Non-stable K-theory and extremally rich C*-algebras

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The joint papers of Gert K. Pedersen and LGB:

C^*-algebras of real rank zero, Journal of Functional Analysis 99 (1991), 131–149.

Interpolation by projections in C^*-algebras, “Operator Algebras” (Oslo, 2004), Abel Symposia 1 (2006), 1–13.

Series of six related papers:

On the geometry of the unit ball of a C^*-algebra, Journal für die reine und angewandte Mathematik 469 (1995), 113–147.

Approximation and convex decomposition by extremals in C^*-algebras, Mathematica Scandinavica 81 (1997), 69–85.

Extremal $K-$theory and index for C^*-algebras, $K-$Theory 20 (2000), 201–241.

Ideal structure and C^*-algebras of low rank, Mathematica Scandinavica, 100 (2007), 5-34.

Limits and C^*-algebras of low rank or dimension, Journal of Operator Theory, to appear (Arxiv no. 0708.2727).

Non-stable $K-$theory and extremally rich C^*-algebras, submitted 2007, (Arxiv no. 0708.3078).

tsr$(A) = 1 \Rightarrow A$ is isometrically rich $\Rightarrow A$ is extremally rich.

Isometric richness and extremal richness are equivalent for prime C^*-algebras.

A simple C^*-algebra is extremally rich if and only if it is either purely infinite or has stable rank one. Much of the success in the classification of simple C^*-algebras has been for these classes of algebras.

Since stable rank 1 implies stable finiteness, one regards extremal richness as a generalization of stable rank 1 suitable for infinite, not necessarily simple, algebras.

Rieffel (1983): tsr$(A) = 1$ if and only if invertibles are dense in $\widetilde{A}$.

B–Pedersen (1995): A is extremally rich if and only if quasi-invertible elements are dense in $\widetilde{A}$.

A is isometrically rich if and only if one-sided invertibles are dense in $\widetilde{A}$.

Finally, a is quasi-invertible if there are ideals I and J such that $IJ = 0$ and a is left invertible mod I and right invertible mod J.

Kadison's criterion for extreme points:

$$(\mathbf{1} - uu^*)A(\mathbf{1} - u^*u) = 0.$$

Also, a is quasi-invertible if and only if it has closed range and the partial isometry appearing in its polar decomposition is in $\mathcal{E}(A)$, the set of extreme points of the unit ball of A. This explains the "extremal" in "extremally rich".

Analogies: quasi-invertible is to extremal as invertible is to unitary as left inertible is to isometry as right invertible is to co-isometry.

Rørdam (1988): For A unital, tsr(A) $= 1$ if and only if the closed unit ball of A is the convex hull of its unitaries.

B–Pedersen (1995): For A unital, A is extremally rich if and only if the closed unit ball of A is the convex hull of its extreme points.

A C^*-algebra A has *weak cancellation* if whenever p and q are projections in A which generate the same (closed, two-sided) ideal I and have the same class in $K_0(I)$, then p is Murray-von Neumann equivalent to q ($p \sim q$). Of course, $p \sim q$ implies that they generate the same ideal I and that $[p] = [q]$ in $K_0(I)$. Nevertheless, it was observed by Rieffel that C^*-algebras of stable rank one satisfy a stronger property: If $[p] = [q]$ in $K_0(A)$, then $p \sim q$. Of course this stronger property can't hold in infinite algebras. Cuntz showed that if A is purely infinite simple, if p and q are both non-zero, and if $[p] = [q]$ in $K_0(A)$, then $p \sim q$; and the concept of weak cancellation was designed to specialize to this property in the simple case.

Does every extremally rich C^*-algebra have weak cancellation?

It is possible to reformulate weak cancellation in a way that does not mention $K-$theory. Note that if p, q, and r are projections in an ideal I and if p generates I as an ideal, then $[q]_{K_0(I)} = [r]_{K_0(I)}$ if and only if $q \oplus np \sim r \oplus np$ for sufficiently large n. Thus the hypotheses, id(p) =id(q) = I and $[p]_{K_0(I)} = [q]_{K_0(I)}$, can be replaced by, $p \oplus nq \sim (n+1)q$ and $q \oplus np \sim (n+1)p$ for sufficiently large n.

We say A has K_1-*surjectivity* if the map from $\mathcal{U}(\widetilde{A})/\mathcal{U}_0(\widetilde{A})$ to $K_1(A)$ is surjective, K_1-*injectivity* if this map is injective, and K_1-*bijectivity* if it is bijective. Cuntz showed that purely infinite simple C^*-algebras have K_1-bijectivity and Rieffel showed the same for algebras of stable rank one.

In the third paper of the series we defined the extremal $K-$set of A, $K_e(A)$, which is an analogue of $K_1(A)$ that uses extremals instead of unitaries (equivalently, quasi-invertibles instead of invertibles).

Another analogy: K_1 is to the extremal $K-$set as index theory for Fredholm operators is to index theory for semi-Fredholm operators.

The concepts K_e-surjectivity, K_e-injectivity, and K_e-bijectivity are defined analogously to the corresponding K_1- concepts.

We say that a (non-unital) C^*-algebra K has *good index theory* if whenever K is embedded as an ideal in a unital C^*-algebra A and u is a unitary in A/K such that $\partial_1([u]_{K_1}) = 0$ in $K_0(K)$, there is a unitary in A which lifts u.

The boundary map is often regarded as an index map, so this definition abstracts a basic property of the Fredholm index.

Although the concept has been studied before, we made up the name and contend that the concept is deserving of emphasis.

Pimsner, Popa, and Voiculescu proved that $C(X) \otimes \mathbb{K}$ has good index theory when X is compact. Mingo showed that $C(X)$ can be replaced by an arbitrary unital C^*-algebra and asked whether every stable C^*-algebra has good index theory. Shortly after, G. Nagy proved that csr(K) ≤ 2 implies that K has good index theory. Here csr denotes connected stable rank (Rieffel), and csr(K) ≤ 2 also implies K_1-surjectivity for K. It had already been proved by Sheu and Nistor (independently) that csr(K) ≤ 2 for all stable K. So Nagy completed the affirmative answer to Mingo's question. Since Rieffel proved that csr(K) $\leq$tsr(K)$+$ 1, Nagy also established good index theory for C^*-algebras of stable rank one. The fact that purely infinite simple C^*-algebras have good index theory should also be considered previously known, but we haven't found a precise reference for it.

For the proof of our K_1- injectivity result it is necessary to work with concepts of K_0-surjectivity.

We say that A has *(strong) K_0-surjectivity* if the group $K_0(A)$ is generated by $\{[p] \mid p$ is a projection in $A\}$. Zhang's Riesz decomposition result shows that C^*-algebras of real rank zero have strong K_0-surjectivity. Cuntz had previously shown that purely infinite simple C^*-algebras satisfy a still stronger property, which, however, is too strong for our purposes. We say that A has *weak K_0-surjectivity* if $\mathsf{S}A$ has K_1-surjectivity. Note that Bott periodicity implies that $K_0(A) \cong K_1(\mathsf{S}A)$. The Bott map can be used to show that strong K_0-surjectivity implies weak K_0-surjectivity. Since Rieffel showed that csr$(A) \leq 2$ implies K_1-surjectivity for A and also that tsr$(A) \leq 1$ implies csr$(\mathsf{S}A) \leq 2$, we see that all C^*-algebras of stable rank one have weak K_0-surjectivity.

Theorem 3.5. For an extremally rich C^*–algebra A the following conditions are equivalent:

(i) A has weak cancellation;

(ii) If $B = pAp$ for some projection p in A and $u \in \mathcal{E}(\mathbb{M}_2(B))$, there is a projection q in B such that

$$q \oplus 0 \sim (p \oplus p) - uu^* \quad (\text{in} \quad \mathbb{M}_2(B)) \; ;$$

(iii) If $B = pAp$ for some projection p in A, and $\{u_1, \ldots, u_n\}$ is a finite subset of $\mathcal{E}(B)$ there is a projection q in B such that

$$q \oplus 0 \sim \bigoplus_{i=1}^{n} (p - u_i u_i^*) \quad (\text{in} \quad \mathbb{M}_n(B)) \; .$$

Corollary 3.6. Every isometrically rich C^*-algebra has stable weak cancellation.

In the fourth paper of the series we defined the defect ideal of A, denoted $\mathcal{D}(A)$, as the ideal generated by all defect projection of elements of $\mathcal{E}(\widetilde{A})$. When A is extremally rich, $\mathcal{D}(A)$ is the smallest ideal such that the quotient has stable rank one.

Corollary 3.7. If A is an extremally rich C^*-algebra such that $\mathcal{D}(I) = I$ whenever I is the left defect ideal of an element of $\mathcal{E}(\widetilde{A})$, then A has weak cancellation. In particular, this applies if every defect projection is properly infinite or if A is purely infinite.

Theorem 3.10. Every extremally rich C^*-algebra A of real rank zero has stable weak cancellation.

Proposition 3.15. If A is an extremally rich C^*–algebra with weak cancellation, then:

(i) There is for every projection p in $\mathcal{D}(A \otimes \mathbb{K})$ $(= \mathcal{D}(A) \otimes \mathbb{K})$ a projection q in $\mathcal{D}(A)$ such that $p \sim q$.

(ii) If p is a projection in $\mathcal{D}(A) \otimes \mathbb{K}$, then there is an infinite set $\{p_n\}$ of mutually orthogonal projections in $\mathcal{D}(A)$ such that $p_n \sim p, \forall n$.

(iii) If $\mathcal{D}(A)$ is $\sigma-$unital, then $\mathcal{D}(A)$ has a full, hereditary, stable, $\sigma-$unital C^*-subalgebra B.

Corollary 3.17. If A is extremally rich with weak cancellation and if p is a projection in A such that $\mathcal{D}(pAp) = pAp$ (equivalently $\mathcal{D}(I) = I$ where $I = id(p)$), then p is properly infinite. In particular, the hypotheses of Corollary 3.7 imply that every defect projection is properly infinite.

Lemma 3.11. Let A be an extremally rich C^*–algebra and I a closed ideal of A. Assume that both I and A/I have weak cancellation and that eAe/eIe has K_1–surjectivity for every projection e in A. Then A has weak cancellation.

Theorem 4.5. In the category of extremally rich C^*–algebras the subcategory of algebras that also have weak cancellation is stable under the formation of quotients, hereditary C^*–subalgebras (in particular ideals), matrix tensoring, Rieffel-Morita equivalence, arbitrary extensions, and inductive limits. Also if the extremally rich C^*-algebra A has a composition series of ideals, $\{I_\alpha\}$, such that each $I_{\alpha+1}/I_\alpha$ has weak cancellation, then A has weak cancellation.

Theorem 4.4. Every extremally rich C^*–algebra with weak cancellation has K_1–surjectivity.

Theorem 4.7. Every extremally rich C^*-algebra with weak cancellation has K_e-surjectivity.

Theorem 5.1. Every extremally rich C^*-algebra with weak cancellation has good index theory.

Corollary 5.3. If A is extremally rich with weak cancellation, then $K_e(A) = \mathcal{E}_\infty(A)$.

Theorem 5.5. Let K be a closed ideal in a unital C^*–algebra A and assume that K is extremally rich with weak cancellation. If $\overline{u} \in \mathcal{E}(A/K)$, $v \in \mathcal{E}(\mathbb{M}_n(A))$, and $[\pi_n(v)]_{\mathcal{E}_\infty} = [\overline{u}]_{\mathcal{E}_\infty}$, where $\pi_n : \mathbb{M}_n(A) \to \mathbb{M}_n(A/K)$ is the quotient map, then there is $u \in \mathcal{E}(A)$ such that $\pi_1(u) = \overline{u}$ and $[u]_{\mathcal{E}_\infty} = [v]_{\mathcal{E}_\infty}$.

Theorem 6.7. If A is an extremally rich C^*-algebra with weak cancellation, then A has K_1-injectivity and weak K_0-surjectivity.

Theorem 6.10. If A is an extremally rich C^*-algebra, then, under any one of the hypotheses listed below, A has weak cancellation, K_1-bijectivity, good index theory, K_e-bijectivity, and weak K_0-surjectivity. Moreover, $\mathcal{D}(A)$ has strong K_0-surjectivity.

(i) A has real rank zero.

(ii) If I is the left defect ideal of an element of $\mathcal{E}(\widetilde{A})$, then $\mathcal{D}(I) = I$. In particular, this applies if A is purely infinite or if every defect projection is properly infinite.

(iii) A is isometrically rich.

There are additional interplays between the concepts.

Proposition 6.6. Let I be an ideal of a C^*-algebra A. Then:

(i) If I and A/I have K_1-injectivity and A/I has weak K_0-surjectivity, then A has K_1-injectivity.

(ii) If I and A/I have weak K_0-surjectivity and I has K_1-injectivity, then A has weak K_0-surjectivity.

Proposition 7.2.4. Let I be an ideal of a C^*-algebra A. Then:

(i) If If I and A/I have K_1-surjectivity and I has good index theory, then A has K_1-surjectivity.

(ii) If I and A/I have good index theory and A/I has K_1-surjectivity, then A has good index theory.