Bost-Connes-Marcolli systems for Shimura varieties

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Four parts :

1. Quantum statistical mechanics.
2. Bost-Connes systems.
3. Bost-Connes-Marcolli systems.
4. Back to Bost-Connes.

First part :
Quantum statistical mechanics.

	CLASSICAL	QUANTUM
Observables	$a \in C^\infty(X)$ (X, ω) symplect. var.	$a \in \mathcal{A}$ $\mathcal{A}$: C*-alg, a=a*
Hamiltonian	$H\colon X \to \mathbb{R}$	H auto-adj unbounded on $\mathcal{H}$ $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$
Time evol^n	solution of the hamilt field ξ_H $dH + \omega(\xi_H, .) = 0$	$\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ $e^{itH}\pi(a)e^{-itH} = \pi(\sigma_t(a))$
States	Proba measure μ sur X $\phi(a) = \int_X a \mathrm{d}\mu$	linear functional of norm 1 $\phi : \mathcal{A} \to \mathbb{C}$
Partition function	$Z(\beta) = \int_X e^{-\beta H} d\Omega$ $\Omega = \omega^{\wedge n}$	$Z(\beta) = \mathrm{Tr}(e^{-\beta H})$
Equilibrium states	Canon. Ens. : $\mu = \frac{e^{-\beta H} d\Omega}{Z(\beta)}$	KMS : $\Phi(a) = \frac{\mathrm{Tr}(ae^{-\beta H})}{Z(\beta)}$ KMS : $\Phi(ab) = \Phi(\sigma_{i\beta}(b)a)$

PHASE TRANSITION

WITH SPONTANEOUS SYMMETRY BREAKING.

An arbitrary small perturbation of the temperature

T

induces

a radical change

of equilibrium states.

FOR EXAMPLE

 $\rightsquigarrow$ 

Second part :
Bost-Connes systems

1. CLASSICAL BOST-CONNES SYSTEMS.[a]

Groupoid :
$G = \{(g,\rho) \in \mathbb{Q}_+^\times \times \widehat{\mathbb{Z}} \mid g\rho \in \widehat{\mathbb{Z}}\}$

Algebra : $\mathcal{A} := C_c(G)$ with convolution,
$(f_1 * f_2)(g,\rho) = \sum_{h \in \mathbb{Q}_+^\times} f_1(gh^{-1}, h\rho) f_2(h,\rho)$

Symmetry : $\widehat{\mathbb{Z}}^\times$ acting on the right on $\widehat{\mathbb{Z}}$.

Evolution : $\sigma_t(f)(g,\rho) = g^{it} f(g,\rho)$

Representation : $\forall \rho_0 \in \widehat{\mathbb{Z}}^\times$, $\pi_0 : \mathcal{A} \to \mathcal{B}(\ell^2(\mathbb{N}^\times))$
$(\pi_0(f)(\xi))(n) = \sum_{h \in \mathbb{N}^\times} f(nh^{-1}, h\rho_0)\xi(h)$

Hamiltonian :
$H : \ell^2(\mathbb{N}^\times) \to \ell^2(\mathbb{N}^\times), f(n) \mapsto \log(n) f(n)$

[a] On board.

2. ADELIC BOST-CONNES SYSTEM. [a]

Strong approximation : $\mathbb{A}_f^\times = \mathbb{Q}_+^\times \cdot \widehat{\mathbb{Z}}^\times$.
For $\widehat{\mathbb{Z}}^\natural = \widehat{\mathbb{Z}} \cap \mathbb{A}_f^\times$, we have $\widehat{\mathbb{Z}}^\times \backslash \widehat{\mathbb{Z}}^\natural = \mathbb{N}^\times$.

Shimura variety :
$\mathrm{Sh}(\mathbb{G}_m, \{\pm 1\}) = \mathbb{Q}^\times \backslash \mathbb{A}_f^\times \times \{\pm 1\}$.

Partial action of $g \in \mathbb{A}_f^\times$ on
$Y = \widehat{\mathbb{Z}} \times \mathrm{Sh}(\mathbb{G}_m, \{\pm 1\})$ given by
$y = (\rho, [z, l]) \mapsto gy = (g\rho, [z, lg^{-1}])$.

Corresponding big groupoid :
$U = \{(g, y) \in \mathbb{A}_f^\times \times Y \mid gy \in Y\}$.

Bost-Connes groupoid :

$$\boxed{Z = U/(\widehat{\mathbb{Z}}^\times)^2}$$

where $(\widehat{\mathbb{Z}}^\times)^2$ acts by $(g, y) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 y)$.

Lemma : $G \cong Z$.

[a] On board.

3. Arithmetical QSM in dimension 1.

Bost-Connes systems : $(\mathcal{A}, \sigma_t)$.

GL_1-system with :

- partition function=Riemann zeta
- spontaneous symmetry breaking at $T = 1$
- disorder at high temperature (1 equi. state)
- $\widehat{\mathbb{Z}}^\times$-symmetry at small temperature

Relation with class field theory :

- rational subalgebra $A_\mathbb{Q} \subset \mathcal{A}$ defined by the reciprocity law
- values of extremal KMS_∞ states [a] on $A_\mathbb{Q}$ generate $\mathbb{Q}^{\mathrm{ab}}$

[a] irreductible=factorial

4. Problematic of arithmetical QSM.

1. Bost-Connes for number fields ?
- works of Cohen, Arledge-Laca-Raeburn, Van Frankenhuijsen-Laca, Harari-Leichtnam, Laca.

right partition function (Dedekind zeta)	*or*	right symmetry (Galois group)

restricted to class number 1 (excl. Cohen)
- January 2005 : Connes-Marcolli-Ramachandran, quadratic imaginary fields.

2. Connes-Marcolli for other groups ?
- same problem as for GL_1.
- [Connes-Marcolli] tricks that are difficult to generalize.

NEW RESULTS.

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1. definition of Bost-Connes for number fields with the right partition function *and* the right symmetry

2. definition and formal properties of Bost-Connes-Marcolli for Shimura data

Third part :
Bost-Connes-Marcolli systems.

1. ADELIC CONNES-MARCOLLI.[a]

Strong approximation :
$\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2(\mathbb{Q})_+ \cdot \mathrm{GL}_2(\widehat{\mathbb{Z}})$.

Shimura variety :
$\mathrm{Sh}(\mathrm{GL}_2, \mathbb{H}^\pm) = \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm$.

Partial action of $g \in \mathrm{GL}_2(\mathbb{A}_f)$ on
$Y = \mathrm{M}_2(\widehat{\mathbb{Z}}) \times \mathrm{Sh}(\mathrm{GL}_2, \mathbb{H}^\pm)$ given by
$y = (\rho, [z, l]) \mapsto gy = (g\rho, [z, lg^{-1}])$.

Corresponding big groupoid :
$U = \{(g, y) \in \mathrm{GL}_2(\mathbb{A}_f) \times Y \mid gy \in Y\}$.

Connes-Marcolli groupoid :

$$\boxed{Z = U/\mathrm{GL}_2(\widehat{\mathbb{Z}})^2}$$

where $\mathrm{GL}_2(\widehat{\mathbb{Z}})^2$ acts by $(g, y) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 y)$.

[a]On board.

2. Arithmetic QSM in dimension 2.

Connes-Marcolli system : $(\mathcal{A}(\mathrm{GL}_2), \sigma_t)$.
$\mathcal{A}(\mathrm{GL}_2) = C_c(Z)$, $\sigma_t(f)(g,y) = \det(g)^{it} f(g,y)$.

New viewpoint :
NC space of $\mathbb{Q}$-lattices mod commensurability

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathbb{H}^{\pm} \times \mathrm{M}_2(\mathbb{A}_f)$$

GL_2-system with :
- partition function=$\zeta(s)\zeta(s-1)$
- spontaneous symmetry breaking for $T = 1, 2$
- big dissorder at high temperature (no eq.)
- $\mathbb{Q}^{\times}\backslash\mathrm{GL}_2(\mathbb{A}_f)$-symmetry at low temperature

Relation to Shimura's reciprocity law :
- rational subalgebra $A_{\mathbb{Q}}(\mathrm{GL}_2) \subset \mathcal{A}(\mathrm{GL}_2)$
- values of generic extremal KMS_{∞} states on $A_{\mathbb{Q}}(\mathrm{GL}_2)$ generate the modular field.

3. SHIMURA VARIETIES.[a]

Shimura datum : triple (G, X, h) with

- G connected reductive over $\mathbb{Q}$
- X a $G(\mathbb{R})$-homogeneous space
- $h : X \to \mathrm{Hom}(\mathbb{C}^\times, G_\mathbb{R})$, $G(\mathbb{R})$-equivariant

Axioms :

1. X : presymmetric hermitian. (...)

2. Griffiths' transversality.

Classical datum :

3. the center of G is small enough.

Shimura variety : $K \subset G(\mathbb{A}_f)$ compact open

$$\mathrm{Sh}(G, X) = \varprojlim_K G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$$

Example : F number field,

$$(\mathbb{G}_{m,F}, X_F = \mathbb{G}_{m,F}(\mathbb{R})/\mathbb{G}_{m,F}(\mathbb{R})^+).$$

modular Shimura variety,

$$\mathrm{Sh}(\mathrm{GL}_2, \mathbb{H}^\pm) \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f).$$

[a] On board.

4. BCM SYSTEMS.[a]

ADELIC CONNES-MARCOLLI GROUPOID :

Algebraic datum :

- $(\mathrm{GL}_2, \mathbb{H}^{\pm}) \rightsquigarrow (G, X)$
- $\mathrm{M}_2 \rightsquigarrow M$ envelopping semigroup[b] [c] $M^{\times} = G$

Level structure :

- $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \rightsquigarrow K \subset G(\mathbb{A}_f)$ compact open
- $\mathrm{M}_2(\widehat{\mathbb{Z}}) \rightsquigarrow K_M \subset M(\mathbb{A}_f)$ compact open

TIME EVOLUTION :

Rational determinant for $g \in G(\mathbb{A}_f)$:

- $G \to \mathrm{GL}(V)$ representation, $G \subset M \subset \mathrm{End}(V)$.
- $L \subset V$, K-"stable" lattice

RESULTS :

- Hamiltonian, time evolution, partition function, symmetries.
- Construction of extremal KMS states corresponding to points in $\mathrm{Sh}(G, X)$.

[a] on board.

[b] Ramachandran : symplect.

[c] Drinfeld, Vinberg : classification

TO DO.

- Complete caracterisation of KMS states at zero temperature (technical trick)

- Definition of the rational subalgebra with help of Milne-Shi's reciprocity law

- Explanation of equidistribution results (Clozel, Ullmo) on Hecke operators in QSM terms

Fourth part : Bost-Connes systems for number fields

$$(G, X) = (\mathbb{G}_{m,F}, X_F)$$
$$M = \mathrm{M}_{1,F}$$

$$K = \widehat{\mathcal{O}}_F^{\times}$$
$$K_M = \widehat{\mathcal{O}}_F$$

1. THE BC GROUPOID.[a]

Denote $C_F = F^\times \backslash \mathbb{A}_F^\times$.
partial action of $\mathbb{A}_{f,F}^\times$ on

$$Y_F = \widehat{\mathcal{O}}_F \times \pi_0(C_F),$$

$\mathcal{U} \subset \mathbb{A}_{f,F}^\times \times Y_F$,
$\mathcal{U} = \{(g, y = (\rho, [z, l]))| g\rho \in \widehat{\mathcal{O}}_F\}$.

BC stack groupoid [b] :

$$\boxed{Z := [\mathcal{U}/(\widehat{\mathcal{O}}_F^\times)^2]}$$

où $\gamma_1, \gamma_2 \in (\widehat{\mathcal{O}}_F^\times)^2$ acts by
$(g, \rho, [z, l]) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, [z, l\gamma_2^{-1}])$.

Composition law :
Si $y_1 = g_2 y_2$, $(g_1, y_1) \circ (g_2, y_2) = (g_1 g_2, y_2)$.

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[a] On board.
[b] Not algebraic in general.

2. HAMILTONIAN, PARTITION, KMS STATES.[a]

Denote $\widehat{\mathcal{O}}_F^\natural := \mathbb{A}_{f,F}^\times \cap \widehat{\mathcal{O}}_F$, $\mathcal{H} = \ell^2(\widehat{\mathcal{O}}_F^\times \backslash \widehat{\mathcal{O}}_F^\natural)$.
Let $y = (\rho, [z, l]) \in Y_F$ with ρ invertible.

Representation :
$\pi_y : \mathcal{H}_F \to \mathcal{B}(\mathcal{H})$.

Hamiltonian on $\mathcal{H}$:
$(H_y\xi)(g) = \log(\mathrm{Nm}(g)).\xi(g)$.

Partition function :
$\zeta_F(s) = \sum_{\widehat{\mathcal{O}}_F^\times \backslash \widehat{\mathcal{O}}_F^\natural} \mathrm{Nm}(g)^{-s}$

Extremal KMS states at low temperature :
$\phi_{\beta,y}(f) := \frac{\mathrm{Trace}(\pi_y(f)e^{-\beta H})}{\zeta_F(\beta)}$

[a] On board.

3. SYMMETRIES.[a]

$\widehat{\mathcal{O}}_F^\natural \times \mathbb{G}_{m,F}(\mathbb{R})$ acts by symmetries on $(\mathcal{H}_F, \sigma_t)$.

$$(m, r) : f(g, \rho, [z, l]) \mapsto f(g, \rho m, [zr, l])$$

Proposition :
This gives an exterior action of

$$\pi_0(F^\times \backslash \mathbb{A}_F^\times) \overset{\mathrm{rec}}{\cong} \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

on the system.

[a] On board.