Universal coefficient theorems for Kirchberg's bivariant K-theory I

Ralf Meyer

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1 Non-commutative topology

1.1 The role of Kasparov theory and K-theory

Definition 1. A *non-commutative homology theory* is a functor on a category of (separable) C^*-algebras (with extra structure) that is

- C^*-stable (Morita invariant)
- split-exact
- homotopy invariant
- has Puppe exact sequence for mapping cones

Example 2. K-theory is a non-commutative homology theory for C^*-algebras. It maps separable C^*-algebras to the category $\mathfrak{Ab}_{\mathrm{c}}^{\mathbb{Z}/2}$ of $\mathbb{Z}/2$-graded countable Abelian groups.

Example 3. KK^G is a non-commutative homology theory for C^*-algebras with a G-action.

Theorem 4 (Joachim Cuntz and Nigel Higson). *Bivariant* KK*-theory is the universal C^*-stable, split-exact functor on the category of separable C^*-algebras.*

That is, a functor from the category of separable C^*-algebras to some additive category factors through KK *if and only if it is C^*-stable and split-exact, and this factorisation is unique if it exists.*

Equivariant versions of KK are characterised by analogous universal properties.

Corollary 5. *C^*-stability and split-exactness $\Longrightarrow$ homotopy invariance, Bott periodicity, Connes–Thom Isomorphism, …*

The central role of K-theory

Theorem 6. *Let X and Y be locally compact spaces with*

$$\mathrm{K}_*(X) \cong \mathrm{K}_*(Y).$$

Then $\mathcal{C}_0(X)$ and $\mathcal{C}_0(Y)$ are KK*-equivalent.*

Thus $F\Big(\mathcal{C}_0(X)\Big) \cong F\Big(\mathcal{C}_0(Y)\Big)$ if F is C^*-stable split-exact.

Proof. This follows from the Universal Coefficient Theorem. □

Bad news
All the intricacies of the stable homotopy category disappear in the non-commutative setting.

Good news
All the intricacies of the stable homotopy category disappear in the non-commutative setting.

Equivariant Kasparov theory

- KK has become a *tool* instead of an *object* of study.
- You can prove theorems *with* KK, but not *about* KK.
- For each additional structure C^*-algebras can carry, there is an appropriate version of KK:
  - group actions of locally compact groups
  - groupoid actions of locally compact groupoids
  - coactions of locally compact quantum groups
  - $C(X)$-algebras
  - C^*-algebra bundles over non-Hausdorff spaces
    This situation generates *Kirchberg's bivariant* K*-theory.*
- These *equivariant* bivariant K-theories are more intricate—you may also prove theorems *about* them.

1.2 Commutative versus non-commutative topology

Why is non-commutative topology so effective?

Question
Some results in topology can, so far, be proved *only* with C^*-algebra methods.

How can that be?

Possible answer
Applications of C^*-algebras in topology usually involve spaces with an extra structure like a group action.

The interaction of the extra structure with the topology of the space becomes *much simpler* in the non-commutative setting.

A hopeless task

Exercise
Classify simplicial complexes up to (stable) homotopy equivalence! — Forget it!

Theorem 7. *K-theory is a complete invariant up to KK-equivalence for C^*-algebras in the bootstrap class.*

Two C^*-algebras in the bootstrap class are KK-equivalent if and only if they have isomorphic K-theory.

Any pair of countable Abelian groups is the K-theory of some C^*-algebra in the bootstrap class.

A ridiculous question

Question
If a G-map $f\colon A \to B$ is a (stable) homotopy equivalence, is it automatically a G-equivariant (stable) homotopy equivalence? — Forget it!

Theorem 8 (Extra strong Baum–Connes property)**.** *Let G be a locally compact group that acts properly by affine isometries on a Hilbert space. If $\alpha \in \mathrm{KK}^G(A, B)$ becomes invertible in KK^H for each compact subgroup $H \subseteq G$, then α is invertible in KK^G.*

Corollary 9 (Homotopic actions are equivalent)**.** *Let G be a torsion-free group with the extra strong Baum–Connes property. Then homotopic G-actions on a separable C^*-algebra A are KK^G-equivalent.*

Example 10. Non-commutative tori of the same dimensionare KK-equivalent.

A general line of inquiry

- We want to study equivariant homology theories on a category of C^*-algebras with some extra structure.
- First we need a bivariant K-theory in this setting.
- Secondly, we need an *invariant* that we use to probe this equivariant KK-category.

Goal
Compute the equivariant KK-theory and other equivariant homology theories using the chosen invariant.

1.3 The rigidity question

What can we know?

Rigidity question
If the invariant $F(A) = 0$ vanishes, does it follow that A is equivariantly KK-equivalent to 0?

Equivalently, if a morphism α induces an isomorphism on the invariant F, is it already invertible?

Definition 11. We call the invariant F *rigid* if $F(A) = 0 \iff A \cong 0$.

Example 12 (Extra strong Baum–Connes property). Let G be a locally compact group, let F be the family of restriction functors $\mathrm{KK}^G \to \mathrm{KK}^H$ for $H \subseteq G$ compact.

This invariant is rigid if and only if the group G has the extra strong Baum–Connes property.

Rigid invariants

Example 13. Let G be a *connected* Lie group with *torsion-free fundamental group.*

- Let $T \subseteq G$ be a maximal torus. The forgetful functor $\mathrm{KK}^G \to \mathrm{KK}^T$ is a rigid invariant.
- The crossed product functor (descent)

$$\mathrm{KK}^G \to \mathrm{KK}, \qquad A \mapsto A \rtimes G$$

  is rigid.

Example 14 (Bootstrap category). There exist separable C^*-algebras with $\mathrm{K}_*(A) = 0$ but $\mathrm{KK}_0(A, A) \neq 0$.

Localisation at the invariant

- We want to focus on the part of our bivariant K-theory that our invariant can detect.
- There is a general process to do this: localisation of triangulated categories
- Often it works as follows.

Theorem 15. *Let $\mathfrak{T}$ be a triangulated category and let F be a stable homological functor.*

- $\mathcal{N} := \{A \mid F(A) = 0\}$
- $\mathcal{N}^{\dashv} := \{A \mid \mathfrak{T}(A, B) = 0 \quad \forall B \in \mathcal{N}$

If $\mathcal{N} \cup \mathcal{N}^{\dashv}$ generates $\mathfrak{T}$, then $\mathfrak{T}/\mathcal{N}$ is equivalent to $\mathcal{N}^{\dashv}$.

Example 16. This works for the K-theory functor on KK, where $\mathcal{N}^{\dashv}$ is the *bootstrap class* or *UCT class.*

1.4 The classification question

Lemma 17. *After localisation, the invariant F becomes rigid, that is, it detects zero objects and isomorphisms: α invertible $\iff$ $F(\alpha)$ invertible*

Classification question

If $F(A) \cong F(B)$, does it follow that $A \cong B$?

If yes, can you also describe the range of F?

Definition 18. We call the invariant F *complete* if the answer to both questions is “Yes” after localisation.

Example 19. K-theory is a complete invariant on KK.

Example 20. The forgetful functor $\mathrm{KK}^{\mathbb{Z}} \to \mathrm{KK}$ is both rigid and complete.

Necessity of a Universal Coefficient Theorem

For most equivariant situations, we do not expect a *manageable* complete invariant to exist.

(The identity invariant is always complete and rigid.)

Open question

Is there a manageable complete invariant for $\mathbb{Z}^2$-actions?

Theorem 21 (No-Go Theorem)**.** *Let* $\mathcal{I} := \ker F$ *be the ideal of morphisms in* $\mathfrak{T}$ *annihilated by the invariant* F*. Assume enough* $\mathcal{I}$*-projective objects.*

$\mathcal{I} \circ \mathcal{I} \neq 0 \implies$ *there are* A, B *with* $F(A) \cong F(B)$ *but* $A \ncong B$*.*

Corollary 22. *Let* F *be a complete, rigid invariant on* $\mathfrak{T}$*. Then there is a natural exact sequence*

$$\mathcal{I}/\mathcal{I}^2(A, B) \rightarrowtail \mathfrak{T}(A, B) \twoheadrightarrow \mathfrak{T}/\mathcal{I}(A, B).$$

Homological algebra in triangulated categories

Questions

When is $\mathcal{I}^2 = \mathcal{I} \circ \mathcal{I} = 0$?

Are $\mathfrak{T}/\mathcal{I}$ and $\mathcal{I}/\mathcal{I}^2$ derived functors?

- There is a general machinery for *homological algebra* in triangulated categories using an ideal such as $\ker F$.
- It yields an *Abelian approximation* $\mathfrak{C}$ to our triangulated category $\mathfrak{T}$ and a homological functor $F' \colon \mathfrak{T} \to \mathfrak{C}$ such that homological algebra in $\mathfrak{C}$ lifts back to $\mathfrak{T}$.
- A universal coefficient exact sequence exists for A if $F'(A)$ has a *projective resolution of length* 1.
- This neet not be the case and may be hard to check.

Difficult problem

Refine an incomplete invariant to make it (more) complete!

2 C^*-algebra bundles over non-Hausdorff spaces

- We will first reformulate the notion of a $C(X)$-algebra in a way suitable for non-Hausdorff spaces.
- This leads to a definition of a C^*-algebra (bundle) over a non-Hausdorff space, which we illustrate by some examples.
- Then we introduce Kirchberg's bivariant KK-theory for such C^*-algebra bundles.

2.1 Equivalent definitions in the Hausdorff case

Theorem 23. *Let X be a Hausdorff topological space and A a C^*-algebra. The following additional structures on A are equivalent:*

- *a non-degenerate *-homomorphism from $\mathcal{C}_0(X)$ to the centre of the multiplier algebra of A*
- *a non-degenerate *-homomorphism $\mathcal{C}_0(X, A) \to A$;*

  it has a class in $\mathrm{KK}(\mathcal{C}_0(X, A), A)$
- *a continuous map $\mathrm{Prim}(A) \to X$, where $\mathrm{Prim}(A)$ is the primitive ideal space of A*
- *a map from the lattice of open subsets of X to the lattice of ideals in A that commutes with arbitrary suprema and finite infima*

The last two conditions make sense for non-Hausdorff spaces.

2.2 Definition and Examples

Definition 24. Let X be a topological space. A *C^*-algebra over X* is a C^*-algebra A together with a continuous map $\psi\colon \mathrm{Prim}(A) \to X$.

Theorem 25. *The lattice $\mathbb{O}(\mathrm{Prim}\, A)$ of open subsets of $\mathrm{Prim}(A)$ is isomorphic to the lattice $\mathbb{I}(A)$ of ideals in A.*

Lemma 26. *A continuous map $\psi\colon \mathrm{Prim}(A) \to X$ is equivalent to a lattice morphism $\mathbb{O}(X) \to \mathbb{O}(\mathrm{Prim}\, A) \cong \mathbb{I}(A)$ that preserves arbitrary suprema (provided X is sober).*

Finite T_0-spaces and partial orders

- We will concentrate on *finite* topological T_0-spaces here.
- Let $x \preceq y$ if $\overline{\{x\}} \subseteq \overline{\{y\}}$, this is a *partial order* on X.

Lemma 27. *A subset $S \subseteq X$ is closed $\iff$ $x \prec y \in S$ implies $x \in S$.*
A subset $S \subseteq X$ is open $\iff$ $x \succ y \in S$ implies $x \in S$.

Important observation
T_0-topologies and partial orders on finite sets are equivalent.

Example 28. Consider $X_n = \{1, \dots, n\}$ with the total order $1 \succ 2 \succ \cdots \succ n$.

- The open subsets are $U_k := \{1, \dots, k\}$ for $k = 0, \dots, n$.
- A C^*-algebra over X_n is a C^*-algebra A together with an increasing *chain* of ideals

$$\{0\} = A(U_0) \lhd A(U_1) \lhd \cdots \lhd A(U_{n-1}) \lhd A(U_n) = A.$$

- For $n = 2$, we get a chain $I \lhd A$, so that we study C^*-algebra extensions.

Example 29. Consider $Y_n = \{0, 1, \ldots, n\}$ with the partial order $0 \prec 1, 2, \ldots, n$ and no further relation between $1, \ldots, n$.

- The open subsets of Y_n are all subsets of $\{1, \ldots, n\}$ and Y_n itself.
- The lattice of open subsets is already generated by the singletons $U_j := \{j\}$ for $j = 1, \ldots, n$ and $U_0 := Y_n$.
- A C^*-algebra over Y_n is a C^*-algebra A together with n *orthogonal* ideals $A(U_j)$, $j = 1 \ldots, n$, that is, $A(U_j) \cap A(U_k) = \{0\}$.
- This is equivalent to a C^*-algebra extension $I \rightarrowtail A \twoheadrightarrow A/I$ with a direct sum decomposition $I = I_1 \oplus I_2 \oplus \cdots \oplus I_n$.

2.3 Kirchberg's bivariant K-theory

Definition 30. A *morphism* between two C^*-algebras over X is a *-homomorphism $f\colon A \to B$ that maps $A(U)$ to $B(U)$ for all open subsets $U \subseteq X$.

Theorem 31. *If X is Hausdorff and A and B are C^*-algebras over X (equivalently, $\mathcal{C}_0(X)$-algebras), then the morphisms are the $\mathcal{C}_0(X)$-linear *-homomorphisms.*

Definition 32 (Eberhard Kirchberg)**.** Let A and B be C^*-algebras over X and let $(\mathcal{E}, F)$ be a Kasparov cycle for $\mathrm{KK}_*(A, B)$. We call it a *Kasparov cycle over X* if $A(U) \cdot \mathcal{E} \subseteq \mathcal{E} \cdot B(U)$ for all $U \in \mathbb{O}(X)$.

$\mathrm{KK}_*^X(A, B)$ is the group of homotopy classes of such Kasparov cycles over X.

Formal properties

- Since we only impose restrictions on the Hilbert module, the *Kasparov product* works as usual for Kasparov cycles over X.
- There is an *exterior product*

  $$\mathrm{KK}_*^X(A, B) \otimes \mathrm{KK}_*(D, E) \to \mathrm{KK}_*^X(A \otimes D, B \otimes E).$$

  This allows us to carry over properties like *C^*-stability* and *Bott periodicity* from KK to KK^X.
- There are *long exact sequences* in both variables for an extension $I \rightarrowtail E \twoheadrightarrow Q$ of C^*-algebras over X with a completely positive contractive section over X, that is, the section maps $Q(U)$ to $E(U)$ for all $U \in \mathbb{O}(X)$.

  In particular, KK^X is *split-exact* in both variables.

Theorem 33 (Universal property)**.** *Kirchberg's bivariant* K*-theory is the universal split-exact C^*-stable functor on the category of separable C^*-algebras over X.*

3 The bootstrap class

3.1 The classical case

Definition 34. The *bootstrap class* $\mathcal{B}$ in KK is the smallest class of C^*-algebras containing the *generator* $\mathbb{C}$ and closed under the following operations:

- suspensions $A \mapsto \mathcal{C}_0(\mathbb{R}, A)$
- countable direct sums
- KK-equivalence
- mapping cones: if $f\colon A \to B$ is a *-homomorphism and A and B belong to the bootstrap class, so does $\operatorname{cone}(f)$.

Lemma 35. *The bootstrap class $\mathcal{B}$ is also closed under*

- *extensions with completely positive section*
- *inductive limits with completely positive approximations*
- *fibred products of $A \xrightarrow{p} B \leftarrow A'$ if p is surjective with a completely positive section*

Theorem 36. *Let G be a locally compact group with the extra strong Baum–Connes property and let A be a G-C^*-algebra.*
$A \rtimes H$ in $\mathcal{B}$ for all compact subgroups $H \subseteq G \implies A \rtimes G$ in $\mathcal{B}$

Theorem 37. *Let G be a connected Lie group with torsion-free fundamental group, let $T \subseteq G$ be a maximal torus, and let A be a G-C^*-algebra.*
$A \rtimes G$ in $\mathcal{B} \iff A \rtimes T$ in $\mathcal{B} \implies A$ in $\mathcal{B}$

Further properties of the bootstrap class

Question
Are all separable *nuclear* C^*-algebras in the bootstrap class?

Example 38 (Georges Skandalis). If G is a cocompact lattice in $\operatorname{Sp}(n,1)$, then $C^*_{\mathrm{r}}(G)$ does not belong to the bootstrap class.

Reason: The image of $\gamma \in \mathrm{KK}^G_0(\mathbb{C},\mathbb{C})$ in $\mathrm{KK}_0(C^*_{\mathrm{r}}G, C^*_{\mathrm{r}}G)$ acts identically on $\mathrm{K}_*(C^*_{\mathrm{r}}G)$, but it is not invertible.

Exercise
The bootstrap class is not closed under crossed products by actions of $\mathbb{Z}/2$.

Hint (Chris Phillips)
There is an action of $\mathbb{Z}/2$ on a contractible, commutative C^*-algebra with $\mathrm{K}_*(\mathbb{Z}/2 \ltimes A) \neq 0$. Tensor this with Skandalis' counterexample.

3.2 Bootstrap class over a space

Localisation at K-theory over a space X

X: a topological T_0-space.

$F(A)$: $\bigoplus_{U\in\mathbb{O}(X)} \mathrm{K}_*\bigl(A(U)\bigr)$ as functor on $\mathrm{KK}(X)$

$\mathcal{B}(X)^{\dashv}$: $\{A \in\in \mathrm{KK}(X) \mid F(A) = 0\}$

- If $A \in \mathcal{B}(X)^{\dashv}$, then any computation that is based on K-theoretic invariants of A *must give zero.*
- The bootstrap class is supposed to be the *localisation* of $\mathrm{KK}(X)$ at the subcategory $\mathcal{B}(X)^{\dashv}$: it is a "quotient" of $\mathrm{KK}(X)$ where
  - objects of $\mathcal{B}(X)^{\dashv}$ become zero;
  - an element in $\mathrm{KK}_0^X(A, B)$ becomes invertible if it induces an isomorphism $F(A) \cong F(B)$.

Lemma 39. *Let $\mathcal{U}$ be an open covering of X.*
If $\mathrm{K}_*\bigl(A(U)\bigr) = 0$ whenever $U \in \mathbb{O}(X)$ is contained in some $V \in \mathcal{U}$, then $A \in \mathcal{B}(X)^{\dashv}$.

Small open subsets versus fibres

- If X is a *Hausdorff space*, then a C^*-algebra over X has *fibres* $A(x) := A/A(X\setminus\{x\})$ for all $x \in X$.
- For continuous bundles of nuclear C^*-algebras, Marius Dadarlat's lecture showed that we can detect invertibility of a KK^X-morphism by its restrictions to these fibres.
- This suggests that a KK^X-morphism should become invertible in $\mathcal{B}(X)$ once its restrictions to the fibres in $\mathrm{KK}_*\bigl(A(x), B(x)\bigr)$ are invertible for all $x \in X$.

  I do not know how to prove this.
- Such a reduction to fibres is impossible for general non-Hausdorff spaces because bundles over such spaces need not have any "fibres" at all.