Asymptotic Kasparov cycles and applications

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Aim: construct a bivariant K-theory for separable G-C^*-algebras, denoted KE-*theory*, which

- is well adapted to (the index theory of) elliptic operators on manifolds
- is closely related to KK-theory but in some circumstances has a simpler product
- the Bott and Dirac elements of Higson and Kasparov have simple classes in the KE-groups and their products can be computed directly; shed a different light on the proof of the Baum-Connes conjecture for groups with the Haagerup property

It turned out to be an intermediate theory:

$$\mathbf{KK} \longrightarrow \mathbf{KE} \longrightarrow \mathbf{E}$$

Continuous fields of Hilbert modules

$L = [1, \infty)$

$BL = C_0(L, B) = C_0(L) \otimes B$, for any C^*-algebra B

Definition. Given two C^*-algebras A and B, a *continuous field of* (A, B)*-bimodules* is a Hilbert BL-module $\mathcal{E}$, admitting an action $\varphi : A \to \mathcal{L}(\mathcal{E})$.

Note. Such a continuous field $\mathcal{E}$ may be thought of as a family $\{ \mathcal{E}_t \}_{t \in [1,\infty)}$ of Hilbert (A, B)-bimodules (by 'evaluation at t'). An operator F in $\mathcal{L}(\mathcal{E})$ may be thought of as a family $\{ F_t \}_{t \in [1,\infty)}$.

$$\mathcal{E}_t = \mathcal{E} \otimes_{ev_t} B \quad \text{and} \quad F_t = F \otimes_{ev_t} 1.$$

Notation. $(\mathcal{E}, \varphi, F) = \{ (\mathcal{E}_t, \varphi_t, F_t) \}_{t \in [1,\infty)}$.

Note. If $\{\mathcal{E}_t\}_t$=$\{\mathcal{E}_\bullet\}_t$= constant family, then the family $\{ F_t \}_t$ is **-strongly continuous in* $\mathcal{L}(\mathcal{E}_\bullet)$, meaning that for every $\xi \in \mathcal{E}_\bullet$ the functions

$$t \mapsto F_t(\xi) \quad \text{and} \quad t \mapsto F_t^*(\xi)$$

are continuous.

Some useful ideals

Let $\mathcal{E}$ be a Hilbert $\mathbb{B}\mathbb{L}$-module. Consider two ideals:

The *locally compact-valued families of operators*:

$$\mathcal{C}(\mathcal{E}) = \left\{ F \in \mathcal{L}(\mathcal{E}) \;\middle|\; F f \in \mathcal{K}(\mathcal{E}), \text{ for all } f \in C_0(L) \right\}.$$

The *vanishing families of operators*:

$$\mathcal{J}(\mathcal{E}) = \left\{ F \in \mathcal{L}(\mathcal{E}) \;\middle|\; \lim_{t\to\infty} \|F_t\| = 0 \right\}.$$

$$\begin{tikzcd}
 & \mathcal{L}(\mathcal{E}) & \\
\mathcal{C}(\mathcal{E}) \arrow[ur] & & \mathcal{J}(\mathcal{E}) \arrow[ul] \\
 & \mathcal{K}(\mathcal{E}) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Asymptotic Kasparov modules

Definition. Let G be a group; A and B be G-C^*-algebras. An *asymptotic Kasparov* G-(A, B)*-module* is a triple $(\mathcal{E}, \varphi, F)$, where:

- $\mathcal{E}$ is a continuous field of G-(A, B)-modules,
- $\varphi : A \to \mathcal{L}(\mathcal{E})$ is the left-action $*$-morphism,
- $F \in \mathcal{L}(\mathcal{E})$ is an odd self-adjoint operator satisfying:

(aKm1) (asymptotic commutativity with action of A)

$$[F, \varphi(a)] \in \mathcal{J}(\mathcal{E}), \text{ for all } a \in A;$$

(aKm2) (positivity)

$$\varphi(a)\,(\,F^2 - 1\,)\,\varphi(a)^* \geq 0, \text{ modulo } \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E});$$

(aKm3) (asymptotic equivariance)

$$\big(g(F) - F\big)\,\varphi(a) \in \mathcal{J}(\mathcal{E}), \text{ for all } a \in A.$$

Examples

Example. Let M^{2n} be a complete, spin^c-manifold, with spinor bundle $\mathbb{S} = \mathbb{S}_M$, and Dirac operator $D = D_M$. The *fundamental asymptotic Kasparov* $(C_0(M), \mathbb{C})$*-module* is given by

$$\left\{ \left(L^2(M, \mathbb{S}),\ \varphi,\ \chi(\tfrac{1}{t}D) \right) \right\}_{t \in [1,\infty)}$$

where $\varphi : C_0(M) \to \mathcal{L}(L^2(M,\mathbb{S}))$, $\varphi(f) = M_f$, and χ is a normalizing function.

○ F *satisfies* (aKm2). This follows from ellipticity. Let:

$$f(x) = (x \pm i)^{-1} \in C_0(\mathbb{R}), g \in C_c^\infty(M).$$

Then

$$L^2(M,\mathbb{S}) \xrightarrow[\text{Gårding}]{M_g(\frac{1}{t}D \pm i\mathrm{Id})^{-1}} H^1_{\text{supp}(g)}(M,\mathbb{S}) \xrightarrow[\text{Rellich}]{\text{cpct}} L^2(M,\mathbb{S})$$

shows that $M_g\, f(\frac{1}{t}D) \in \mathcal{K}(L^2(M,\mathbb{S})$, for all f, g, t.

∘ F *satisfies* (aKm1). Here Fourier analysis and the properties of wave operator come in handy. Choose χ s.t. $\widehat{\chi}$ is compactly supported, and $s \mapsto s\,\widehat{\chi}(s)$ is smooth. Then:

$$\langle\, \chi(D)u, v\,\rangle = \int_{\mathbb{R}} \langle\, e^{isD}u, v\,\rangle\, \widehat{\chi}(s)\, ds, \text{ for } u, v \in C_c^\infty(M, \mathbb{S}).$$

For $f \in C^\infty(M, S^1)$ (*i.e.* M_f is an unitary operator):

$$\langle [\chi(\tfrac{1}{t}D), f]u, v\rangle = \int_{\mathbb{R}} \langle \big(e^{ist^{-1}f^{-1}Df} - e^{ist^{-1}D}\big)u, \overline{f}v\rangle \widehat{\chi}(s)\, ds$$

$$\Downarrow \; (\|e^{isT_1} - e^{isT_2}\| \leq |s|\; \|T_1 - T_2\|)$$

$$\|\, [\chi(\tfrac{1}{t}D), f]\, \| \leq \tfrac{1}{t}\, \|\nabla f\| \int_{\mathbb{R}} |s\, \widehat{\chi}(s)|\, ds.$$

(see N. Higson and J. Roe, *Analytic* K-*homology*)

KE-theory groups

Definition. The set $\mathrm{KE}_G(A, B)$ is defined as the homotopy equivalence classes of asymptotic Kasparov G-(A, B)-modules. The *addition* of two asymptotic Kasparov (A, B)-modules is

$$[(\mathcal{E}_1, \varphi_1, F_1)] + [(\mathcal{E}_2, \varphi_2, F_2)] = [(\mathcal{E}_1 \oplus \mathcal{E}_2, \varphi_1 \oplus \varphi_2, F_1 \oplus F_2)].$$

Theorem. $\mathrm{KE}_G(A, B)$ *is an abelian group.*

An index theory implication

Theorem. [D] $\mathrm{KE}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{Z}$*, the isomorphism still being given by the Fredholm index.*

Consequence. The Fredholm index is invariant for *-strongly continuous families of operators.

The associative product in KE-theory

Theorem. [D] *There is an* associative product*:*

$$
\begin{aligned}
KE_G(A, D) \otimes KE_G(D, B) &\xrightarrow{\ \sharp_D\ } KE_G(A, B), \\
(x, y) &\longrightarrow x \,\sharp_D\, y \,.
\end{aligned}
$$

In terms of families the external product

$$KE_G(A_1, B_1) \otimes KE_G(A_2, B_2) \to KE_G(A_1 \otimes A_2, B_1 \otimes B_2)$$

$$(\,(\mathcal{E}_1, F_1), (\mathcal{E}_2, F_2)\,) \mapsto (\mathcal{E}, F)$$

reads:

$$
\begin{aligned}
\mathcal{E} &= \{\ \mathcal{E}_{1,t} \otimes \mathcal{E}_{2,t}\ \}_t, \\
F &= \{\ F_{1,t} \otimes 1 + 1 \otimes F_{2,t}\ \}_t.
\end{aligned}
$$

The general case is more complicated.

The map $\mathrm{KK} \to \mathrm{KE}$

Theorem. [D] *Let* A *and* B *be* G-C^**-algebras. Consider* $(\mathcal{E}, \varphi, F) \in kk_G(A, B)$*, and let* $u = \{u_t\}_t$ *be a quasi-central quasi-invariant approximate unit of* $\mathcal{K}(\mathcal{E})$*. The map*

$$\begin{aligned} \Theta\ &: kk_G(A,B) \longrightarrow ke_G(A,B), \\ &\big(\, \mathcal{E}, \varphi, F \,\big) \mapsto \Big\{ \, \big(\mathcal{E}, \varphi, (1-u_t)F(1-u_t)\big) \, \Big\}_{t \in [1,\infty)}, \end{aligned}$$

gives a morphism $\Theta : KK_G(A,B) \to KE_G(A,B)$*.*

Theorem. [D] $\Theta : \mathbf{KK} \longrightarrow \mathbf{KE}$ *is a functor: given* $x \in kk_G(A, D)$*,* $y \in kk_G(D, B)$ *then*

$$\Theta(\, \boldsymbol{x} \sharp_D \boldsymbol{y} \,) = \Theta(\boldsymbol{x}) \sharp_D \Theta(\boldsymbol{y}) \, .$$

Corollary. (a) $\Theta(1_A) = 1_A$.
(b) A KK-equivalence $\boldsymbol{x} \in KK_G(A, B)$ gives, by multiplication with $\Theta(\boldsymbol{x})$, an isomorphism between $KE_G(D, A)$ and $KE_G(D, B)$.
(c) KE-theory satisfies Bott periodicity.

a-T-menable groups

Definition. G is an *a-T-menable group* if it admits an affine, isometric, metrically proper action on a separable real Hilbert space:

$$g \cdot v = \pi(g)v + b(g).$$

Examples.

- no noncompact property (T) groups are a-T-menable
- the amenable groups
- the free groups
- the isometry groups of real and complex hyperbolic spaces
- the (finitely generated) Coxeter groups

etc...

The Bott-Dirac operator

(The constructions are due to Higson & Kasparov.)

A. The finite dimensional case ($V = \mathbb{R}^n$**)**
Define:

$$\boxed{\mathcal{H}(V) = L^2(V, \wedge^*(V))}$$

For $v \in V$, let:

$\mathrm{ext}(v) =$ exterior multiplication by v on $\wedge^*(V)$

$\mathrm{int}(v) =$ interior multiplication by v on $\wedge^*(V)$

$c_\pm(v) = \mathrm{ext}(v) \pm \mathrm{int}(v).$

Definition. The *Bott-Dirac operator on* V is

$$B_{\alpha,V} : \mathfrak{s}(V) \to \mathfrak{s}(V),$$

$$\boxed{B_{\alpha,V} = \sum_{i=1}^{n} \alpha\, c_-(v_i)\, \frac{\partial}{\partial x_i} + c_+(v_i)\, x_i.}$$

B. The infinite dimensional case

Fix $\alpha \in (0, \infty)$. For any linear subspace W of H, let

$$\xi_{\alpha,W}(w) = \text{constant} \cdot \exp(-\|w\|^2/2\alpha), w \in W.$$

For $V' \subset V$ this gives an isometry $\mathcal{H}(V') \hookrightarrow \mathcal{H}(V)$.

Definition. Consider $V_1 \subset V_2 \subset V_3 \subset \ldots H$, linear subspaces with dense union. Let

$$\boxed{\mathcal{H}_\alpha(H) = \varinjlim \mathcal{H}(V_n)}$$

If $H = W_1 \oplus W_2 \oplus \ldots$, the *Bott-Dirac operator on* H is

$$\boxed{B_\alpha = B_{\alpha,W_1} + B_{\alpha,W_2} + \ldots.}$$

It is an essentially self-adjoint operator, with one dimensional kernel.

An application to the Baum-Connes conjecture

Let G be an a-T-menable group acting on H. For $t \in [1, \infty)$, construct the Hilbert spaces $\mathcal{H}_t$ and the operators B_t, as explained above. The action of G on $\mathcal{H}_t$ is $g_t \cdot v = \pi(g)v + \frac{1}{t}b(g)$. Consider $h = \text{diag}(2, 3, 4, \ldots)$ on $V_1 \subset V_2 \subset V_3 \subset \ldots H$, a self-adjoint operator with compact resolvent. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a normalizing function.

Theorem. [D] *The family*

$$\{ (\mathcal{H}_t , \chi(h(B_t))) \}_{t \in [1,\infty)}$$

is an asymptotic Kasparov G-$(\mathbb{C}, \mathbb{C})$*-module. Its class in* $KE_G(\mathbb{C}, \mathbb{C})$ *is* $1_{\mathbb{C}}$*, the unit of the ring.*