

# Universal coefficient theorems for Kirchberg's bivariant K-theory II

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## 1 The bootstrap class

**Locally closed subsets**

**Ideals for open subsets**

Recall that the continuous map  $\text{Prim}(A) \rightarrow X$  that is part of a  $C^*$ -algebra over  $X$  yields a lattice morphism  $\mathbb{O}(X) \rightarrow \mathbb{O}(\text{Prim } A) \cong \mathbb{I}(A)$ , mapping  $U \in \mathbb{O}(X)$  to the ideal  $A(U)$  of elements of  $A$  that *vanish outside*  $U$ .

**Definition 1.**  $Y \subseteq X$  *locally closed*:  $Y = U \setminus V$  with  $U, V \in \mathbb{O}(X)$ ,  $V \subseteq U$

**Definition and Lemma**

Let  $Y \subseteq X$  be locally closed, write  $Y = U \setminus V$  as above. Let  $A$  be a  $C^*$ -algebra over  $X$ .

$$A(Y) := A(U)/A(V)$$

does not depend on the choice of  $U$  and  $V$ .

*Proof.* Use that  $U \mapsto A(U)$  is a lattice morphism. □

**Fibres of  $C^*$ -algebra bundles**

*Example 2* (Fibres of a  $C^*$ -algebra bundle). Let  $X$  be finite and let  $U_x$  for  $x \in X$  be the *minimal open neighbourhood* of  $x$ . Then  $U_x \setminus \{x\}$  is open, so that  $\{x\}$  is locally closed.

View  $A(x) = A_x := A(\{x\})$  as the fibre of  $A$  at  $x$ .

*Example 3* (A space over which bundles have no fibres). Let  $X = [0, 1]$  with the topology

$$\mathbb{O}(X) = \{(t, 1] \mid t \in [0, 1]\}.$$

Then the locally closed subsets are of the form  $(a, b]$  with  $0 \leq a \leq b \leq 1$ . Hence  $\{x\}$  is not locally closed for any  $x \in [0, 1]$ .

$C^*$ -algebra bundles over this space have no well-defined fibres.

## 1.1 The complement of the bootstrap class

### Definition and Lemma

The following conditions for a  $C^*$ -algebra  $A$  over  $X$  are equivalent and define  $\mathcal{B}(X)^\perp$ :

- $K_*\left(A(Y)\right) = 0$  for all locally closed  $Y \subseteq X$ ,
- $K_*\left(A(U)\right) = 0$  for all open  $U \subseteq X$ ,
- each  $x \in X$  has an open neighbourhood  $U$  such that  $K_*\left(A(V)\right) = 0$  for all open  $V \subseteq U$ .

If  $X$  is finite, these are also equivalent to

- $K_*\left(A(U_x)\right) = 0$  for all  $x \in X$ ;
- $K_*\left(A(x)\right) = 0$  for all  $x \in X$ .

*Important ingredient in the proof.* There is a filtration  $\emptyset = \mathfrak{F}_0 X \subset \mathfrak{F}_1 X \subset \dots \subset \mathfrak{F}_\ell X = X$  by open subsets such that the differences  $X_j := \mathfrak{F}_j X \setminus \mathfrak{F}_{j-1} X$  are all *discrete*.  $\square$

## 1.2 Generators of the bootstrap class

**Definition 4.** Given  $x \in X$  and a  $C^*$ -algebra  $A$ , let  $i_x(A)$  be  $A$  together with the constant map  $x: \text{Prim}(A) \rightarrow X$  with value  $x$ . This is a  $C^*$ -algebra over  $X$  with

$$i_x A(U) = \begin{cases} A & \text{for } x \in U, \\ 0 & \text{for } x \notin U. \end{cases}$$

**Lemma 5.** *Let  $X$  be finite. Then  $\text{KK}_*^X(i_x(A), B) \cong \text{KK}_*(A, B(U_x))$ .*

*Proof.* Since  $i_x A(U) = A$  for  $U \supseteq U_x$  open  $i_x A(V) = 0$  for other open subsets, a cycle for  $\text{KK}_*^X(i_x(A), B)$  determines one for  $\text{KK}_*(i_x A(U_x), B(U_x)) = \text{KK}_*(A, B(U_x))$ , and the latter does not have to satisfy any further conditions.  $\square$

### The bootstrap class over finite spaces

**Lemma 6.** *For finite  $X$ , the bootstrap class is (equivalent to) the localising subcategory of  $\text{KK}(X)$  generated by  $i_x \mathbb{C}$  for  $x \in X$ .*

- More precisely, let  $\mathcal{B}(X)$  be the smallest class of separable  $C^*$ -algebras that contains  $i_x \mathbb{C}$  for  $x \in X$  and is closed under suspensions,  $\text{KK}^X$ -equivalence, countable direct sums, and mapping cones.
- $\text{KK}_*^X(A, B) = 0$  if  $A \in \mathcal{B}(X)$  and  $B \in \mathcal{B}(X)^\perp$  because  $\text{KK}_*^X(i_x \mathbb{C}, B) = 0$  for  $B \in \mathcal{B}(X)^\perp$ .
- $\mathcal{B}(X)$  and  $\mathcal{B}(X)^\perp$  together “generate” all of  $\text{KK}(X)$ .  
(They form a complementary pair of localising subcategories.)
- The last statement is non-trivial, but it follows easily from some general machinery.

## Characterisation of the bootstrap class

**Theorem 7.** *Let  $X$  be a finite  $T_0$ -space. For a nuclear separable  $C^*$ -algebra  $A$  over  $X$ , the following are equivalent:*

- $A$  belongs to the bootstrap class over  $X$ ;
- $A(U_x)$  belongs to the bootstrap class for all  $x \in X$
- $A(x) := A(U_x)/A(U_x \setminus \{x\})$  belongs to the bootstrap class in  $\text{KK}$  for all  $x \in X$
- $A(U)$  belongs to the bootstrap class for all  $U \in \mathcal{O}(X)$

## 1.3 Connection to the $C^*$ -algebra classification programme

### Making the fibres simple

**Theorem 8** (Kirchberg). *Any exact  $C^*$ -algebra is  $\text{KK}$ -equivalent to a simple one.*

*The latter can be assumed  $\mathcal{O}_\infty \otimes \mathbb{K}$ -stable, that is, purely infinite and  $C^*$ -stable.*

**Theorem 9.** *Let  $X$  be finite. Any object of the bootstrap class  $\mathcal{B}(X)$  is  $\text{KK}^X$ -equivalent to a separable, nuclear,  $\mathcal{O}_\infty \otimes \mathbb{K}$ -stable one for which the map  $\text{Prim}(A) \rightarrow X$  is a homeomorphism. Then the  $A(x)$  for  $x \in X$  are the simple subquotients of  $A$ .*

**Theorem 10** (Kirchberg). *Let  $A$  and  $B$  be separable, nuclear,  $\mathcal{O}_\infty \otimes \mathbb{K}$ -stable  $C^*$ -algebras with homeomorphisms  $\text{Prim}(A) \cong X \cong \text{Prim}(B)$ . Then any invertible element in  $\text{KK}_*^X(A, B)$  lifts to a  $*$ -isomorphism  $A \rightarrow B$  over  $X$ .*

Hence the representative in the previous theorem is *unique* up to isomorphism over  $X$ .

### Summary from the classification viewpoint

Classifying separable, nuclear, purely infinite, stable  $C^*$ -algebras with finite primitive ideal space  $X$  and simple subquotients in the bootstrap class up to isomorphism over  $X$   $\iff$  classifying objects of  $\mathcal{B}(X)$  up to  $\text{KK}^X$ -equivalence.

### Open question

Is there a  $K$ -theoretic complete invariant that achieves this classification?

## 2 The Universal Coefficient Theorem

### Question

Can we compute  $\text{KK}_*(A, B)$  from  $K_*(A)$  and  $K_*(B)$ ?

**Theorem 11.** *Let  $A$  and  $B$  be separable  $C^*$ -algebras, suppose that  $A$  belongs to the bootstrap class.*

*Then there is a natural short exact sequence of  $\mathbb{Z}/2$ -graded Abelian groups*

$$\text{Ext}\left(K_{*+1}(A), K_*(B)\right) \twoheadrightarrow \text{KK}_*(A, B) \rightarrow \text{Hom}\left(K_*(A), K_*(B)\right).$$

- $\mathcal{I} := \ker K_*$

- Naturality implies that the Kasparov product

$$\mathrm{KK}(A, B) \otimes \mathrm{Ext}(K_*(C, A)) \rightarrow \mathrm{Ext}(K_*(C, B))$$

descends to  $\mathrm{Hom}(K_*(A), K_*(B))$ , that is,  $\mathcal{I} \circ \mathcal{I} = 0$  in the bootstrap class.

## Some applications

**Corollary 12.** *If both  $A$  and  $B$  belong to the bootstrap category, then*

- $\alpha \in \mathrm{KK}_0(A, B)$  is a KK-equivalence if and only if  $K_*(\alpha): K_*(A) \rightarrow K_*(B)$  is invertible.
- Any grading preserving group homomorphism  $K_*(A) \rightarrow K_*(B)$  lifts to an element of  $\mathrm{KK}_*(A, B)$ .
- An isomorphism  $K_*(A) \cong K_*(B)$  lifts to a KK-equivalence  $A \simeq B$ .

**Theorem 13.** *Any pair of countable Abelian groups is the K-theory of some  $C^*$ -algebra in the bootstrap class.*

### Summary

K-theory is a complete invariant for KK-equivalence classes of  $C^*$ -algebras in the bootstrap class.

## 2.1 Proof of the UCT

$\mathfrak{T}$  Kasparov theory, viewed as a category

$\mathfrak{C}$  category  $\mathfrak{Ab}_c^{\mathbb{Z}/2}$  of countable  $\mathbb{Z}/2$ -graded Abelian groups

$F$  functor  $F := K_*: \mathfrak{T} \rightarrow \mathfrak{C}$

$A \in \mathfrak{C}$   $A$  is an object of the category  $\mathfrak{C}$

### Basic Lemma

*If  $A \in \mathfrak{C}$  is projective (free), there is  $F^+(A) \in \mathfrak{T}$  with*

$$\mathfrak{T}(F^+(A), B) \cong \mathrm{Hom}(A, F(B)).$$

*Moreover,  $F \circ F^+(A) \cong A$ .*

Equivalently: the left adjoint of  $F$  is defined on all projective objects of  $\mathfrak{C}$ .

*Proof.*

- Since  $F^+$  is compatible with suspensions and direct sums, it suffices to check this for  $A = \mathbb{Z}$ .
- Take  $F^+(\mathbb{Z}) = \mathbb{C}$  and use

$$\mathrm{Hom}(\mathbb{Z}, K_*(B)) \cong K_0(B) \cong \mathrm{KK}_0(\mathbb{C}, B)$$

and  $K_*(\mathbb{C}) = \mathbb{Z}$ . □

## Exact chain complexes and projective objects

**Definition 14.** A chain complex  $(C_n, d_n)$  with entries in  $\mathfrak{T}$  is *F-exact* if the chain complex  $F(C_n), F(d_n)$  is exact.

**Definition 15.** A separable  $C^*$ -algebra  $A \in \mathfrak{T}$  is *F-projective* if the chain complex

$$\cdots \rightarrow \mathfrak{T}_*(A, C_n) \xrightarrow{\mathfrak{T}_*(A, d_n)} \mathfrak{T}_*(A, C_{n-1}) \rightarrow \cdots$$

is exact for any *F*-exact chain complex  $(C_n, d_n)$ .

**Lemma 16.** *Objects of the form  $F^\perp(A)$  for projective  $A \in \mathfrak{C}$  are F-projective.*

*Proof.*  $\mathfrak{T}(F^\perp(A), B) \cong \mathfrak{C}(A, F(B))$  and  $\mathfrak{C}(A, \square)$  is exact. □

## Description of projective objects

**Theorem 17.** *The functors  $F$  and  $F^\perp$  restrict to an equivalence of categories*

$$\{F\text{-projective objects in } \mathfrak{T}\} \simeq \{\text{projective objects in } \mathfrak{C}\}.$$

*Proof.*

- $F^\perp$  is a functor and  $F \circ F^\perp(A) \cong A$ .
- Let  $B \in \mathfrak{T}$  be *F*-projective, let  $\alpha: A \twoheadrightarrow F(B)$  be a quotient map with projective  $A$ .
- $\mathfrak{C}(A, F(B)) \cong \mathfrak{T}(F^\perp(A), B)$  maps  $\alpha \mapsto \alpha^\perp$
- $\alpha^\perp$  is split epimorphism because  $B$  is *F*-projective and  $F(\alpha^\perp)$  is surjective.
- $B = F^\perp(eA)$  for an idempotent map  $e: A \rightarrow A$  because  $F^\perp$  is fully faithful.
- $eA$  is again projective. □

## Lifting projective resolutions

**Definition 18.** An *F-projective resolution* of  $A \in \mathfrak{T}$  is an *F*-exact chain complex

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0 \rightarrow \cdots$$

with *F*-projective  $P_n$  for  $n \in \mathbb{N}$ .

**Theorem 19** (Geometric resolutions). *A projective resolution of  $F(A)$  in  $\mathfrak{C}$  lifts uniquely up to isomorphism to an F-projective resolution in  $\mathfrak{T}$ .*

*Proof.* Let  $P_n := F^\perp(B_n)$  for a projective resolution  $(B_n)$ , use  $\mathfrak{C}(B_0, F(A)) \cong \mathfrak{T}(P_0, A)$  to get the map  $P_0 \rightarrow A$ . □

## Proof of the UCT for plain Kasparov theory

### Crucial fact

Any object of  $\mathfrak{C} = \mathfrak{Ab}_c^{\mathbb{Z}/2}$  has a free resolution of *length* 1.

- For  $A \in \mathfrak{T}$ , choose a free resolution of  $F(A)$  of length 1 and lift it to an  $F$ -projective resolution  $0 \rightarrow P_1 \xrightarrow{d_1} P_0 \rightarrow A$  in  $\mathfrak{T}$ .
- Since  $\mathfrak{T}$  is *triangulated*, the map  $d_1$  embeds in an *exact triangle*  $P_1 \xrightarrow{d_1} P_0 \rightarrow A' \rightarrow P_1[1]$ .

To get this for  $\mathfrak{T} = \text{KK}$ , view  $d_1 \in \text{KK}_1(\mathcal{C}_0(\mathbb{R}) \otimes P_1, P_0)$  as the equivalence class of a  $C^*$ -algebra extension  $P_0 \twoheadrightarrow A' \twoheadrightarrow \mathcal{C}_0(\mathbb{R}) \otimes P_1$  with a completely positive contractive section.

- An analysis of the long exact sequence for this exact triangle yields a short exact sequence

$$\text{Ext}(\text{K}_{*+1}(A), \text{K}_*(B)) \twoheadrightarrow \text{KK}_*(A', B) \twoheadrightarrow \text{Hom}(\text{K}_*(A), \text{K}_*(B)).$$

because  $\text{KK}_*(P_0, B) \cong \text{Hom}(F(P_0), F(B))$ .

### Proof of the UCT continued

- There is  $f \in \mathfrak{T}(A', A)$  making the following diagram commute:

$$\begin{array}{ccccc} P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & A \\ \parallel & & \parallel & & \uparrow f \\ P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & A' \end{array}$$

- $f$  induces an isomorphism on K-theory.
- The class of  $X \in \mathfrak{T}$  for which  $f$  induces an isomorphism on  $\mathfrak{T}_*(X, \square)$  contains  $\mathbb{C}$  and is closed under isomorphism, suspensions, direct sums, and mapping cones.
- If  $A$  belongs to the bootstrap class, then  $f$  is invertible.

## 2.2 Abstraction

Most parts of the proof of the UCT work in all kinds of equivariant KK-theories.

### Dramatis Personæ

$\mathfrak{T}$  a triangulated category

$\mathfrak{C}$  an Abelian category with a translation automorphism

$F$  a homological functor  $\mathfrak{T} \rightarrow \mathfrak{C}$  that intertwines the translation automorphisms

**Definition 20.** We call  $F$  an *invariant without hidden symmetries* if

- $\mathfrak{C}$  has enough projective objects;
- for  $A \in \mathfrak{C}$  projective, there is  $F^+(A) \in \mathfrak{T}$  with  $\mathfrak{T}(F^+(A), B) \cong \mathfrak{C}(A, F(B))$ ;
- $F \circ F^+(A) \cong A$  for all projective  $A \in \mathfrak{C}$ .

### An example of hidden symmetries

#### Dramatis Personæ

$\mathfrak{T}$   $\mathrm{KK}^G$  for some discrete group  $G$

$\mathfrak{C}$   $\mathfrak{Ab}_c^{\mathbb{Z}/2}$ , the translation automorphism shifts the grading

$F$   $F(A, \alpha) = K_*(A)$ : forget group action and take K-theory

- $\mathrm{KK}_*^G(\mathcal{C}_0(G, A), B) \cong \mathrm{KK}_*(A, B)$  shows that  $F^+(\mathbb{Z}) = \mathcal{C}_0(G)$  works.
- But  $FF^+(\mathbb{Z}) = K_*(\mathcal{C}_0(G)) = \mathbb{Z}[G]$  is too big.
- A poor invariant like this can be refined uniquely to one without hidden symmetries:

**Theorem 21.** *If there are enough  $F$ -projective objects, there is a unique stable homological functor  $F'$  without hidden symmetries and  $\ker F = \ker F'$ .*

### The example rectified

#### Dramatis Personæ

$\mathfrak{T}$   $\mathrm{KK}^G$  for some discrete group  $G$

$\mathfrak{C}$  category of countable  $\mathbb{Z}/2$ -graded  $\mathbb{Z}[G]$ -modules

$F$   $F(A, \alpha) = (K_*(A), K_*(\alpha))$ : take K-theory and *remember the induced action* of  $G$  on  $K_*(A)$

- $\mathrm{KK}_*^G(\mathcal{C}_0(G), B) \cong \mathrm{KK}_*(\mathbb{C}, B) \cong \mathrm{Hom}_G(\mathbb{Z}[G], K_*(B))$  shows that  $F^+(\mathbb{Z}[G]) = \mathcal{C}_0(G)$  works.
- $FF^+(\mathbb{Z}[G]) = K_*(\mathcal{C}_0(G)) \cong \mathbb{Z}[G]$
- As above, this yields  $F^+$  first for all free modules, then for all projective modules.
- This time the functor  $F$  has no hidden symmetries.

## Projective objects and abstract bootstrap class

Define  $F$ -exact chain complexes and  $F$ -projective objects in  $\mathfrak{T}$  as above.

**Lemma 22.**  $F$  and  $F^\perp$  restrict to an equivalence of categories

$$\{F\text{-projectives in } \mathfrak{T}\} \simeq \{\text{projectives in } \mathfrak{C}\}.$$

**Theorem 23.** Projective resolutions of  $F(A)$  lift uniquely up to isomorphism to  $F$ -projective resolutions of  $A$ .

**Definition 24** (Abstract bootstrap category). Let  $\mathcal{B}$  be the localising subcategory of  $\mathfrak{T}$  generated by the  $F$ -projective objects.

Let  $\mathcal{B}^\perp$  consist of all  $A$  with  $F(A) = 0$ .

**Theorem 25.**  $\mathcal{B}$  and  $\mathcal{B}^\perp$  are complementary if  $F$  commutes with  $\oplus$ .

## The abstract UCT

**Theorem 26** (Universal Coefficient Theorem). If  $F(A)$  has a projective resolution of length 1 and  $A \in \mathcal{B}$ ,  $B \in \mathfrak{T}$ , there is a UCT exact sequence

$$\text{Ext}_{\mathfrak{C}}^1(F(A)[1], F(B)) \rightarrow \mathfrak{T}(A, B) \rightarrow \text{Hom}_{\mathfrak{C}}(F(A), F(B))$$

**Theorem 27.** Any object of  $\mathfrak{C}$  with a projective resolution of length 1 is  $F(A)$  for some  $A \in \mathcal{B}$ .

## Open problem

The range of  $F$  may be smaller than  $\mathfrak{C}$ . Is it always an *exact* subcategory of  $\mathfrak{C}$ ? This would often allow to describe the range of  $F$ .

## What happens for long projective resolutions?

- Without projective resolutions of length 1, the short exact sequence becomes a *spectral sequence*.
- Its higher differentials provide *obstructions* for lifting a map  $F(A) \rightarrow F(B)$  back to  $\mathfrak{T}$ .
- We should expect that there are  $A, B \in \mathcal{B}$  with  $F(A) \cong F(B)$  but  $A \not\cong B$ .

**Theorem 28.** Let  $\alpha \in \mathfrak{T}(A, B)$ . If  $F(\alpha)$  is invertible and  $A, B \in \mathcal{B}$ , then  $\alpha$  is invertible.

Equivalently,  $F$  restricts to a *rigid* invariant on  $\mathcal{B}$ .

## Summary: Homological algebra

- If our invariant  $F$  has *hidden symmetries*, there is a unique way to make these explicit, replacing  $F$  by an invariant without hidden symmetries.
- If  $F: \mathfrak{T} \rightarrow \mathfrak{C}$  has no hidden symmetries, then the *classification problem* can be answered using *homological algebra in  $\mathfrak{C}$* .
- The bootstrap class and the object-kernel of  $F$  are *complementary* If  $F$  commutes with direct sums.

Thus *localisation* at  $F$  is easy to understand.

- *Projective resolutions* in  $\mathfrak{C}$  lift uniquely.
- If  $F(A)$  has a projective resolution of *length 1* and  $A$  belongs to the bootstrap class for  $F$ , then  $A$  satisfies a *Universal Coefficient Theorem*.
- Its ingredients are the *derived functors* in the target category of  $F$ .

## 3 An invariant for Kirchberg's bivariant K-theory

### Question

Which invariant should we use to study  $\mathrm{KK}^X$  for a (finite) topological space  $X$ ?

Is there a *complete invariant*?

- $\bigoplus_{x \in X} \mathrm{K}_*(A(U_x))$  is as *rigid* as possible: it gives the right bootstrap class.
- But it is never complete unless  $X$  is discrete.
- $\bigoplus_{U \in \mathbb{O}(X)} \mathrm{K}_*(A(U))$  is not better: it still fails to be complete for the non-Hausdorff two-point space.

**Definition 29.** *Filtrated K-theory* is the invariant without hidden symmetries associated to  $F(A) := \bigoplus_{Y \in \mathrm{LC}(X)} \mathrm{K}_*(A(Y))$ , where  $\mathrm{LC}(X)$  denotes the *locally closed* subsets of  $X$ .

### Filtrated K-theory

**Theorem 30.** *Filtrated K-theory is a complete invariant if  $\mathbb{O}(X)$  is totally ordered.*

**Theorem 31.** *Filtrated K-theory is not complete for  $Y_3 = \{0, 1, 2, 3\}$  with  $0 \prec 1, 2, 3$  and no relation among  $\{1, 2, 3\}$ .*

*For this space, adding another K-theory group to  $F$  refines filtrated K-theory to a complete invariant.*

### Question

Can we always refine filtrated K-theory to a manageable and complete invariant?

### What I expect

This should become *impossible*, in a sense to be made precise, for sufficiently complicated spaces, maybe already for  $Y_n$  defined like  $Y_3$  with  $n \geq 7$ .

## Where problems could come from

- In the examples I know, the K-theory functors that are used to build a complete invariant correspond to *irreducible modules* over the quiver algebra of the partial order  $\prec$  on  $X$ .
- In this algebraic context, there are three cases:
  - finite** there are only *finitely many* irreducible modules
  - tame** there are infinitely many irreducible modules, but they can still be classified
  - wild** no classification of irreducible modules is possible
- The examples where things work all correspond to the finite case.

## How to find the hidden symmetries?

### Problem

Find the functor without hidden symmetries attached to  $F(A) := \bigoplus_{Y \in \text{LC}(X)} K_*(A(Y))!$

- Find a *representing object* for this functor or, equivalently, for the functors  $A \mapsto K_*(A(Y))$ , that is,

$$\text{KK}_*^X(\mathcal{R}_Y, A) \cong K_*(A(Y)).$$

- Then  $\mathcal{R} := \bigoplus_{Y \in \text{LC}(X)} \mathcal{R}_Y$  satisfies  $\text{KK}_*^X(\mathcal{R}, A) \cong F(A)$ .
- The graded ring  $\mathcal{NT}_* := \text{KK}_*(\mathcal{R}, \mathcal{R})$  acts naturally on  $F(A)$  by Kasparov product.

**Theorem 32.** *Representing objects  $\mathcal{R}_Y$  exist and can be constructed explicitly, and filtered K-theory is the functor  $A \mapsto \bigoplus_{Y \in \text{LC}(X)} K_*(A(Y))$ , viewed as a functor to the category  $\mathfrak{C}$  of countable ( $\mathbb{Z}/2$ -graded)  $\mathcal{NT}_*$ -modules.*