The Representation Ring

K — compact group

$$\mathrm{Rep}(K) = \left\{ \begin{array}{c} \text{finite dimensional} \\ \text{representations of } K \end{array} \right\} \Big/ \cong$$

$\mathrm{Rep}(K)$ is a semiring with unit:

$\oplus$ — direct sum

$\otimes$ — tensor product

$\mathbb{1}$ — trivial representation on $\mathbb{C}$.

Can extend it to a ring if we can institute a difference $\ominus$.

K-Index of a Fredholm intertwiner

Consider data:

$$\begin{array}{ccc} \mathcal{H}_+ & \overset{F}{} & \mathcal{H}_- \\ \\ \pi_+(\mathsf{K}) & & \pi_-(\mathsf{K}) \end{array}$$

where

1. π_+, π_- are unitary representations of K
2. $F : \mathcal{H}_+ \to \mathcal{H}_-$ is a Fredholm intertwiner (we will relax this condition soon)

$\Longrightarrow$ $\ker F$ and $\operatorname{coker} F$ are fin. dim. rep'ns.

Think of F as instituting their difference:

$$\text{K-Index}(F) \;=\; [\ker F] \ominus [\operatorname{coker} F]$$

Kasparov Representation Ring

G — locally compact group.

A *Fredholm representation* of G is

$$\mathcal{H}_+ \qquad\qquad \mathcal{H}_-$$

$$\pi_+(G) \qquad\qquad \pi_-(G)$$

- $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ — graded Hilbert space
- $\pi = \pi_+ \oplus \pi_-$ — rep'n of G on $\mathcal{H}$ (by even operators)
- F — odd, self-adjoint Fredholm op.

s.t.

(i) $g \mapsto \pi(g)F\pi(g^{-1})$ is norm-cts

(ii) $F^2 - 1 \in \mathcal{K}(\mathcal{H})$

(iii) $[F, \pi(g)] \in \mathcal{K}(\mathcal{H}) \qquad (\forall g \in \mathsf{G})$

Kasparov representation ring $R(\mathsf{G})$

Defn: $R(\mathsf{G}) = \{\text{Fredholm rep'ns}\}/\text{homotopy}.$

ie, $R(\mathsf{G}) = KK^{\mathsf{G}}(\mathbb{C}, \mathbb{C}).$

Theorem (Kasparov)**.** $R(\mathsf{G})$ *admits a product which makes it into a ring.*

Remark: For K compact, $R(\mathsf{K})$ is isomorphic to the classical representation ring.

Two Theorems

Theorem 1. (Kasparov)

Let $\theta \in KK^{\mathsf{G}}(C(\mathcal{X}), \mathbb{C})$. If

$$\theta \mapsto 1 \in KK^{\mathsf{K}}(\mathbb{C}, \mathbb{C}),$$

then

$$\theta \mapsto \gamma \in KK^{\mathsf{G}}(\mathbb{C}, \mathbb{C}).$$

Theorem 2. (Bernstein-Gelfand-Gelfand)

There is a differential complex comprised of

- (section spaces of) direct sums of G-homogeneous line bundles over $\mathcal{X}$,
- G-invariant differential operators between them,

which resolves the trivial representation.

Some geometry

G — complex semisimple Lie group

Defn. A subgroup $P \leq G$ is called **parabolic** if G/P is compact.

Theorem. There is a 1-1 correspondence between

1. subsets of the set Σ of simple roots of G,
2. parabolic subgroups (up to conjugacy),
3. compact G-homogeneous spaces.
4. G-equivariant fibrations of $\mathcal{X} = G/B$.

Harmonic Decompositions

Every G-homogeneous line bundle E over $\mathcal{X}$ admits a decomposition:

$$L^2(\mathcal{X}; E) \cong \bigoplus_{\pi \in \widehat{\mathsf{K}}} V^{\pi *} \otimes (V^{\pi})_{\mu}$$

where $(V^{\pi})_{\mu} = \mu$-weight space of π, for some weight μ.

Notation.

$S \subseteq \Sigma$ — subset of simple roots

P_S — associated parabolic subgroup

$\mathsf{K}_S := \mathsf{P}_S \cap \mathsf{K}$

For each irrep σ of K_S,

$p_\sigma :=$ proj onto subspace of $L^2(\mathcal{X}; E)$ of *right* K_S-type σ.

Spectrally Proper Operators

$T : L^2(\mathcal{X}; E_1) \to L^2(\mathcal{X}; E_2)$ — bounded op.

Fix S and write T as a matrix w.r.t. harmonic decomposition $\{p_\sigma\}_{\sigma \in \widehat{\mathsf{K}}_S}$.

Definition.

- T is S-**spectrally finite** if this matrix has finitely many nonzero entries.
- T is S-**spectrally proper** if this matrix has finitely many nonzero entries in each row and column.

$$\mathcal{A}_S := \overline{\{S\text{-spectrally proper op's }\}}^{\|\cdot\|}$$

$$\mathcal{K}_S := \overline{\{S\text{-spectrally finite op's }\}}^{\|\cdot\|}$$

C^*-algebras associated to the fibrations

Definition. $$\mathcal{A} := \bigcap_{S \subseteq \Sigma} \mathcal{A}_S$$

Proposition. ($\mathsf{G} = \mathsf{SL}(n, \mathbb{C})$.) The following operators all belong to $\mathcal{A}$:

- $f \in C(\mathcal{X})$
- $\alpha(g)$ for $g \in \mathsf{G}$
- $F_\alpha := \frac{X_\alpha}{|X_\alpha|}$, where X_α is a BGG operator along the fibration corresp. to $\alpha \in \Sigma$.

Moreover, $[F_\alpha, f]$ and $[F_\alpha, \alpha(g)]$ belong to $\mathcal{K}_{\{\alpha\}}$.

Lattice of ideals

Definition.

$$\mathcal{J}_S := \mathcal{K}_S \cap \mathcal{A}$$

Theorem. $(\mathsf{G} = \mathsf{SL}(n, \mathbb{C}))$

1. $S \subset T \subset \Sigma \quad \Rightarrow \quad \mathcal{J}_T \triangleleft \mathcal{J}_S$.
2. For any $S, T \subseteq \Sigma$, $\mathcal{J}_S \cap \mathcal{J}_T = \mathcal{J}_{S \cup T}$.
3. $\mathcal{J}_\Sigma = \mathcal{K}$ (compact operators).

‘Normalized’ BGG complex

Proposition. The phases $F_\alpha = \frac{X_\alpha}{|X_\alpha|}$ of the simple BGG operators can be completed to produce a complex modulo $\sum\limits_{\alpha \in \Sigma} \mathcal{J}_{\{\alpha\}}$.