The fundamental group $\mathcal{F}(M)$ of a type II_1 factor M was introduced by Murray and von Neumann in 1943 in connection with their notion of continuous dimension. It measures the extent to which "amplifications" of M are isomorphic to M (e.g., if the algebra of 2×2 matrices over M is isomorhic to M then $2 \in \mathcal{F}(M)$). It is a puzzling and still poorly understood invariant.

We will present results providing the first examples of factors M with trivial fundamental group. Thus, if G is the arithmetic group $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ and $M = L(G)$ is the associated group von Neumann algebra then $\mathcal{F}(M) = \{1\}$. The proof uses in a crucial way the "weak amenability" of $\Gamma = SL(2, \mathbb{Z})$ (i.e. Haagerup's approximation property or equivalently Gromov's a-T menability) and the relative property (T) of Kazhdan-Margulis of the inclusion $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma$. The combination of these two properties makes it possible to prove a unique decomposition of M as a cross-product $M = L^\infty(\mathbb{T}^2) \rtimes \Gamma$, thus allowing us to define ℓ^2-Betti number invariants $\beta_n(M)$ from the ℓ^2-Betti numbers $\beta_n(\mathcal{R}_\Gamma)$ (defined by Gaboriau in 2001) of the equivalence relation $\mathcal{R}_\Gamma$ induced by Γ on $\mathbb{T}^2$.