplus 0, F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$$

The Fredholm module **Dirac** is the bounded formulation of the Dirac operator on $\mathbf{T}^2$:

$$\mathbf{Dirac} = (H, \pi, F)$$

$$H = l^2(\mathbf{Z}^2) \oplus l^2(\mathbf{Z}^2)$$

$$\pi = \pi_0 \oplus \pi_0$$

$$\pi_0(U)e_{m,n} = e_{m+1,n}, \quad \pi_0(V)e_{m,n} = e_{m,n+1}$$

$$F = \begin{pmatrix} 0 & F_0 \\ F_0{}^* & 0 \end{pmatrix}$$

$$F_0 e_{m,n} = \frac{m + in}{(m^2 + n^2)^{1/2}} e_{m,n}$$

Odd K-homology of $C(\mathbf{T}^2)$

$C(\mathbf{T}^2)$ is a crossed product $C(\mathbf{T}) \times_{id} \mathbf{Z}$ via a trivial action of $\mathbf{Z}$.

Taking $C(\mathbf{T}) \cong C^*(V)$, and then $C(\mathbf{T}) \cong C^*(U)$, with the trivial $\mathbf{Z}$-action being implemented by U and then V respectively, gives the Fredholm modules

$$\mathbf{z}_U = (l^2(\mathbf{Z}), \pi_U, F),$$
$$\mathbf{z}_V = (l^2(\mathbf{Z}), \pi_V, F)$$

with

$$\pi_U(U) = S, \quad \pi_U(V) = I,$$
$$\pi_V(U) = I, \quad \pi_V(V) = S$$

S is the shift $Se_n = e_{n+1}$, F the diagonal operator

$$Fe_n = sign(n)e_n$$

Identification of K-theory and K-homology

In this situation, we can identify

$$K^i(C(\mathbf{T}^2)) \cong K_i(C(\mathbf{T}^2))$$

via the Fourier transform.

$$K^i(C(\mathbf{T}^2)) \leftrightarrow K_i(C(\mathbf{T}^2))$$
$$\mathbf{z}_0 \leftrightarrow \pm[1]$$
$$\mathbf{z}_U \leftrightarrow \pm[U]$$
$$\mathbf{z}_V \leftrightarrow \pm[V]$$
$$\mathbf{Dirac} \leftrightarrow \pm[Bott]$$

K-homology of the rotation algebras A_θ

First studied - as *Ext* - by Popa and Rieffel

The commutative algebra $C(\mathbf{T}^2)$ corresponds to A_θ for $\theta = 0$. The K-groups (and the K-homology groups) are unchanged by the passage to noncommutativity, they are all $\mathbf{Z}^2$ for all values of θ.

Three of the generators of the K-theory of $A_0 = C(\mathbf{T}^2)$, $[1] \in K_0(A_0)$ and $[U]$, $[V] \in K_1(A_0)$ are still generators of $K_*(A_\theta)$ for $\theta \neq 0$. However the second generator of $K_0(A_\theta)$ is the Powers-Rieffel projection $p \in A_\theta$, which satisfies $\tau(p) = \theta$.

The situation for K-homology is similar. The Fredholm modules $\mathbf{z}_U$, $\mathbf{z}_V$ and **Dirac** are impervious to the transition to noncommutativity. However the canonical even Fredholm module $\mathbf{z}_0$ corresponding to the *-homomorphism

$$A_0 \to \mathbf{C}, \quad U, V \mapsto 1$$

vanishes.

My paper *K-homology of the rotation algebras* A_θ uses Pimsner and Voiculescu's work on AF-embeddings of the A_θ to study this missing generator.

Back to $C^*(H_3)$

Via the quotient map $q_0 : C^*(H_3) \to C(\mathbf{T}^2)$ Fredholm modules over $C(\mathbf{T}^2)$ pull back to Fredholm modules over $C^*(H_3)$.

$$q_0{}^* : K^i(C(\mathbf{T}^2)) \to K^i(C^*(H_3))$$
$$(H, \pi, F) \mapsto (H, \pi \circ q_0, F)$$

The Fredholm module $\mathbf{z}_0$ pulls back to the canonical even Fredholm module $\mathbf{z}_0$ corresponding to the *-homomorphism

$$C^*(H_3) \to \mathbf{C}, \quad U, V, W \mapsto 1$$

The Fredholm modules $\mathbf{z}_U$ and $\mathbf{z}_V$ pull back to Fredholm modules which we also denote $\mathbf{z}_U$, $\mathbf{z}_V$:

$$\mathbf{z}_U = (l^2(\mathbf{Z}), \pi_U, F)$$
$$\pi_U(U) = S, \quad \pi_U(V) = I = \pi_U(W)$$

$$\mathbf{z}_V = (l^2(\mathbf{Z}), \pi_V, F)$$
$$\pi_V(U) = I, \quad \pi_V(V) = S, \quad \pi_V(W) = I$$

The Fredholm module **Dirac** pulls back to a Fredholm module $q_0{}^*(\textbf{Dirac})$ which is a trivial element of K-homology.

Six term cyclic sequence for K-homology of crossed products by Z

Given a crossed product $A \times_\alpha \mathbf{Z}$ there is a short exact sequence of C*-algebras, the Pimsner-Voiculescu "Toeplitz extension"

$$0 \to A \otimes \mathbf{K} \to T_\alpha \to A \times_\alpha \mathbf{Z} \to 0$$

T_α is the C*-subalgebra of $(A \times_\alpha \mathbf{Z}) \otimes T$ generated by $a \otimes 1$, $a \in A$ and $V \otimes f$.

Here V is the unitary implementing the automorphism α, and f is the non-unitary isometry generating the ordinary Toeplitz algebra T,

$$f \in \mathbf{B}(l^2(\mathbf{N})), \quad fe_n = e_{n+1}.$$

This extension defines the Toeplitz element

$$\mathbf{x} \in KK^1(A \times_\alpha \mathbf{Z}, A)$$

Applying the K-functor gives the Pimsner-Voiculescu six term cyclic sequence for K-theory.

Six term cyclic sequence for K-homology

$$\begin{array}{ccccc} K^0(A) & \xleftarrow{id-\alpha^*} & K^0(A) & \xleftarrow{i^*} & K^0(A\times_\alpha\mathbf{Z}) \\ \delta_0\downarrow & & & & \delta_1\uparrow \\ K^1(A\times_\alpha\mathbf{Z}) & \xrightarrow{i^*} & K^1(A) & \xrightarrow{id-\alpha^*} & K^1(A) \end{array}$$

Inclusion map $i : A \hookrightarrow A \times_\alpha \mathbf{Z}$.

The vertical maps δ_0 and δ_1 are given by taking the Kasparov product with the Toeplitz element $\mathbf{x} \in KK^1(A \times_\alpha \mathbf{Z}, A)$:

$$\delta_i : \mathbf{z} \mapsto \mathbf{x}\hat{\otimes}_A\mathbf{z}$$

Let A be a unital C*-algebra with an automorphism α, $\phi : A \to \mathbf{C}$ a nonzero *-homomorphism, and $\mathbf{z}_0$ and $\mathbf{z}_V$ the canonical even and odd Fredholm modules associated to ϕ.

Proposition: Under the map

$$\delta_0 : K^0(A) \to K^1(A\times_\alpha\mathbf{Z})$$

we have $\delta_0(\mathbf{z}_0) = \mathbf{z}_V$.

Proof: The Toeplitz element $\mathbf{x} \in KK^1(A \times_\alpha \mathbf{Z}, A)$ is represented by the Kasparov triple (E_1, π_1, F_1)

$E_1 = l^2(\mathbf{Z}, A)\hat{\otimes}\mathbf{C}_1$ is a Hilbert module over $A\hat{\otimes}\mathbf{C}_1$.

Here $\mathbf{C}_1$ is the Clifford algebra of the 1-dimensional complex vector space, generated by elements 1, ϵ, with $\epsilon^2 = 1$, and $\hat{\otimes}$ is the graded tensor product.

$$\pi_1(x)(\xi\hat{\otimes}\omega) = (\pi_1{}'(x)\xi)\hat{\otimes}\omega$$

for $x \in A \times_\alpha \mathbf{Z}$, where

$$\pi_1{}'(a)\xi(n) = \alpha^n(a)\xi(n)$$

$$\pi_1{}'(V)\xi(n) = \xi(n+1)$$

for $a \in A$, $\xi \in l^2(\mathbf{Z}, A)\hat{\otimes}\mathbf{C}_1$.

$F_1 = F\hat{\otimes}\epsilon$, with $(F\xi)(n) = sign(n)\xi(n)$

Calculation of the product $\mathbf{x}\hat{\otimes}_A\mathbf{z}_0$ comes down to careful book-keeping. $\square$

For the discrete Heisenberg group we have $A = C(\mathbf{T}^2)$, generated by commuting unitaries U and W. The automorphism α acts by

$$\alpha(U) = WU, \quad \alpha(W) = W.$$

Then $A \times_\alpha \mathbf{Z} \cong C^*(H_3)$.

$$\begin{array}{ccccc}
K^0(A) & \xleftarrow{id-\alpha^*} & K^0(A) & \xleftarrow{i^*} & K^0(C^*(H_3)) \\
\delta_0 \downarrow & & & & \delta_1 \uparrow \\
K^1(C^*(H_3)) & \xrightarrow{i^*} & K^1(A) & \xrightarrow{id-\alpha^*} & K^1(A)
\end{array}$$

Note that we are now considering $A = C(\mathbf{T}^2) = C^*(U, W)$ as a subalgebra, rather than a quotient, of $C^*(H_3)$.

$K^0(A)$ is generated by Fredholm modules $\mathbf{z}_0$ and **Dirac**.

$K^1(A)$ is generated by Fredholm modules $\mathbf{z}_U$ and $\mathbf{z}_W$.

Highlights of a diagram chase

1. For the map $(id-\alpha^*) : K^1(A) \to K^1(A)$ we have

$$(id-\alpha^*)(\mathbf{z}_U) = \mathbf{0}, \quad (id-\alpha^*)(\mathbf{z}_W) = -\mathbf{z}_U.$$

2. $(id-\alpha^*) : K^0(A) \to K^0(A)$ is the zero map.
In particular $\alpha^*(\textbf{Dirac}) = \textbf{Dirac}$ as elements of $K^0(A)$.

3. The image of $\delta_0 : K^0(A) \to K^1(C^*(H_3))$ is a copy of $\mathbf{Z}^2$, and $\delta_0(\mathbf{z}_0) = \mathbf{z}_V$.

4. $K^1(C^*(H_3))$ is generated (as a $\mathbf{Z}$-module) by the Fredholm modules $\mathbf{z}_U$, $\mathbf{z}_V$ and $\delta_0(\textbf{Dirac})$.

5. For $i^* : K^1(C^*(H_3)) \to K^1(A)$, we have

$$i^*(\mathbf{z}_V) = \mathbf{0}, \quad i^*(\mathbf{z}_U) = \mathbf{z}_U$$

Highlights ... continued

6. For the map $\delta_1 : K^1(A) \to K^0(C^*(H_3))$ we have $\delta_1(\mathbf{z}_U) = \mathbf{0}$.
The Fredholm module $\delta_1(\mathbf{z}_W)$ is a nontrivial element of $K^0(C^*(H_3))$ not in the span of $\mathbf{z}_0$.

7. The map $i^* : K^0(C^*(H_3)) \to K^0(A)$ is surjective. Let $\mathbf{Dirac}'$ be a generator of $K^0(C^*(H_3))$ satisfying $i^*(\mathbf{Dirac}') = \mathbf{Dirac}$.

8. The Fredholm modules $\mathbf{z}_0$, $\mathbf{Dirac}'$ and $\delta_1(\mathbf{z}_W)$ generate $K^0(C^*(H_3)) \cong \mathbf{Z}^3$.

The Fredholm modules $\mathbf{z}_0$, $\mathbf{Dirac}'$ and $\delta_1(\mathbf{z}_W)$ generate $K^0(C^*(H_3)) \cong \mathbf{Z}^3$.

Even case	$\mathbf{z}_0$	$\mathbf{Dirac}'$	$\delta_1(\mathbf{z}_W)$
$[1]$	1	0	0
$[P_a]$	1	1	0
$[P_b]$	1	0	1

The Fredholm modules $\mathbf{z}_U$, $\mathbf{z}_V$ and $\delta_0(\mathbf{Dirac})$ generate $K^1(C^*(H_3)) \cong \mathbf{Z}^3$.

Odd case	$\mathbf{z}_U$	$\mathbf{z}_V$	$\delta_0(\mathbf{Dirac})$
$[U]$	1	0	0
$[V]$	0	1	0
$[V_a]$	0	1	1

For $C^*(H_3)$ the pairing between K-theory and K-homology via the Chern character is faithful.

Correspondence between K-theory and K-homology

$$K^i(C^*(H_3)) \leftrightarrow K_i(C^*(H_3))$$

$$\mathbf{z}_0 \leftrightarrow [1]$$

$$\mathbf{z}_U \leftrightarrow [U]$$
$$\mathbf{z}_V \leftrightarrow [V]$$

$$\mathbf{Dirac}' \leftrightarrow [P_a]$$
$$\delta_1(\mathbf{z}_W) \leftrightarrow [P_b]$$

$$\delta_0(\mathbf{Dirac}) \leftrightarrow [V_a]$$

We have been able to give a detailed description of the even and odd K-homology of $C^*(H_3)$.

The Fredholm module $\delta_0(\mathbf{Dirac})$ is our candidate for the fundamental class of this "noncommutative manifold".

If we were to try to build a three-dimensional noncommutative geometry over $C^*(H_3)$, we would start with the corresponding spectral triple and try to find the required extra structures.

It is not clear whether in general this "fundamental class" should exist at all, or whether it should be unique.

References

1. *The noncommutative geometry of the discrete Heisenberg group.*
Houston Journal of Mathematics, Vol 29, No. 2 (2003).
#arXiv math.OA/0201093

2. *K-homology of the rotation algebras A_θ.*
To appear in the Canadian Journal of Mathematics.
#arXiv math.OA/0112235