THE NONCOMMUTATIVE TORUS

The noncommutative torus as a twisted convolution

An ordinary two-torus $\mathbb{T}^2$ with coordinate functions given by

$$U_1 = \mathrm{e}^{2\pi \mathrm{i} x_1}, \qquad U_2 = \mathrm{e}^{2\pi \mathrm{i} x_2}, \tag{1}$$

where $x_1, x_2 \in [0, 1]$.

By Fourier expansion, $C^\infty(\mathbb{T}^2)$ consists of all power series

$$a = \sum_{(m,n)\in\mathbb{Z}^2} a_{mn}\, U_1^m\, U_2^n \ , \tag{2}$$

with $\{a_{mn}\} \in S(\mathbb{Z}^2)$ a complex-valued Schwartz function on $\mathbb{Z}^2$; the sequence of complex numbers $\{a_{m,n} \in \mathbb{C} \mid (m, n) \in \mathbb{Z}^2\}$ decreases rapidly at "infinity": for any $k \in \mathbb{N}_0$ bounded semi-norms

$$\|a\|_k = \sup_{(m,n)\in\mathbb{Z}^2} |a_{m,r}| \Big(1 + |m| + |r|\Big)^k < \infty \ . \tag{3}$$

Then fix a real number θ.

The algebra $\mathcal{A}_\theta = C^\infty(\mathbb{T}^2_\theta)$ of smooth functions on the noncommutative torus is the associative algebra made up of all elements of the form (2), but now the two generators U_1 and U_2 satisfy

$$U_2 \, U_1 = \mathrm{e}^{2\pi \mathrm{i} \theta} \; U_1 \, U_2 \ . \tag{4}$$

An involution, making $\mathcal{A}_\theta$ into a $*$-algebra is defined by

$$U_1^\dagger := U_1^{-1} \ , \quad U_2^\dagger := U_2^{-1} \ . \tag{5}$$

From (3) with $k = 0$ one gets a C^*-norm and the corresponding closure of $\mathcal{A}_\theta$ in this norm is the universal C^*-algebra A_θ generated by two unitaries with the relation (4); the algebra $\mathcal{A}_\theta$ is dense in A_θ and is thus a pre-C^*-algebra.

A one-to-one correspondence between elements of the noncommutative torus algebra $\mathcal{A}_\theta$ and the commutative torus algebra $C^\infty(\mathbb{T}^2)$ given by the Weyl map Ω,

$$\Omega : C^\infty(\mathbb{T}^2) \to \mathcal{A}_\theta$$

$$\Omega\left(\sum_{(m_1,m_2)\in\mathbb{Z}^2} f_{m_1 m_2}\ \mathsf{e}^{2\pi\mathsf{i}(m_1\,x_1+m_2\,x_2)}\right) := \sum_{(m_1,m_2)\in\mathbb{Z}^2} f_{m_1 m_2}\ \mathsf{e}^{\pi\mathsf{i}m_1 m_2\,\theta}\ U_1^{m_1}\,U_2^{m_2}\ ;$$

it is a vector space isomorphism

The map Ω is used to deform the commutative product on the algebra $C^\infty(\mathbb{T}^2)$ into a noncommutative star-product:

$$f \star g := \Omega^{-1}\Big(\Omega(f)\,\Omega(g)\Big)\ , \quad f,g \in C^\infty\left(\mathbb{T}^2\right)\ . \tag{6}$$

A straightforward computation gives

$$f \star g = \sum_{(r_1,r_2)\in\mathbb{Z}^2} (f \star g)_{r_1,r_2} \, \mathrm{e}^{2\pi \mathrm{i}(r_1 x_1 + r_2 x_2)} \,, \tag{7}$$

with the coefficients given by a twisted convolution

$$(f \star g)_{r_1,r_2} = \sum_{(s_1,s_2)\in\mathbb{Z}^2} f_{s_1,s_2} \, g_{r_1-s_1,r_2-s_2} \; \mathrm{e}^{\pi \mathrm{i}(r_1 s_2 - r_2 s_1)\,\theta}; \tag{8}$$

it reduces to the usual Fourier convolution product in the limit $\theta = 0$. Up to isomorphism, the deformed product depends only on the cohomology class in the group cohomology $H^2(\mathbb{Z}^2, U(1))$ of the $U(1)$-valued two-cocycle on $\mathbb{Z}^2$ given by

$$\lambda(r,s) := \mathrm{e}^{\pi \mathrm{i}(r_1 s_2 - r_2 s_1)\,\theta} \tag{9}$$

with $r = (r_1, r_2), s = (s_1, s_2) \in \mathbb{Z}^2$.

The unique integral and the toric action

From now on we take θ an irrational number. On the algebra $\mathcal{A}_\theta$ there is a unique normalized, positive definite trace,

$$\int\!\!\!\!\!- : \mathcal{A}_\theta \to \mathbb{C},$$

given by

$$\begin{aligned} \int\!\!\!\!\!- \sum_{(m_1,m_2)\in\mathbb{Z}^2} a_{m_1 m_2}\, U_1^{m_1} U_2^{m_2} &:= a_{00} \qquad (10) \\ &= \int_{\mathbb{T}^2} \mathrm{d}x_1\, \mathrm{d}x_2\, \Omega^{-1} \left(\sum_{(m_1,m_2)\in\mathbb{Z}^2} a_{m_1 m_2}\, U_1^{m_1} U_2^{m_2} \right)(x_1, x_2) \end{aligned}$$

Then, for any $a, b \in \mathcal{A}_\theta$,

$$\int\!\!\!\!\!- a\, b = \int\!\!\!\!\!- b\, a\ , \quad \int\!\!\!\!\!- \mathbb{I} = 1\ , \quad \int\!\!\!\!\!- a^\dagger a > 0\ , \quad a \neq 0\ , \qquad (11)$$

with $⨍ a^\dagger a = 0$ if and only if $a = 0$ (i.e. the trace is faithful).

This trace is invariant under the natural action of the commutative torus $\mathbb{T}^2$ on $\mathcal{A}_\theta$ whose infinitesimal form is generated by two commuting derivations ∂_1, ∂_2 acting as

$$\partial_i(U_j) = 2\pi \mathrm{i}\ \delta_{ij} U_j \ , \quad i,j = 1,2 \ . \tag{12}$$

Invariance is just the statement that $⨍ \partial_i(a) = 0$ for any $a \in \mathcal{A}_\theta$.

The cyclic 2-cocycle which 'integrates' 2-forms is

$$\Psi(a_0, a_1, a_2) = \frac{\mathrm{i}}{2\pi} ⨍ \epsilon^{ij} a_0 \partial_i a_1 \partial_j a_2 \tag{13}$$

the normalization such that for any projection $p \in \mathcal{A}_\theta$, the quantity $\Psi(p,p,p)$ is an integer: it is the index of a Fredholm operator.

The conformal class of a general constant metric is parametrized by a complex number $\tau \in \mathbb{C}$, $\mathrm{Im}\,\tau > 0$. Then, up to a conformal factor, the metric is given by

$$g = (g_{ij}) = \begin{pmatrix} 1 & \mathrm{Re}\,\tau \\ \mathrm{Re}\,\tau & |\tau|^2 \end{pmatrix} \ . \tag{14}$$

Clearly $\sqrt{\det g} = \mathrm{Im}\,\tau$ and the inverse metric is found to be

$$g^{-1} = (g^{ij}) = \frac{1}{(\mathrm{Im}\,\tau)^2} \begin{pmatrix} |\tau|^2 & -\mathrm{Re}\,\tau \\ -\mathrm{Re}\,\tau & 1 \end{pmatrix} \ . \tag{15}$$

Using the two derivations ∂_1, ∂_2 in (14) we may think of "the complex torus" $\mathbb{T}^2$ as acting on the noncommutative torus $\mathcal{A}_\theta$ and construct two associated derivations of $\mathcal{A}_\theta$:

$$\partial_{(\tau)} = \frac{1}{(\tau - \overline{\tau})} \, (-\overline{\tau}\partial_1 + \partial_2) \ , \quad \overline{\partial}_{(\tau)} = \frac{1}{(\tau - \overline{\tau})} \, (\tau\partial_1 - \partial_2) \ . \tag{16}$$

The above endows the noncommutative torus $\mathcal{A}_\theta$ with a complex structure. Also,

$$\partial_{(\tau)}\overline{\partial}_{(\tau)} = \overline{\partial}_{(\tau)}\partial_{(\tau)} = \frac{1}{4} g^{ij}\partial_i\partial_j = \frac{1}{4}\Delta \ , \tag{17}$$

$\Delta = g^{ij}\partial_i\partial_i$ is the Laplacian corresponding to the metric (16).

By working with this metric, the positive Hochschild cocycle Φ associated with the cyclic one (15) is given by

$$\Phi(a_0, a_1, a_2) = \frac{2}{\pi} ⨍ a_0 \partial_{(\tau)} a_1 \overline{\partial}_{(\tau)} a_2 \ . \tag{18}$$

A construction of this cocycle as the conformal class of a general constant metric on the torus can be found in [Connes-book].

The spectral geometry of the torus

The algebra $\mathcal{A}_\theta$ is represented faithfully as operators on a separable Hilbert space $\mathcal{H}_0$, the GNS rep space $\mathcal{H}_0 = L^2(\mathcal{A}_\theta \,, \fint)$ defined as the completion of $\mathcal{A}_\theta$ itself in the Hilbert norm

$$\|a\|_{\mathsf{GNS}} := \left(\fint a^\dagger \, a \right)^{1/2} . \tag{19}$$

Since the trace is faithful, the map $\mathcal{A}_\theta \ni a \mapsto \widehat{a} \in \mathcal{H}_0$ is injective and the faithful GNS representation $\pi : \mathcal{A}_\theta \to \mathcal{B}(\mathcal{H}_0)$ is

$$\pi(a)\widehat{b} = \widehat{a \, b} \,, \qquad a, b \in \mathcal{A}_\theta. \tag{20}$$

The vector $1 = \widehat{\mathbb{I}}$ of $\mathcal{H}_0$ is cyclic (i.e. $\pi(\mathcal{A}_\theta)1$ is dense in $\mathcal{H}_0$) and separating (i.e. $\pi(a)1 = 0$ implies $a = 0$). Furthermore, the state $(1, \pi(a)1)$ is tracial. Thus, the Tomita involution is just

$$J_0(\widehat{a}) = \widehat{a^\dagger}\,, \qquad \forall\, \widehat{a} \in \mathcal{H}_0\,. \tag{21}$$

The C^*-algebra norm on $\mathcal{A}_\theta$ given in (3) with $k = 0$ coincides with the operator norm in (21) when $\mathcal{A}_\theta$ is represented on the Hilbert space $\mathcal{H}_0$, and with the L^∞-norm in the Wigner representation.

A two dimensional noncommutative geometry $(\mathcal{A}_\theta, \mathcal{H}, D, \gamma, J)$

The Hilbert space $\mathcal{H}$ is just $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0$ on which $\mathcal{A}_\theta$ acts diagonally with two copies of the representation π in (22). Then

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad J = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix} . \tag{22}$$

The Dirac operator D is

$$D = \begin{pmatrix} 0 & \overline{\partial}_{(\tau)} \\ \partial_{(\tau)} & 0 \end{pmatrix} . \tag{23}$$

For the particular choice $\tau = \mathrm{i}$ this reduces to $D = \partial_1 \sigma_1 + \partial_2 \sigma_2$ with σ_1, σ_2, two of the Pauli matrices.

The Hochschild 2-cycle c giving the orientation and volume form:

$$c = \frac{1}{(2\mathrm{i}\pi)^2} \frac{1}{(\tau - \overline{\tau})} \left(U_2^{-1} U_1^{-1} \otimes U_1 \otimes U_2 - U_1^{-1} U_2^{-1} \otimes U_2 \otimes U_1 \right) . \quad (24)$$

All properties of a real spectral geometry are satisfied.

We only mention that the "area" $⨍ D^{-2}$ of the torus depends on the complex parameter τ; with Φ the cyclic 2-cocycle in (15), one finds that

$$⨍ D^{-2} = \langle \Phi | c \rangle = \frac{1}{2\pi \ \mathrm{Im}\, \tau} . \quad (25)$$

Modules over the noncommutative torus

When θ is not rational, every finitely generated projective module over the algebra $\mathcal{A}_\theta$ which is not free is isomorphic to a Heisenberg module [Connes,Rieffel]. Any such a module $\mathcal{E}_{r,q}$ is characterized by two integers r, q. If $q = 0$, $\mathcal{E}_{r,q} = \mathcal{A}_\theta^{|r|}$. Otherwise, they can be taken to be relatively coprime with $q > 0$ (a similar construction being possible for $q < 0$), or $r = 0$ and $q = 1$.

As a vector space

$$\mathcal{E}_{r,q} = \mathcal{S}(\mathbb{R} \times \mathbb{Z}_q) \ \cong \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^q \ , \tag{26}$$

the space of Schwartz functions of one continuous variable $s \in \mathbb{R}$ and a discrete one $k \in \mathbb{Z}_q$ (such a variable is defined modulo q).

With $\varepsilon = r/q - \theta$, the space $\mathcal{E}_{r,q}$ is made into a *right* $\mathcal{A}_\theta$-module,

$$(\xi\ U_1)(s,k) := \xi(s-\varepsilon, k-r)\ ,$$
$$(\xi\ U_2)(s,k) := \mathrm{e}^{2\pi\mathrm{i}(s-k/q)}\xi(s,k)\ , \qquad \forall \quad \xi \in \mathcal{E}_{r,q}. \quad (27)$$

On the module $\mathcal{E}_{r,q}$ one defines an $\mathcal{A}_\theta$-valued hermitian structure:

$$\langle\xi|\eta\rangle_\theta := \sum_{m,n}\sum_{k=0}^{q-1}\int \mathrm{d}s\ \overline{\xi(s-m\,\varepsilon, k-mp)}\ \eta(s,k)\mathrm{e}^{-2\pi\mathrm{i}n(s-k/q)}\ U_1^m\ U_2^n$$

for all $\xi, \eta \in \mathcal{E}_{r,q}$, $a \in \mathcal{A}_\theta$; antilinearity is in the first variable.

The endomorphism algebra $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}_{r,q})$, acting on the left on $\mathcal{E}_{r,q}$, can be identified with another copy of the noncommutative torus $\mathcal{A}_\alpha$. The parameter α is 'uniquely' determined by θ.

Since r and q are coprime, there exist integer $a, b \in \mathbb{Z}$ such that $ar + bq = 1$. Then, the transformed parameter is

$$\alpha = \frac{a\theta + b}{-q\theta + r} \,. \tag{28}$$

Given any two other integers $a', b' \in \mathbb{Z}$ such that $a'r + b'q = 1$, one finds that $\alpha' - \alpha \in \mathbb{Z}$ so that $\mathcal{A}_{\alpha'} \cong \mathcal{A}_\alpha$.

Thus, the algebra $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}_{r,q})$ is generated by two operators Z_1, Z_2 acting on the *left* on $\mathcal{E}_{r,q}$,

$$\begin{aligned} (Z_1\xi)(s,k) &:= \xi(s - 1/q, k-1) \ , \\ (Z_2\xi)(s,k) &:= \mathrm{e}^{2\pi \mathrm{i} \ (s/\varepsilon - ak)/q} \ \xi(s,k) \ , \end{aligned} \tag{29}$$

and one verifies the defining relations of the nct algebra $\mathcal{A}_\alpha$:

$$Z_2 \; Z_1 = \mathrm{e}^{2\pi \mathrm{i} \alpha} Z_1 \; Z_2 \; . \tag{30}$$

The $\mathcal{A}_\alpha$-$\mathcal{A}_\theta$-bimodule $\mathcal{E}_{r,q}$ is a *Morita equivalence* between $\mathcal{A}_\alpha$ and $\mathcal{A}_\theta$; there exists also an $\mathcal{A}_\alpha$-valued hermitian structure on $\mathcal{E}_{r,q}$,

$$\langle \cdot | \cdot \rangle_\alpha : \mathcal{E}_{r,q} \times \mathcal{E}_{r,q} \longrightarrow \mathcal{A}_\alpha \; , \tag{31}$$

compatible with the $\mathcal{A}_\theta$-valued one $\langle \cdot | \cdot \rangle_\theta$, in the sense that

$$\langle \xi | \eta \rangle_\alpha \; \zeta = \xi \; \langle \eta | \zeta \rangle_\theta \; , \qquad \forall \quad \xi, \eta, \zeta \in \mathcal{E}_{r,q} \tag{32}$$

The second hermitian structure is

$$\langle \xi | \eta \rangle_\alpha := \frac{1}{|q\varepsilon|} \sum_{m,n} \sum_{k=0}^{q-1} \int_{-\infty}^{+\infty} \mathrm{d}s \; \xi(s,k) \; \overline{\eta(s - m/q, k - m)} \; \times$$
$$\times \mathrm{e}^{-2\pi \mathrm{i} n/q[(s-m/q)/\varepsilon - ak]} \; U_1^m \; U_2^n \; .$$

Gauge connections

On the right $\mathcal{A}_\theta$-module $\mathcal{E}_{r,q}$ is given by two covariant derivatives

$$\nabla_i : \mathcal{E}_{r,q} \to \mathcal{E}_{r,q} \ , \quad i = 1, 2 \ , \tag{33}$$

which satisfy a right Leibniz rule

$$\nabla_i(\xi \ a) = (\nabla_i \xi) a + \xi(\partial_i a) \ , \quad i = 1, 2 \ . \tag{34}$$

Compatibility with the $\mathcal{A}_\theta$-valued hermitian structure

$$\partial_i(\langle \xi | \eta \rangle_\theta) = \langle \nabla_i \xi | \eta \rangle_\theta + \langle \xi | \nabla_i \eta \rangle_\theta \ , \quad i = 1, 2 \ . \tag{35}$$

The operators ∇_i are used to define derivations on the endomorphism algebra $\text{End}_{\mathcal{A}_\theta}(\mathcal{E}_{r,q})$:

$$\delta_i(T) := \nabla_i \circ T - T \circ \nabla_i \ , \quad i = 1, 2 \ , \qquad T \in \text{End}_{\mathcal{A}_\theta}(\mathcal{E}_{r,q}). \quad (36)$$

By using the compatibility (37) and the right Leibniz rule (36), one easily proves that the connection ∇_i is compatible with the derivations δ_i and the hermitian structure $\langle\cdot|\cdot\rangle_\alpha$, that is,

$$\delta_i(\langle\xi|\eta\rangle_\alpha) = \langle\nabla_i\xi|\eta\rangle_\alpha + \langle\xi|\nabla_i\eta\rangle_\alpha \ , \quad i = 1, 2 \ . \qquad (37)$$

There is also a left Leibniz rule,

$$\nabla_i(T\xi) = T(\nabla_i\xi) + (\delta_i T)\xi \ , \quad i = 1, 2 \ . \qquad (38)$$

Constant curvature connections

A particular connection on the right $\mathcal{A}_\theta$-module $\mathcal{E}_{r,q}$:

$$(\nabla_1\xi)(s,k) := \frac{2\pi \mathrm{i}}{\varepsilon} s \; \xi(s,k) \ , \quad (\nabla_2\xi)(s,k) := \frac{\mathrm{d}\xi}{\mathrm{d}s}(s,k) \ , \tag{39}$$

the discrete index k is not touched.

This connection is of constant curvature,

$$F_{12} := [\nabla_1, \nabla_2] - \nabla_{[\partial_1,\partial_2]} = -\frac{2\pi \mathrm{i}}{\varepsilon} \; \mathbb{I}_{\mathcal{E}_{r,q}} \ , \tag{40}$$

with $\mathbb{I}_{\mathcal{E}_{r,q}}$ the identity operator on $\mathcal{E}_{r,q}$.

Used to compute the first Chern number of the module $\mathcal{E}_{r,q}$:

$$c_1(\mathcal{E}_{r,q}) := -\frac{1}{2\pi \mathrm{i}} \int\!\!\!\!\!- : \mathrm{tr}\,(F_{12}) = \frac{1}{\varepsilon} : \mathrm{tr}\,(\mathbb{I}_{\mathcal{E}_{r,q}}) = \frac{q}{r - q\theta}|r - q\theta| = \pm q \,.$$

By remembering that $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}_{r,q}) \cong \mathcal{A}_\alpha$, one also proves that the derivations δ_i on $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}_{r,q})$ determined by the constant curvature connection (41) are proportional to the generators of the infinitesimal action of the commutative torus $\mathbb{T}^2$ on $\mathcal{A}_\alpha$:

$$\delta_i(Z_j) = \frac{2\pi \mathrm{i}}{|q\varepsilon|} \, \delta_i^j Z_j \ , \quad i,j = 1,2 \ . \tag{41}$$

Any other constant curvature connection will have the form

$$\widetilde{\nabla}_i = \nabla_i + a_i$$

where a_1 and a_2 are constants.

A complex (or better holomorphic) structure on the module $\mathcal{E}_{r,q}$ is given by the covariant derivatives $\nabla_{(\tau)}, \overline{\nabla}_{(\tau)}$ lifting the complex derivations $\partial_{(\tau)}, \overline{\partial}_{(\tau)}$ in (18); they are given by

$$\nabla_{(\tau)} = \frac{1}{(\tau - \overline{\tau})} (-\overline{\tau}\nabla_1 + \nabla_2) , \quad \overline{\nabla}_{(\tau)} = \frac{1}{(\tau - \overline{\tau})} (\tau\nabla_1 - \nabla_2) \ . \quad (42)$$

Higher dimensional tori

Take $\theta = (\theta_{jk})$ a real antisymmetric $n \times n$ matrix. The noncommutative torus $\mathbb{T}^n_\theta$ of dimension n and twist θ is the "quantum space" whose algebra of polynomial functions is generated by n independent unitaries $U_1, \ldots, U_n$, subject to the commutation relations:

$$U_j U_k = \mathrm{e}^{2\pi \mathrm{i} \theta_{jk}} U_k U_j \ . \qquad (43)$$

The corresponding C^*-algebra of continuous functions is the universal C^*-algebra with the same generators and relations, $A_\theta = C(\mathbb{T}^n_\theta)$. Elements of A_θ are convergent sums

$$a = \sum_{r \in \mathbb{Z}^n} a_r U^r \,, \qquad (44)$$

with $a_r \in \mathbb{C}$ and

$$U^r := \mathrm{e}^{-\pi \mathrm{i} r_j \theta_{jk} r_k} \, U_1^{r_1} U_2^{r_2} \ldots U_n^{r_n} \ . \qquad (45)$$

These unitary elements $\{U^r : r \in \mathbb{Z}^n\}$ form a Weyl system

$$U^r\, U^s = \mathrm{e}^{\pi \mathrm{i} r_j \theta_{jk} s_k}\, U^{r+s}. \tag{46}$$

The phase factors

$$\rho_\theta(r,s) := \exp\{\pi \mathrm{i} r_j \theta_{jk} s_k\} \tag{47}$$

form a 2-cocycle for the group $\mathbb{Z}^n$, which is skew (i.e., $\rho_\theta(r,r) = 1$) since θ is antisymmetric. Thus A_θ may be defined as the twisted group C^*-algebra $A_\theta = C(\mathbb{Z}^n, \rho_\theta)$.

An action τ of the ordinary torus $\mathbb{T}^n$. With $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{T}^n$, the action is given by

$$\tau(\mathrm{e}^{2\pi \mathrm{i}\alpha}) : U_j \longmapsto \mathrm{e}^{2\pi \mathrm{i}\alpha_j}\, U_j. \tag{48}$$

The smooth subalgebra $\mathcal{A}_\theta = C^\infty(\mathbb{T}^n_\theta)$ of A_θ under this action consists of rapidly convergent Fourier series of the form

$\sum_{r\in\mathbb{Z}^n} a_r u^r$; that is the coefficient function $\mathbb{Z}^n \ni r \mapsto a_r$ is an element of $\mathcal{S}(\mathbb{Z}^n)$, the Schwartz space of $\mathbb{Z}^n$.

From the action of $\mathbb{T}^n$ we also get an infinitesimal action of its Lie algebra $L = \mathbb{R}^n$ as derivations on $\mathcal{A}_\theta$ explicitly given as

$$\partial_X(U^r) = 2\pi\mathsf{i}\,\langle X|r\rangle\, U^r, \tag{49}$$

with $X \in L$ and $\langle X|r\rangle$ is the scalar product in $\mathbb{R}^n$. The above map in then extended as a derivation. One can repeat all the constructions of the two-dimensional case. Given any right $\mathcal{A}_\theta$-module $\mathcal{E}$, a connection on $\mathcal{E}$ is a linear map $\nabla : L \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{E})$ which, for all $\eta \in \mathcal{E}$ and $a \in \mathcal{A}_\theta$ satisfies a Leibniz rule,

$$\nabla_X(\eta a) = (\nabla_X \eta)a + \eta\partial_X(a). \tag{50}$$

A connection has constant curvature if there exists a skew-symmetric bilinear map $\Omega : L \times L \to \mathbb{C}$ such that the curvature

$$F(X,Y) := [\nabla_X, \nabla_Y] = \Omega(X,Y)\mathbb{I}_{\mathcal{E}} \ . \tag{51}$$

When $\mathcal{E}$ has an $\mathcal{A}_\theta$-Hermitian structure, one also demands that the connection be Hermitian, i.e. one requires the additional rule

$$\partial_X \langle \eta|\xi\rangle = \langle \nabla_X \eta|\xi\rangle + \langle \eta|\nabla_X \xi\rangle \ . \tag{52}$$

For each $X \in L$, the connection induces a derivation δ_X on $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E})$ by

$$\delta_X(T) = \nabla_X \circ T - T \circ \nabla_X.$$

For a constant curvature connection, the map $X \mapsto \delta_X$ is a Lie algebra homomorphism from L to the space of derivations $\mathrm{Der}(\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}))$ of $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E})$.