SPECTRAL MEASURES OF SMALL INDEX PRINCIPAL GRAPHS

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Abstract. The principal graph X of a subfactor with finite Jones index is one of the important algebraic invariants of the subfactor. If Δ is the adjacency matrix of X we consider the equation $\Delta = U + U^{-1}$. When X has square norm ≤ 4 the spectral measure of U can be averaged by using the map $u \to u^{-1}$, and we get a probability measure ε on the unit circle which does not depend on U. We find explicit formulae for this measure ε for the principal graphs of subfactors with index ≤ 4, the (extended) Coxeter-Dynkin graphs of type A, D and E. The moment generating function of ε is closely related to Jones' Θ-series.

Introduction

The Coxeter-Dynkin graphs of type A, D_{even}, E_6, E_8, and the extended Coxeter-Dynkin graphs of type ADE appear in the theory of subfactor as basic invariants for inclusions of II_1 factors with Jones index ≤ 4 ([6], [10], [11], [12], see also [4], [5]). They are fusion graphs of subfactor representations and capture the algebraic information contained in the standard invariant of the subfactor $N \subset M$ (see e.g. [2], [12], [4], [5]). Such a graph X is bipartite and has a distinguished vertex 1. Of particular interest in subfactor theory are the number of length $2k$ loops on X based at 1, since these are the dimensions of the higher relative commutants associated to $N \subset M$. If combined in a formal power series $f(z)$ we obtain the Poincaré series of X.

A related series $\Theta(u)$, with $z^{1/2} = u + u^{-1}$, is considered by Jones in [8]. Jones made the remarkable discovery that if the subfactor has index > 4 then the coefficients of this series are necessarily *positive* integers. The series f and Θ are natural invariants of the subfactor. In this paper we compute explicitly measures whose generating series of moments are (essentially) these two power series in the case the graphs have square norm ≤ 4. These measures can then be regarded as invariants of the subfactors.

For a graph X of square norm ≤ 4 we have the following measure-theoretic version of Θ. Consider the equation $\Delta = U + U^{-1}$, where Δ is the adjacency matrix of X. We can average the spectral measure of U by using the map $u \to u^{-1}$, and we get a probability measure ε on the unit circle which does not depend on U.

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We find that for the (extended) Coxeter-Dynkin graphs of type A and D this measure ε is given by very simple formulae as follows:

$$\begin{array}{rcl} A_{n-1} & \to & \alpha\, d_n \\ D_{n+1} & \to & \alpha\, d'_n \\ A_\infty & \to & \alpha\, d \\ A_{2n}^{(1)} & \to & d_n \\ A_{-\infty,\infty} & \to & d \\ D_{n+2}^{(1)} & \to & (d'_1 + d_n)/2 \\ D_\infty & \to & (d'_1 + d)/2 \end{array}$$

Here d, d_n, d'_n are the uniform measures on the unit circle, the uniform measure supported on the $2n$-th roots of unity, and the uniform measure supported on the $4n$-th roots of unity of odd order. The fundamental density α is given by $\alpha(u) = 2Im(u)^2$ and corresponds via $x = u + u^{-1}$ to the semicircle law from [14].

For the graphs of type E_6, E_7, E_8, $E_6^{(1)}$, $E_7^{(1)}$, and $E_8^{(1)}$, ε is given by the following formulae:

$$\begin{array}{rcl} E_6 & \to & \alpha\, d_{12} + (d_{12} - d_6 - d_4 + d_3)/2 \\ E_7 & \to & \varepsilon_7 \\ E_8 & \to & \varepsilon_8 \\ E_6^{(1)} & \to & \alpha\, d_3 + (d_2 - d_3)/2 \\ E_7^{(1)} & \to & \alpha\, d_4 + (d_3 - d_4)/2 \\ E_8^{(1)} & \to & \alpha\, d_6 + (d_5 - d_6)/2 \end{array}$$

Here ε_7, ε_8 denote certain exceptional measures for which we do not have closed formulae (we do compute their moment generating series though). These results are obtained by explicit computations. They could also be obtained by using planar algebras methods, see [8], [9], [13] for related work.

The measure ε computed in this article should be viewed as an analytic invariant for subfactors of index ≤ 4. It is unclear what the appropriate generalization of ε to subfactors of index > 4 should be. However, in light of Jones' work in [8], the measure ε should be related to certain representations of planar algebras. It should shed some light on the structure of subfactor planar algebras (or equivalently standard invariants) arising from subfactors with index > 4. We intend to come back to this question in future work.

Similar considerations make sense for quantum groups. The hope would be that an analytic invariant for quantum groups would emerge from the Weingarten formula in [1], and work here is in progress. This in turn might be related to the results in the present article via Di Francesco's formula in [3].

The paper is dedicated to the proofs of the above results and is organized as follows. Section 1 fixes the notation and contains some preliminaries. In sections 2, 3 and 4 we divide the graphs of type A and D into three classes – circulant graphs, graphs with A_n tails, and graphs with fork tails – and we compute ε. In sections 5, 6 and 7 we discuss the graphs of type E using a key lemma in section 3.

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1. Spectral measures and the Jones series

We collect in this section several known results about spectral measures associated to graphs and their Stieltjes transforms. We relate them to a natural power series discovered by Jones in the context of planar algebras and their representation theory ([7], [8]).

Let X be a (possibly infinite) bipartite graph with distinguished vertex labeled by "1". Since X is bipartite, its adjacency matrix Δ is given by

$$\Delta = \begin{pmatrix} 0 & M \\ M^t & 0 \end{pmatrix}$$

where M is a rectangular matrix with non-negative integer entries. If we let $L = MM^t$ and $N = M^tM$, then

$$\Delta^2 = \begin{pmatrix} L & 0 \\ 0 & N \end{pmatrix}$$

For a matrix T with entries labeled by the vertices of X, we use the following notation ("1" is the label of the distinguished vertex):

$$\int T = T_{11}$$

We call T_{11} the integral of the matrix T.

Definition 1.1. *The spectral measure of X is the probability measure μ on $\mathbb{R}$ satisfying*

$$\int_{-\infty}^{\infty} \varphi(x)\, d\mu(x) = \int \varphi(\Delta)$$

for any continuous function $\varphi : \mathbb{R} \to \mathbb{C}$.

Note that the spectral measure of Δ can be regarded as an invariant of X. The spectral measure is uniquely determined by its moments. The generating series of these moments is called the Stieltjes transform σ of μ, i.e.

$$\sigma(z) = \int_{-\infty}^{\infty} \frac{1}{1 - zx}\, d\mu(x).$$

This is related to the Poincaré series of X, which appears as the generating function of the numbers $\mathrm{loop}(2k)$, counting loops of length $2k$ on X based at 1. The Poincaré series of X is defined as

$$f(z) = \sum_{k=0}^{\infty} \mathrm{loop}(2k) z^k.$$

We have then the following well-known result.

Proposition 1.1. *Let X be a bipartite graph with spectral measure μ and Poincaré series f. The Stieltjes transform σ of μ is given by $\sigma(z) = f(z^2)$.*

Proof. We compute $\sigma(z)$ using the fact that the integral of Δ^l is loop(l).

$$\sigma(z) = \int \frac{1}{1 - z\Delta} = \sum_{l=0}^{\infty} \mathrm{loop}(l) z^l$$

Since loop$(l) = 0$ for l odd, we have

$$f(z^2) = \sum_{k=0}^{\infty} \mathrm{loop}(2k) z^{2k} = \sum_{l=0}^{\infty} \mathrm{loop}(l) z^l.$$

This proves the statement. □

We assume now that the graph X has norm ≤ 2, that is the matrix Δ has norm ≤ 2. Thus the support of μ, which is contained in the spectrum of Δ, is contained in $[-2, 2]$.

Let $\mathbb{T}$ be the unit circle, and consider the following map $\Phi : \mathbb{T} \to [-2, 2]$, defined by $\Phi(u) = u + u^{-1}$. Any probability measure ε on $\mathbb{T}$ produces a probability measure $\mu = \Phi_*(\varepsilon)$ on $[-2, 2]$, according to the following formula, valid for any continuous function $\varphi : [-2, 2] \to \mathbb{C}$:

$$\int_{\mathbb{R}} \varphi(x)\, d\mu(x) = \int_{\mathbb{T}} \varphi(u + u^{-1})\, d\varepsilon(u)$$

We can obtain in this way all probability measures on $[-2, 2]$. Given μ, there is a unique probability measure ε on $\mathbb{T}$ satisfying $\Phi_*(\varepsilon) = \mu$ with the normalisation $d\varepsilon(u) = d\varepsilon(u^{-1})$.

Definition 1.2. *The spectral measure of X (on $\mathbb{T}$) is the probability measure ε on $\mathbb{T}$ given by*

$$\int_{\mathbb{T}} \varphi(u + u^{-1})\, d\varepsilon(u) = \int \varphi(\Delta)$$

for any continuous function $\varphi : [-2, 2] \to \mathbb{C}$, with the normalisation $d\varepsilon(u) = d\varepsilon(u^{-1})$.

The generating series of the moments of ε (the Stieltjes transform) is given by

$$S(q) = \int_{\mathbb{T}} \frac{1}{1 - qu}\, d\varepsilon(u).$$

Following Jones ([7], [8]), given a subfactor planar algebra ($P = (P_0^{\pm}, (P_k)_{k\geq 1})$ with parameter δ, the associated Poincaré series is defined as

$$f(z) = \frac{1}{2}(\dim P_0^+ + \dim P_0^-) + \sum_{k=1}^{\infty} \dim P_k z^k.$$

Jones introduced in [8] an associated series Θ, which is essentially obtained from the Poincaré series by a change of variables $z \to \frac{q}{1+q^2}$. In this paper we will call this series the *Jones series.* If $\delta > 2$, then the Jones series is the dimension generating function for the multiplicities of certain Temperley-Lieb modules which appear in the decomposition of the planar algebra P viewed as a module for the Temperley-Lieb planar algebra ([8], see also [9], [13]). Using the formula for the Θ-series from [8] we define the *Jones series of the graph* X by

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

where $f(z)$ is the Poincaré series of X. With these notations, we have then the following result.

Proposition 1.2. *Let X be a bipartite graph with spectral measure ε (on $\mathbb{T}$) and Jones series Θ. The Stieltjes transform of ε is given by $2S(q) = \Theta(q^2) - q^2 + 1$.*

Proof. We compute S in terms of ε.

$$\begin{aligned} 2S(q) &= \int_{\mathbb{T}} \frac{1}{1-qu}\, d\varepsilon(u) + \int_{\mathbb{T}} \frac{1}{1-qu^{-1}}\, d\varepsilon(u) \\ &= 1 + \int_{\mathbb{T}} \frac{1-q^2}{1-q(u+u^{-1})+q^2}\, d\varepsilon(u) \end{aligned}$$

We compute now Θ in terms of μ.

$$\begin{aligned} \Theta(q^2) - q^2 &= \frac{1-q^2}{1+q^2} f\left(\frac{q^2}{(1+q^2)^2}\right) \\ &= \int_{-\infty}^{\infty} \frac{1-q^2}{1-qx+q^2}\, d\mu(x) \end{aligned}$$

This formula in the statement follows now from the definition of ε. □

In the next sections we compute explicitly the spectral measure (on $\mathbb{T}$) for graphs which appear in the classification of subfactors with index ≤ 4 (the so-called *principal graphs*). We use standard notation for these graphs (see for instance [5]).

2. Circulant graphs

Consider the circulant graph $A_{2n}^{(1)}$, that is $A_{2n}^{(1)}$ is the $2n$-gon, and choose any vertex as the distinguished vertex 1.

Theorem 2.1. *Let X be the circulant graph $A_{2n}^{(1)}$. The spectral measure of X (on $\mathbb{T}$) is given by*

$$d\varepsilon(u) = d_n u$$

where d_n is the uniform measure on $2n$-th roots of unity.

Proof. We identify X with the group $\{w^k\}_{0\leq k\leq 2n-1}$ of $2n$-th roots of unity, where $w = e^{i\pi/n}$. The adjacency matrix of X acts on functions $f \in \mathbb{C}(X)$ in the following way.

$$\Delta f(w^s) = f(w^{s-1}) + f(w^{s+1})$$

Consider the following operators U and U^{-1}:

$$Uf(w^s) = f(w^{s+1})$$
$$U^{-1}f(w^s) = f(w^{s-1})$$

We have $\Delta = U + U^{-1}$. The moments of the spectral measure of U are obtained as follows:

$$\int U^k =< U^k\delta_1, \delta_1 >=< \delta_{w^k}, \delta_1 >= \delta_{w^k,1}$$

We compute now the moments of the measure in the statement.

$$\int_{\mathbb{T}} u^k \, d\varepsilon(u) = \int_{\mathbb{T}} u^k \, d_n u = \delta_{w^k,1}$$

Thus ε is the spectral measure of U, and together with the identity $d\varepsilon(u) = d\varepsilon(u^{-1})$, we get the result. □

The graph $A_{-\infty,\infty}$ is the set $\mathbb{Z}$ with consecutive integers connected by edges. Choose any vertex as the distinguished vertex labeled 1.

Theorem 2.2. *Let X be the graph $A_{-\infty,\infty}$. The spectral measure of X (on $\mathbb{T}$) is given by*

$$d\varepsilon(u) = du$$

where du is the uniform measure on the unit circle.

Proof. This follows from Theorem 2.1 by letting $n \to \infty$, or from a direct loop count. □

3. Graphs with tails

The Coxeter-Dynkin graph of type A_n, $n \geq 2$, has n vertices and the distinguished vertex 1 labels a vertex at one end of the graph, i.e. A_n is bipartite graph of the form

$$1 - \alpha_1 - 2 - \alpha_2 - 3 - \alpha_3 \ ::: \ \beta$$

where 1, 2, 3, ... labels the even vertices and α_1, α_2, ... labels the odd vertices. β is either even or odd depending on the parity of n.

Consider a sequence of graphs X_k obtained by adding A_{2k} tails to a finite graph Γ. We let

$$X_0 = 1 \ ::: \ \Gamma$$
$$X_1 = 1 - \alpha - 2 \ ::: \ \Gamma$$
$$X_2 = 1 - \alpha - 2 - \beta - 3 \ ::: \ \Gamma$$

where 1 denotes the distinguished vertex, α, 2, β and 3, ... denote vertices connected by single edges as indicated and Γ denotes a finite graph connected by a single edge to the preceding vertex. For instance, X_1 is obtained by attaching A_3 to Γ (since the vertex 2 of A_3 is identified with one of the vertices of Γ, we have attached an A_2-tail to Γ), X_2 by attaching A_5 to Γ etc. Similarly we define the graph X_k.

We denote by L_0 the matrix appearing on the top left of the square of the adjacency matrix of X_0, that is:

$$\Delta_0^2 = \begin{pmatrix} L_0 & 0 \\ 0 & N_0 \end{pmatrix}$$

We compute the Jones series of each X_k in the next lemma.

Lemma 3.1. *The Jones series of the graphs X_k, $k \geq 0$, is given by*

$$\frac{\Theta(q) - q}{1 - q} = \frac{1 - Pq^{2k}}{1 - Pq^{2k+1}}$$

where P is defined by the formula

$$P = \frac{P_1 - q^{-1}P_0}{P_1 - qP_0}$$

and $P_i = P_i(y) = \det(y - K_i)$, $i = 0, 1$, with $y = 2 + q + q^{-1}$, and with K_0, K_1 being the following matrices:
– *K_0 is obtained from L_0 by deleting the first row and column.*
– *K_1 is obtained from L_0 by adding 1 to the first entry.*

Proof. We use the notation fixed in section 1. The matrix M_k in the adjacency matrix of X_k is given by a row vector w and a matrix M, as follows:

$$M_0 = \begin{pmatrix} w \\ M \end{pmatrix} \qquad M_1 = \begin{pmatrix} 1 & \\ 1 & w \\ & M \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ & 1 & w \\ & & M \end{pmatrix}$$

The corresponding matrices L_k are given by the real number $a = ww^t + 1$, the row vector $u = wM^t$, the column vector $v = Mw^t$, and the matrix $N = MM^t$ as follows:

$$L_0 = \begin{pmatrix} w \\ M \end{pmatrix} \begin{pmatrix} w^t & M^t \end{pmatrix} = \begin{pmatrix} a-1 & u \\ v & N \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 & \\ 1 & w \\ & M \end{pmatrix} \begin{pmatrix} 1 & 1 & \\ & w^t & M^t \end{pmatrix} = \begin{pmatrix} 1 & 1 & \\ 1 & a & u \\ & v & N \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ & 1 & w \\ & & M \end{pmatrix} \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & w^t & M^t \end{pmatrix} = \begin{pmatrix} 1 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & a & u \\ & & v & N \end{pmatrix}$$

It is now clear what the form of the matrices M_k and L_k is for general k. Consider the matrix K_k, $k \geq 0$, obtained from L_k by deleting the first row and column, in other words

$$K_0 = (N) \qquad K_1 = \begin{pmatrix} a & u \\ v & N \end{pmatrix} \qquad K_2 = \begin{pmatrix} 2 & 1 & & \\ 1 & a & u \\ & v & N \end{pmatrix}$$

and similarly for general k.

We have the following formula for the Poincaré series of X_k, with $y = z^{-1}$:

$$f_k(z) = \int \frac{1}{1 - zL_k} = \frac{\det(1 - zK_k)}{\det(1 - zL_k)} = y \cdot \frac{\det(y - K_k)}{\det(y - L_k)}$$

The characteristic polynomials $P_k = \det(y - K_k)$ and $Q_k = \det(y - L_k)$ satisfy the following two identities, obtained by developing determinants at the top left.

$$P_{k+1} = (y - 2)P_k - P_{k-1}$$
$$Q_k = (y - 1)P_k - P_{k-1}$$

We consider the first identity and the second identity minus the first, with the change of variables $y = 2 + q + q^{-1}$, and obtain

$$P_{k+1} = (q + q^{-1})P_k - P_{k-1}$$
$$Q_k = P_{k+1} + P_k$$

If we let $P_+ = P_1 - qP_0$ and $P_- = P_1 - q^{-1}P_0$, the solutions of these equations can be written as follows

$$P_k = \frac{q^{-k}P_+ - q^k P_-}{q^{-1} - q}$$

$$Q_k = \frac{q^{-k}P_+ - q^{k+1}P_-}{1 - q}$$

We can compute now the series f_k by using the variables $z = y^{-1} = q(1 + q)^{-2}$.

$$\begin{aligned} f_k(z) &= \frac{(1+q)^2}{q} \cdot \frac{P_k}{Q_k} \\ &= \frac{(1+q)^2}{q} \cdot \frac{1-q}{q^{-1}-q} \cdot \frac{q^{-k}P_+ - q^k P_-}{q^{-k}P_+ - q^{k+1}P_-} \\ &= (1+q)\,\frac{1 - Pq^{2k}}{1 - Pq^{2k+1}} \end{aligned}$$

And finally we obtain the Jones series as

$$\Theta_k(q) - q = \frac{1-q}{1+q}\, f_k(z) = (1-q)\frac{1 - Pq^{2k}}{1 - Pq^{2k+1}}$$

This proves the statement. □

We are now ready to compute the spectral measure for A_n (on $\mathbb{T}$).

Theorem 3.1. *Let X be the Coxeter-Dynkin graph A_{n-1}, $n \geq 1$, with $n-1$ vertices. The spectral measure of X (on $\mathbb{T}$) is given by*

$$d\varepsilon(u) = \alpha(u)\, d_n u$$

where $\alpha(u) = 2Im(u)^2$, and d_n is the uniform measure on $2n$-th roots of unity.

Proof. We use Lemma 3.1 to compute the Jones series of $X_k = A_{2k+2}$ by letting

$$M_0 = \begin{pmatrix} 1 \end{pmatrix} \quad L_0 = \begin{pmatrix} 1 \end{pmatrix} \quad K_0 = \begin{pmatrix} \end{pmatrix} \quad K_1 = \begin{pmatrix} 2 \end{pmatrix}$$

$$P_0 = 1$$
$$P_1 = q + q^{-1}$$
$$P = q^2$$

Then

$$\frac{\Theta(q) - q}{1-q} = \frac{1 - q^{2k+2}}{1 - q^{2k+3}}.$$

Similarly, Lemma 3.1 gives the Jones series of $X_k = A_{2k+3}$ by letting

$$M_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad L_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad K_0 = \begin{pmatrix} 1 \end{pmatrix} \quad K_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P_0 = 1 + q + q^{-1}$$
$$P_1 = 1 + q + q^{-1} + q^2 + q^{-2}$$
$$P = q^3$$

Then

$$\frac{\Theta(q) - q}{1-q} = \frac{1 - q^{2k+3}}{1 - q^{2k+4}}.$$

The above two formulae give the Jones series for A_{n-1} as

$$\frac{\Theta(q) - q}{1-q} = \frac{1 - q^{n-1}}{1 - q^{n}}.$$

We use the following formula, valid for $m = 2nk + r$ with $r = 0, 1, \ldots, 2n-1$.

$$\int_{\mathbb{T}} \frac{u^{-m}}{1-qu}\, d_n u \quad = \quad \frac{q^r}{1-q^{2n}}$$

We can compute now the Stieltjes transform of ε.

$$\begin{aligned} 2S(q) - 1 &= -1 + \int_{\mathbb{T}} \frac{2 - u^2 - u^{-2}}{1-qu}\, d_n u \\ &= -1 + \frac{2 - q^{2n-2} - q^2}{1 - q^{2n}} \\ &= \frac{1 + q^{2n} - q^{2n-2} - q^2}{1 - q^{2n}} \\ &= \frac{(1-q^2)(1-q^{2n-2})}{1-q^{2n}} \end{aligned}$$

Thus we have $2S(q) - 1 = \Theta(q^2) - q^2$, and we are done. □

We compute next the Jones series for the Coxeter-Dynkin graph of type D_n, $n \geq 3$. The graph D_n has n vertices. It consists of two vertices connected to one another vertex, and this vertex in turn is connected to an A-tail, ending at the distinguished vertex 1.

Theorem 3.2. *Let X be the Coxeter-Dynkin graph D_{n+1}, $n \geq 2$, with $n+1$ vertices. The spectral measure of X (on $\mathbb{T}$) is given by*

$$d\varepsilon(u) = \alpha(u)\, d_n' u$$

where $\alpha(u) = 2Im(u)^2$, and d_n' is the uniform measure on $4n$-th roots of unity of odd order.

Proof. We use again Lemma 3.1 to compute the Jones series of $X_k = D_{2k+3}$ by letting

$$M_0 = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad L_0 = \begin{pmatrix} 2 \end{pmatrix} \quad K_0 = \begin{pmatrix} \end{pmatrix} \quad K_1 = \begin{pmatrix} 3 \end{pmatrix}$$
$$P_0 = 1$$
$$P_1 = -1 + q + q^{-1}$$
$$P = -q$$

Thus

$$\frac{\Theta(q) - q}{1 - q} = \frac{1 + q^{2k+1}}{1 + q^{2k+2}}.$$

Similarly, Lemma 3.1 allows us to compute the Jones series of $X_k = D_{2k+4}$ by letting

$$M_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad L_0 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad K_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad K_1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
$$P_0 = 2 + 2q + 2q^{-1} + q^2 + q^{-2}$$
$$P_1 = q + q^{-1} + 2q^2 + 2q^{-2} + q^3 + q^{-3}$$
$$P = -q^2$$

Thus

$$\frac{\Theta(q) - q}{1 - q} = \frac{1 + q^{2k+2}}{1 + q^{2k+3}}.$$

From the above two formulae we deduce the Jones series of D_{n+1} as

$$\frac{\Theta(q) - q}{1 - q} = \frac{1 + q^{n-1}}{1 + q^n}.$$

We use now the following identity.

$$\begin{aligned}
\frac{1 + q^{n-1}}{1 + q^n} &= \frac{(1 - q^n)(1 + q^{n-1})}{(1 - q^n)(1 + q^n)} \\
&= \frac{1 - q^n + q^{n-1} - q^{2n-1}}{1 - q^{2n}} \\
&= \frac{2(1 - q^{2n-1}) - (1 + q^n)(1 - q^{n-1})}{1 - q^{2n}}
\end{aligned}$$

$$= \quad 2\,\frac{1-q^{2n-1}}{1-q^{2n}} - \frac{1-q^{n-1}}{1-q^n}$$

Multiplying by $1-q$ and adding q makes appear the Jones series Θ_m for A_{m-1} computed in Theorem 3.1, and we obtain

$$\Theta(q) = 2\Theta_{2n}(q) - \Theta_n(q).$$

We compute now the Stieltjes transform of ε. Denote by S_m the Stieltjes transform of the measure for A_{m-1}. Then

$$\begin{aligned} 2S(q)-1 &= 2(2S_{2n}(q)-S_n(q))-1 \\ &= 2(S_{2n}(q)-1)-(2S_n(q)-1) \\ &= 2(\Theta_{2n}(q^2)-q^2)-(\Theta_n(q^2)-q^2) \\ &= (2\Theta_{2n}(q^2)-\Theta_n(q^2))-q^2 \end{aligned}$$

Thus we have $2S(q)-1 = \Theta(q^2)-q^2$ as claimed. □

Theorem 3.3. *For $X = A_\infty$ the spectral measure is given by*

$$d\varepsilon(u) = \alpha(u)\,du$$

where $\alpha(u) = 2Im(u)^2$, and du is the uniform measure on the unit circle.

Proof. This follows from Theorems 3.1 or 3.2 with $n \to \infty$, or from a direct loop count. □

4. Graphs with fork tails

Consider a sequence of graphs X_k obtained by adding D_{2k+2} tails to a given graph (compare with section 3). We let

$$X_0 = \frac{1}{2} > \alpha ::: \Gamma$$

$$X_1 = \frac{1}{2} > \alpha - 3 - \beta ::: \Gamma$$

$$X_2 = \frac{1}{2} > \alpha - 3 - \beta - 4 - \gamma ::: \Gamma$$

As before we let L_0 be the matrix appearing on the top left of the adjacency matrix of X_0, i.e.

$$\Delta_0^2 = \begin{pmatrix} L_0 & 0 \\ 0 & N_0 \end{pmatrix}.$$

Lemma 4.1. *The Jones series of the graphs X_k is given by*

$$(\Theta(q)-q)(1+q) = \frac{1-Pq^{2k+1}}{1+Pq^{2k}}$$

where P is defined by

$$P = \frac{P_1 - q^{-1}P_0}{P_1 - qP_0}$$

where $P_i = P_i(y) = \det(y - J_i)$, $i = 0,\, 1$, *with* $y = 2 + q + q^{-1}$, *and with* J_0, J_1 *being the following matrices:*

– J_0 *is obtained from* L_0 *by deleting the first two rows and columns.*

– J_1 *is obtained from* L_0 *by deleting the first row and column, then adding* 1 *to the first entry.*

Proof. The matrix M_k associated to the adjacency matrix of X_k is described by a column vector w, and a matrix M, as follows:

$$M_0 = \begin{pmatrix}1 & \\ 1 & \\ w & M\end{pmatrix} \qquad M_1 = \begin{pmatrix}1 & & \\ 1 & & \\ 1 & 1 & \\ & w & M\end{pmatrix} \qquad M_2 = \begin{pmatrix}1 & & & \\ 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & w & M\end{pmatrix}$$

The corresponding matrices L_k make appear the matrix $N = ww^t + MM^t$ and are given by

$$L_0 = \begin{pmatrix}1 & \\ 1 & \\ w & M\end{pmatrix}\begin{pmatrix}1 & 1 & w^t \\ & & M^t\end{pmatrix} = \begin{pmatrix}1 & 1 & w^t \\ 1 & 1 & w^t \\ w & w & N\end{pmatrix}$$

$$L_1 = \begin{pmatrix}1 & & \\ 1 & & \\ 1 & 1 & \\ & w & M\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & \\ & & 1 & w^t \\ & & & M^t\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 & \\ 1 & 1 & 1 & \\ 1 & 1 & 2 & w^t \\ & & w & N\end{pmatrix}$$

$$L_2 = \begin{pmatrix}1 & & & \\ 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & w & M\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & w^t \\ & & & & M^t\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 2 & 1 & \\ & & 1 & 2 & w^t \\ & & & w & N\end{pmatrix}$$

It is now clear what the form of the matrices M_k and L_k is for general k. Consider the matrix K_k, obtained from L_k by deleting the first row and column, that is

$$K_0 = \begin{pmatrix}1 & w^t \\ w & N\end{pmatrix} \qquad K_1 = \begin{pmatrix}1 & 1 & \\ 1 & 2 & w^t \\ & w & N\end{pmatrix} \qquad K_2 = \begin{pmatrix}1 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 2 & w^t \\ & & w & N\end{pmatrix}$$

and similarly for general k. Consider also the matrix J_k, obtained from L_k by deleting the first two rows and columns, i.e.

$$J_0 = \begin{pmatrix}N\end{pmatrix} \qquad J_1 = \begin{pmatrix}2 & w^t \\ w & N\end{pmatrix} \qquad J_2 = \begin{pmatrix}2 & 1 & \\ 1 & 2 & w^t \\ & w & N\end{pmatrix}$$

We have then the following formula for the Poincaré series of X_k, with $y = z^{-1}$.

$$f_k(z) = \int \frac{1}{1 - zL_k} = \frac{\det(1 - zK_k)}{\det(1 - zL_k)} = y \cdot \frac{\det(y - K_k)}{\det(y - L_k)}$$

The characteristic polynomials $P_k = \det(y - J_k)$, $Q_k = \det(y - K_k)$ and $R_k = \det(y - L_k)$ satisfy the following relations, obtained by developing the determinant at top left:

$$\begin{aligned} P_{k+1} &= (y-2)P_k - P_{k-1} \\ Q_k &= (y-1)P_k - P_{k-1} \\ R_k &= (y-1)Q_k - P_k - (y+1)P_{k-1} \end{aligned}$$

The solutions of these equations can be written in terms of $P_+ = P_1 - qP_0$ and $P_- = P_1 - q^{-1}P_0$, namely

$$\begin{aligned} P_k &= \frac{q^{-k}P_+ - q^k P_-}{q^{-1} - q} \\ Q_k &= \frac{q^{-k}P_+ - q^{k+1}P_-}{1-q} \\ R_k &= \frac{q^{-k}P_+ + q^k P_-}{q/(1+q)^2} \end{aligned}$$

We can compute now f_k by using the variables $z = y^{-1} = q(1+q)^{-2}$.

$$\begin{aligned} f_k(z) &= \frac{(1+q)^2}{q} \cdot \frac{Q_k}{R_k} \\ &= \frac{(1+q)^2}{q} \cdot \frac{q/(1+q)^2}{1-q} \cdot \frac{q^{-k}P_+ - q^{k+1}P_-}{q^{-k}P_+ + q^k P_-} \\ &= \frac{1}{1-q} \cdot \frac{1 - Pq^{2k+1}}{1 + Pq^{2k}} \end{aligned}$$

And finally we obtain the Jones series for D_{n+1} as

$$\Theta_k(q) - q = \frac{1-q}{1+q} f_k(z) = \frac{1}{1+q} \cdot \frac{1 - Pq^{2k+1}}{1 + Pq^{2k}}$$

This proves the claim. □

We proceed now with the computation of the spectral measure for the extended Coxeter-Dynkin graph $D_n^{(1)}$. The graph $D_n^{(1)}$, $n \geq 4$, has $n+1$ vertices. It consists of the distinguished vertex 1, connected to a triple point, connected in turn to another vertex and to an A-tail ending at another triple point. This triple point is connected to two other vertices.

Theorem 4.1. *Let X be the extended Coxeter-Dynkin graph of type $D_{n+2}^{(1)}$, $n \geq 2$ ($n+3$ vertices). The spectral measure of X (on $\mathbb{T}$) is given by*

$$d\varepsilon(u) = \frac{\delta_i + \delta_{-i}}{4} + \frac{d_n u}{2},$$

where $d_n u$ is the uniform measure on $2n$-th roots of unity and δ_w is the Dirac measure at $w \in \mathbb{T}$.

Proof. We use Lemma 4.1 to compute the Jones series of $X_k = D^{(1)}_{2k+4}$ by letting

$$M_0 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \quad L_0 = \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix} \quad J_0 = \begin{pmatrix}1&1\\1&1\end{pmatrix} \quad J_1 = \begin{pmatrix}2&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

$$P_0 = 2 + 2q + 2q^{-1} + q^2 + q^{-2}$$
$$P_1 = q + q^{-1} + 2q^2 + 2q^{-2} + q^3 + q^{-3}$$
$$P = -q^2$$

We obtain

$$(\Theta(q) - q)(1+q) = \frac{1+q^{2k+3}}{1-q^{2k+2}}.$$

Similarly, Lemma 4.1 gives the Jones series of $X_k = D^{(1)}_{2k+5}$ by letting

$$M_0 = \begin{pmatrix}1&0&0\\1&0&0\\1&1&1\end{pmatrix} \quad L_0 = \begin{pmatrix}1&1&1\\1&1&1\\1&1&3\end{pmatrix} \quad K_0 = \begin{pmatrix}3\end{pmatrix} \quad K_1 = \begin{pmatrix}2&1\\1&3\end{pmatrix}$$

$$P_0 = -1 + q + q^{-1}$$
$$P_1 = 1 - q - q^{-1} + q^2 + q^{-2}$$
$$P = -q^3$$

We obtain

$$(\Theta(q) - q)(1+q) = \frac{1+q^{2k+4}}{1-q^{2k+3}}.$$

From the above two formulae we deduce that Jones series for $D^{(1)}_{n+2}$ satisfies

$$(\Theta(q) - q)(1+q) = \frac{1+q^{n+1}}{1-q^{n}}.$$

Hence we get the following explicit formula for the Jones series for $D^{(1)}_{n+2}$:

$$\begin{aligned}
\Theta(q) - q &= \frac{1}{1+q}\cdot\frac{1-q^n+q^n+q^{n+1}}{1-q^n}\\
&= \frac{1}{1+q}\left(1+\frac{q^n(1+q)}{1-q^n}\right)\\
&= \frac{1}{1+q}+\frac{q^n}{1-q^n}\\
&= \frac{1}{1+q}+\frac{1}{1-q^n}-1
\end{aligned}$$

We compute now the Stieltjes transform of the associated spectral measure ε.

$$\begin{aligned}
2S(q) &= \frac{1}{2}\left(\frac{1}{1-qi}+\frac{1}{1+qi}\right)+\int_{\mathbb{T}}\frac{1}{1-qu}\,d_nu\\
&= \frac{1}{1+q^2}+\frac{1}{1-q^{2n}}
\end{aligned}$$

Thus we have $2S(q) = \Theta(q^2) - q^2 + 1$, and we are done. □

The spectral measure for the infinite Coxeter-Dynkin graph D_∞ follows now. Recall that the graph D_∞ has a triple point, connected to the distinguished vertex 1 and to another vertex, and to an A_∞-tail.

Theorem 4.2. *The spectral measure (on $\mathbb{T}$) of the Coxeter-Dynkin graph D_∞ is given by*

$$d\varepsilon(u) = \frac{\delta_i + \delta_{-i}}{4} + \frac{du}{2},$$

where du is the uniform measure on the unit circle.

Proof. This follows from Theorem 4.1 by letting $n \to \infty$. □

5. Exceptional graphs

In this chapter we computer the spectral measures for the Coxeter-Dynkin graphs of type E. These graphs are

$$\begin{aligned} E_6 &= F(2,1,2) \\ E_7 &= F(2,1,3) \\ E_8 &= F(2,1,4) \\ E_6^{(1)} &= F(2,2,2) \\ E_7^{(1)} &= F(3,1,3) \\ E_8^{(1)} &= F(2,1,5) \end{aligned}$$

Here we denote by $F(a,b,c)$ the graph with $a+b+c+1$ vertices, consisting of a triple point which is connected to an A_a tail, to an A_b tail, and to an A_c tail ending at the distinguished vertex 1. All of these graphs, with the exception of E_7, appear as principal graphs of subfactors with Jones index ≤ 4 (see for instance [4], [5]).

We use the following notation. Assume that ε is a probability measure on the unit circle which is even, in the sense that all its odd moments are 0. The Stieltjes transform $S(q)$ of ε is then a series in q^2, and the following definition makes sense.

Definition 5.1. *The T series of an even measure ε is given by*

$$T(q) = \frac{2S(q^{1/2}) - 1}{1-q}$$

where S is the Stieltjes transform of ε.

It follows from Propositions 1.1 and 1.2 that the spectral measure of a graph is even, and that we have the formula

$$T(q) = \frac{\Theta(q) - q}{1-q}$$

where Θ is the Jones series.

In this section we compute the T series of exceptional graphs of type E.

Theorem 5.1. *The T series of the Coxeter-Dynkin graphs E_6, E_7 and E_8 are given by the following formulae:*

$$\begin{aligned} T_6(q) &= \frac{(1-q^6)(1-q^8)}{(1-q^3)(1-q^{12})} \\ T_7(q) &= \frac{(1-q^9)(1-q^{12})}{(1-q^4)(1-q^{18})} \\ T_8(q) &= \frac{(1-q^{10})(1-q^{15})(1-q^{18})}{(1-q^5)(1-q^9)(1-q^{30})} \end{aligned}$$

The proof of these results uses techniques from section 3. We combine the proof with the proof of next result.

Theorem 5.2. *The T series of the extended Coxeter-Dynkin graphs $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ are given by the following formulae:*

$$\begin{aligned} T_6^{(1)}(q) &= \frac{1-q^{12}}{(1-q^3)(1-q^4)(1-q^6)} \\ T_7^{(1)}(q) &= \frac{1-q^{18}}{(1-q^4)(1-q^6)(1-q^9)} \\ T_8^{(1)}(q) &= \frac{1-q^{30}}{(1-q^6)(1-q^{10})(1-q^{15})} \end{aligned}$$

Proof. We compute first the T series of $E_6, E_7, E_8, E_8^{(1)}$, which are all of the form $F(2,1,n)$. We use Lemma 3.1 to compute the T series of $X_k = F(2,1,2k)$ by letting

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad L_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad K_0 = \begin{pmatrix} 1 \end{pmatrix} \quad K_1 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P_0 = 1 + q + q^{-1}$$

$$P_1 = q^2 + q^{-2}$$

$$P = -q\,\frac{1+q-q^3}{1-q^2-q^3}$$

We obtain

$$T(q) = \frac{(1-q^2-q^3) + q^{2k+1}(1+q-q^3)}{(1-q^2-q^3) + q^{2k+2}(1+q-q^3)}.$$

Similarly, Lemma 3.1 gives the T series of $X_k = F(2,1,2k+1)$ by letting

$$M_0 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \quad L_0 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad K_0 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad K_1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$P_0 = 1 + q + q^{-1} + q^2 + q^{-2}$$

$$P_1 = -1 + q^2 + q^{-2} + q^3 + q^{-3}$$

$$P = -q^2\,\frac{1+q-q^3}{1-q^2-q^3}$$

We obtain

$$T(q) = \frac{(1-q^2-q^3) + q^{2k+2}(1+q-q^3)}{(1-q^2-q^3) + q^{2k+3}(1+q-q^3)}.$$

To simplify this expression we introduce the following polynomials:

$$Q_n = (1-q^2-q^3) + q^n(1+q-q^3)$$

From the above formulae we get then the T series of $F(2,1,n)$ in terms of these polynomials as

$$T(q) = \frac{Q_{n+1}}{Q_{n+2}}.$$

The polynomials Q_k needed for E_6, E_7, E_8, and $E_8^{(1)}$ are all cyclotomic:

$$\begin{aligned}
Q_3 &= \frac{(1-q^2)(1-q^8)}{1-q^4} \\
Q_4 &= \frac{(1-q^2)(1-q^3)(1-q^{12})}{(1-q^4)(1-q^6)} \\
Q_5 &= \frac{(1-q^2)(1-q^3)(1-q^{18})}{(1-q^6)(1-q^9)} \\
Q_6 &= \frac{(1-q^2)(1-q^3)(1-q^5)(1-q^{30})}{(1-q^6)(1-q^{10})(1-q^{15})} \\
Q_7 &= (1-q^2)(1-q^3)(1-q^5)
\end{aligned}$$

The formulae for the T series of E_6, E_7, E_8 and $E_8^{(1)}$ follow now.

We use again Lemma 3.1 to compute the T series of $X_k = F(2,2,2k)$ by letting

$$M_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad L_0 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad K_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad K_1 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$P_0 = 3 + 2q + 2q^{-1} + q^2 + q^{-2}$$
$$P_1 = -1 + q^2 + q^{-2} + q^3 + q^{-3}$$
$$P = -q\,\frac{1+q-q^2}{1-q-q^2}$$

We obtain

$$T(q) = \frac{(1-q-q^2) + q^{2k+1}(1+q-q^2)}{(1-q-q^2) + q^{2k+2}(1+q-q^2)}.$$

Factorising the numerator N and denominator D for $k=1$, we get

$$N = 1 - q - q^2 + q^3 + q^4 - q^5 = (1-q)(1-q^2+q^4),$$
$$D = 1 - q - q^2 + q^4 + q^5 - q^6 = (1-q)(1-q^2)(1-q^3).$$

Rewrite these expressions as

$$N = \frac{(1-q)(1-q^2)(1-q^{12})}{(1-q^4)(1-q^6)}$$

$$D \quad = \quad (1-q)(1-q^2)(1-q^3)$$

and the formula for the T series of $E_6^{(1)}$ follows.

We use next Lemma 3.1 to compute the T series of $X_k = F(3,1,2k+1)$ by letting

$$M_0 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \quad L_0 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad K_0 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad K_1 = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$P_0 = 2 + 2q + 2q^{-1} + 2q^2 + 2q^{-2} + q^3 + q^{-3}$$
$$P_1 = -2 - q - q^{-1} + q^2 + q^{-2} + 2q^3 + 2q^{-3} + q^4 + q^{-4}$$
$$P = -q^2\, \frac{1+q^2-q^3}{1-q-q^3}$$

We obtain

$$T(q) = \frac{(1-q-q^3) + q^{2k+2}(1+q^2-q^3)}{(1-q-q^3) + q^{2k+3}(1+q^2-q^3)}.$$

Factorising the numerator N and denominator D for $k=1$ gives

$$N = 1 - q - q^3 + q^4 + q^6 - q^7 = (1-q)(1-q^3+q^6),$$
$$D = 1 - q - q^3 + q^5 + q^7 - q^8 = (1-q)(1-q^3)(1-q^4).$$

Rewrite these expressions as

$$N \quad = \quad \frac{(1-q)(1-q^3)(1-q^{18})}{(1-q^6)(1-q^9)}$$
$$D \quad = \quad (1-q)(1-q^3)(1-q^4)$$

and the formula for the T series of $E_7^{(1)}$ follows. □

6. Spectral measures for exceptional graphs

We compute in this section the spectral measures of the exceptional graphs E_6 and $E_{6,7,8}^{(1)}$. We will express the T series computed in the previous section as linear combinations of elementary T series computed in the next two Lemmas. The notation for the measures is as in sections 3 and 4.

Lemma 6.1. *The T series of the measures $\alpha\, d_n$, $\alpha\, d_n'$, d_n, d_n' are given by the following identities:*

$$T(\alpha\, d_n) \quad = \quad \frac{1-q^{n-1}}{1-q^n}$$
$$T(\alpha\, d_n') \quad = \quad \frac{1+q^{n-1}}{1+q^n}$$
$$T(d_n) \quad = \quad \frac{1}{1-q} \cdot \frac{1+q^n}{1-q^n}$$
$$T(d_n') \quad = \quad \frac{1}{1-q} \cdot \frac{1-q^n}{1+q^n}$$

Proof. The first two identities appear in proof of Theorems 3.1 and 3.2. For the third one we use that

$$\int_{\mathbb{T}} u^k d_n u = (2n|k),$$

where $(2n|k)$ is defined to be 1 when $2n$ divides k, and 0 otherwise. This gives the following identity for the Stieltjes transform:

$$S(q) = \sum_{s=0}^{\infty} q^{2ns} = \frac{1}{1-q^{2n}}.$$

We can then compute the T series as in Definition 5.1 by

$$\begin{aligned} T(d_n) &= \frac{2S(q^{1/2})-1}{1-q} \\ &= \frac{1}{1-q}\left(\frac{2}{1-q^n}-1\right) \\ &= \frac{1}{1-q}\cdot\frac{1+q^n}{1-q^n} \end{aligned}$$

For the fourth identity in the lemma we use that $d_n' = 2d_{2n} - d_n$. Hence

$$\begin{aligned} T(d_n') &= 2T(d_{2n}) - T(d_n) \\ &= \frac{1}{1-q}\left(2\cdot\frac{1+q^{2n}}{1-q^{2n}} - \frac{1+q^n}{1-q^n}\right) \\ &= \frac{1}{1-q}\cdot\frac{1-2q^n+q^{2n}}{1-q^{2n}} \\ &= \frac{1}{1-q}\cdot\frac{(1-q^n)^2}{1-q^{2n}} \end{aligned}$$

After simplification we obtain the formula in the statement. □

Theorem 6.1. *The spectral measures of $E_{6,7,8}^{(1)}$ (on $\mathbb{T}$) are given by*

$$\begin{aligned} \varepsilon_6^{(1)} &= \alpha\, d_3 + (d_2 - d_3)/2 \\ \varepsilon_7^{(1)} &= \alpha\, d_4 + (d_3 - d_4)/2 \\ \varepsilon_8^{(1)} &= \alpha\, d_6 + (d_5 - d_6)/2 \end{aligned}$$

where $\alpha(u) = 2Im(u)^2$, and $d_n u$ is the uniform measure on $2n$-th roots of unity.

Proof. The T series in Theorem 5.2 can be written as

$$\begin{aligned} T_6^{(1)} &= \frac{1+q^6}{(1-q^3)(1-q^4)} \\ T_7^{(1)} &= \frac{1+q^9}{(1-q^4)(1-q^6)} \\ T_8^{(1)} &= \frac{1+q^{15}}{(1-q^6)(1-q^{10})} \end{aligned}$$

Factoring by $1+q^2$, $1+q^3$ resp. $1+q^5$ gives

$$\begin{aligned} T_6^{(1)} &= \frac{1-q^2+q^4}{(1-q^2)(1-q^3)} \\ T_7^{(1)} &= \frac{1-q^3+q^6}{(1-q^3)(1-q^4)} \\ T_8^{(1)} &= \frac{1-q^5+q^{10}}{(1-q^5)(1-q^6)} \end{aligned}$$

We get then the following formula, with $k=2,3,5$ corresponding to $n=6,7,8$:

$$T_n^{(1)} = \frac{1-q^k+q^{2k}}{(1-q^k)(1-q^{k+1})}.$$

We can rewrite this series in the following way:

$$\begin{aligned} T_n^{(1)} &= \frac{1-2q^k+q^k}{(1-q^k)(1-q^{k+1})} + \frac{q^k}{(1-q^k)(1-q^{k+1})} \\ &= \frac{1-q^k}{1-q^{k+1}} + \frac{1}{1-q}\cdot\frac{q^k-q^{k+1}}{(1-q^k)(1-q^{k+1})} \\ &= \frac{1-q^k}{1-q^{k+1}} + \frac{1}{1-q}\left(\frac{1}{1-q^k} - \frac{1}{1-q^{k+1}}\right) \end{aligned}$$

We can then write $T_n^{(1)}$ as a linear combination of the elementary T series from Lemma 6.1 as follows

$$T_n^{(1)} = T(\alpha\, d_{k+1}) + (T(d_k) - T(d_{k+1}))/2.$$

By using linearity of the Stieltjes transform, hence of the T series, we get the formulae in the statement of the theorem. ☐

Theorem 6.2. *The spectral measure of E_6 (on $\mathbb{T}$) is given by*

$$\varepsilon_6 = \alpha\, d_{12} + (d_{12} - d_6 - d_4 + d_3)/2$$

where $\alpha(u) = 2Im(u)^2$, and d_n is the uniform measure on $2n$-th roots of unity.

Proof. The T series of E_6 can be written as

$$\begin{aligned} T_6 &= \frac{(1+q^3)(1-q^8)}{1-q^{12}} \\ &= \frac{1-q^{11}}{1-q^{12}} + \frac{q^3-q^8}{1-q^{12}}. \end{aligned}$$

Note that we have the following identity:

$$\begin{aligned} \frac{q^3-q^8}{1-q^{12}} &= \frac{1}{1-q}\cdot\frac{q^3-q^4-q^8+q^9}{1-q^{12}} \\ &= \frac{1}{1-q}\left(\frac{1}{1-q^{12}} - \frac{1+q^6}{1-q^{12}} - \frac{1+q^4+q^8}{1-q^{12}} + \frac{1+q^3+q^6+q^9}{1-q^{12}}\right) \\ &= \frac{1}{1-q}\left(\frac{1}{1-q^{12}} - \frac{1}{1-q^6} - \frac{1}{1-q^4} + \frac{1}{1-q^3}\right). \end{aligned}$$

It follows now easily that T_6 can be written as a linear combination of the elementary T series in Lemma 6.1, namely

$$T_6 = T(\alpha\, d_{12}) + (T(d_{12}) - T(d_6) - T(d_4) + T(d_3))/2.$$

This gives the formula for the spectral measure ε_6 in the statement of the theorem. □

7. Exceptional measures: E_7 and E_8

Note that all spectral measures of the finite ADE graphs computed so far are linear combinations of measures of type d_n and $\alpha\, d_n$ (observe that we have $d'_n = 2d_{2n} - d_n$).

Definition 7.1. *A discrete measure supported by roots of unity is called* cyclotomic *if it is a linear combination of measures of type d_n, $n \geq 1$, and $\alpha\, d_n$, $n \geq 2$.*

Note that we require $n \geq 2$ for the measure $\alpha\, d_n$. This is simply because $\alpha\, d_1$ is the null measure.

Theorem 7.1. *The spectral measures of E_7, E_8 (on $\mathbb{T}$) are not cyclotomic.*

Proof. From Theorem 5.1 we obtain the T series of E_7 as

$$T_7 = \frac{(1-q^9)(1+q^4+q^8)}{1-q^{18}}.$$

This shows that the corresponding spectral measure ε_7 is supported by 36-th roots of unity. Assume now that ε_7 is cyclotomic. Then

$$\varepsilon_7 \in \text{span}\{d_n,\, \alpha\, d_m \mid n \geq 1,\, m \geq 2,\, n, m|18\}.$$

Using the linearity of the T transform (with respect to the measure) this means

$$T_7 \in \text{span}\{T(d_n),\, T(\alpha\, d_m) \mid n \geq 1,\, m \geq 2,\, n, m|18\}.$$

We multiply the relevant T series by $(1-q)(1-q^{18})$, i.e. we consider the following degree 18 polynomials, where n, m are as above:

$$\begin{aligned} P_n &= (1-q)(1-q^{18})\, T(d_n) \\ Q_m &= (1-q)(1-q^{18})\, T(\alpha\, d_m) \\ R_7 &= (1-q)(1-q^{18})\, T_7 \end{aligned}$$

The assumption that ε_7 is cyclotomic becomes then

$$R_7 \in \text{span}\{P_n,\, Q_m \mid n \geq 1,\, m \geq 2,\, n, m|18\}.$$

The above formula of T and Lemma 6.1 lead to the following expessions:

$$\begin{aligned} P_n &= (1+q^n) \cdot \frac{1-q^{18}}{1-q^n} \\ Q_m &= (1-q)(1-q^{m-1}) \cdot \frac{1-q^{18}}{1-q^m} \\ R_7 &= (1-q)(1-q^9)(1+q^4+q^8) \end{aligned}$$

The coefficients c_k of each of these polynomials satisfy $c_k = c_{18-k}$, so in order to solve the system of linear equations resulting from our assumption, we can restrict attention to coefficients c_k with $k = 0, 1, \ldots, 9$. Thus we have 10 equations, and the unknowns are the coefficients of P_n, Q_m with $n \geq 1$, $m \geq 2$ and $n, m | 18$.

The matrix of the system is given below. The rows correspond to the polynomials appearing, and the columns correspond to coefficients of q^k, with $k = 0, 1, \ldots, 9$:

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
P_1	1	2	2	2	2	2	2	2	2	2
P_2	1		2		2		2		2	
P_3	1			2			2			2
P_6	1						2			
P_9	1									2
P_{18}	1									
Q_2	1	−2	2	−2	2	−2	2	−2	2	−2
Q_3	1	−1	−1	2	−1	−1	2	−1	−1	2
Q_6	1	−1				−1	2	−1		
Q_9	1	−1							−1	2
Q_{18}	1	−1								
R_7	1	−1			1	−1			1	−2

Comparing the c_2 and c_4 columns shows that this system of equations has no solution. This contradicts our assumption that ε_7 is cyclotomic.

The same method applies to E_8. From Theorem 5.1 we get the T series for E_8 as

$$T_8 = \frac{(1+q^5)(1+q^9)(1-q^{15})}{1-q^{30}}.$$

Thus ε_8 is supported by 60-th roots of unity. Now by using degree 30 polynomials P_n, Q_m and R_8 defined as above, we get again a linear system of equations with the following matrix of coefficients:

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}
P_1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
P_2	1		2		2		2		2		2		2		2	
P_3	1			2			2			2			2			2
P_5	1					2					2					2
P_6	1						2						2			
P_{10}	1										2					
P_{15}	1															2
P_{30}	1															
Q_2	1	−2	2	−2	2	−2	2	−2	2	−2	2	−2	2	−2	2	−2
Q_3	1	−1	−1	2	−1	−1	2	−1	−1	2	−1	−1	2	−1	−1	2
Q_5	1	−1			−1	2	−1			−1	2	−1			−1	2
Q_6	1	−1				−1	2	−1				−1	2	−1		
Q_{10}	1	−1								−1	2	−1				
Q_{15}	1	−1													−1	2
Q_{30}	1	−1														
R_8	1	−1				1	−1			1	−1				1	−2

Assume that ε_8 is cyclotomic. This means that R_8 appears as linear combination of P_n, Q_m. Now, comparing the c_2 and c_4 columns shows that the coefficient of Q_5 must be zero, and comparing the c_6 and c_{12} columns shows that the coefficient of Q_5 must be non-zero. Thus our assumption that ε_8 is cyclotomic is wrong. $\square$

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