II.

Zeta functions of schemes and motivic L-functions

K. Consani – Johns Hopkins University

Vanderbilt University, May 2006

(1) Zeta functions of schemes over finite fields

(2) Zeta functions of arithmetic schemes

(3) Motivic L-functions

(1) **Zeta functions of schemes over finite fields**

$X_{/k}$ scheme of finite type, $k = \mathbb{F}_q$ finite field

Main Example

$$X = \{\underline{a} = (a_1, \ldots, a_n) \in k^n : f_i(\underline{a}) = 0,\ i = 1, \ldots r\}$$

$$f_i(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$$

affine variety (e.g. $X = \mathbb{A}^n_k$)

$\underline{a} = (a_1, \ldots, a_n) \in k^n,\ f_i(\underline{a}) = 0\ \forall i$ k-**rational point**

$$X(k) := \{x \in X :\ x = (a_1, \ldots, a_n) \in k^n\}, \quad (|\mathbb{A}^n_k(k)| = q^n)$$

More in general: $X \to Spec(k), \quad d \in \mathbb{N}$

$\boxed{X(\mathbb{F}_{q^d}) := Mor_k(Spec(\mathbb{F}_{q^d}), X)}$ **$\mathbb{F}_{q^d}$-rational point**

Fact: $k_d := \mathbb{F}_{q^d}, \quad N_d := |X(\mathbb{F}_{q^d})| < \infty$

$\bar{X} := \{x \in X : \kappa(x)/k \text{ finite}\}, \quad \kappa(x) = \text{residue field}$

$N(x) := \#\kappa(x) = q^{\deg(x)}, \quad \deg(x) := [\kappa(x) : k]$

$n_l := \#\{x \in \bar{X} : \deg(x) = l\} < \infty, \quad N_d = \sum_{l|d} l n_l$

$N_d = |X(k_d)|$ Diophantine invariant of $X_{/k}$

$$\boxed{Z(X_{/k}, T) := \exp(\sum_{d \geq 1} N_d \frac{T^d}{d}) \in \mathbb{Q}[[T]]}$$

Zeta-function of $X_{/k}$

$s \in \mathbb{C}$, $\boxed{\zeta_X(s) := Z(X_{/k}, q^{-s})}$ **Hasse-Weil zeta**

carries the "complete package" of the Diophantine information associated to the set $\{N_d : d \in \mathbb{N}\}$

Examples

1) $\mathbb{P}^1_{/\mathbb{F}_q}$, $\quad N_d = q^d + 1$

$Z(\mathbb{P}^1, T) = \exp(\sum_{d\geq 1}(q^d+1)\frac{T^d}{d}) = \frac{1}{(1-qT)(1-T)} \in \mathbb{Q}(T)$

$\zeta_{\mathbb{P}^1}(s) = (1 - q^{-s})^{-1}(1 - q^{-(s-1)})^{-1}$

2) $\mathbb{P}^m_{/\mathbb{F}_q}$, $\quad N_d = \dfrac{q^{d(m+1)} - 1}{q^d - 1} = q^{md} + \ldots + q^{2d} + q^d + 1$

$$Z(\mathbb{P}^m, T) = \frac{1}{(1 - q^m T)\cdots(1 - qT)(1 - T)} \in \mathbb{Q}(T)$$

$$\zeta_{\mathbb{P}^m}(s) = \prod_{n=0}^{m}(1 - q^{-(s-n)})^{-1}$$

3) $\mathbb{A}^m_{/\mathbb{F}_q}$, $\quad N_d = q^{md}$

$Z(\mathbb{A}^m, T) = \exp(\sum_{d\geq 1} q^{md}\ \frac{T^d}{d}) = (1 - q^m T)^{-1} \in \mathbb{Q}(T)$

$\zeta_{\mathbb{A}^m}(s) = (1 - q^{-(s-m)})^{-1}$

Main Facts

$$(1) \quad Z(X_{/k}, T) = \prod_{x \in \bar{X}} (1 - T^{deg(x)})^{-1}$$

absolutely convergent in $Re(s) > \dim X$

(2) **Theorem** [Dwork, Grothendieck 1959-64] The zeta function of a scheme of finite type over a finite field **is rational**

$$Z(X,T) = \frac{\prod_i (1-\alpha_i T)}{\prod_j (1-\beta_j T)} \in \mathbb{Q}(T), \qquad \alpha_i, \beta_j \in \mathbb{C}$$

$$F : X(\bar{k}) \to X(\bar{k}), \quad F(\underline{a}) = \underline{a}^q \quad \underline{a} = (a_i),\ a_i \in \bar{k}$$

Frobenius morphism

$N_d = \sharp\{x \in X(\bar{k}) : \ F^d(\underline{a}) = \underline{a}\}$ fixed points of F^d

$\underline{a} =$ description in local coordinates of x

Theorem [Grothendieck 1964] $X_{/k}$ scheme of finite type, smooth and proper over $k = \mathbb{F}_q$

$$N_d = \sum_{i=0}^{2\dim X} (-1)^i Tr((F^d)^*; H^i_{et}(X_{\bar{k}}, \mathbb{Q}_\ell)) \qquad \Rightarrow$$

$$\boxed{Z(X_{/k}, q^{-s}) = \prod_{i=0}^{2\dim X} \det(1 - F^* q^{-s}; H^i_{et}(X_{\bar{k}}, \mathbb{Q}_\ell))^{(-1)^{i+1}}}$$

in $\mathbb{Q}_\ell[[q^{-s}]]$, $X_{\bar{k}} := X \times_k \bar{k}$, $(\ell, q) = 1$, $\ell =$ prime

in 1964 it was not known in general (although expected) that

$$\det(1 - F^* q^{-s}; H^i_{et}(X_{\bar{k}}, \mathbb{Q}_\ell)) \in \mathbb{Q}[q^{-s}]$$

independently of the auxiliary choice of the prime ℓ

Theorem [Deligne 1974] Assume $X_{/k}$ is smooth, and proper ($\dim X = m$)

$$(1) \quad Z(X_{/k}, T) = \frac{P_1(T) \cdots P_{2m-1}(T)}{P_0(T) \cdots P_{2m}(T)} \quad \text{in} \quad \mathbb{Q}(T)$$

$$P_i(T) := \det(1 - F^* T; H^i(X_{\bar{k}}, \mathbb{Q}_\ell)) \in \mathbb{Q}[T]$$

In particular

$$P_0(T) = 1 - T, \quad P_{2m}(T) = 1 - q^m T$$

(2) (functional equations)

$$P_{2m-i}(T) = (-1)^{B_i} \frac{q^{mB_i} T^{B_i}}{\det(F^*; H^i_{et})} P_i(\frac{1}{q^m T})$$

$$B_i := \dim H^i(X_{\bar{k}}, \mathbb{Q}_\ell)$$

$$Z(\frac{1}{q^m T}) = \pm q^{mE/2} T^E Z(T), \qquad E := \sum (-1)^i B_i$$

(3) Riemann Hypothesis

$$P_i(T) = \prod_j (1 - \alpha_{i_j} T) \in \mathbb{Z}[T], \quad \alpha_{i_j} \in \bar{\mathbb{Q}}, \quad |\alpha_{i_j}| = q^{i/2}$$

Example

$E_{/k}$ smooth, proper elliptic curve

$$Z(E,T) = \frac{1 - aT + qT^2}{(1-T)(1-qT)}, \quad \text{in } \mathbb{Q}(T)$$

$$1 - aT + qT^2 = (1 - \alpha_{1_1} T)(1 - \alpha_{1_2} T), \quad |\alpha_{1_i}| = q^{1/2}$$

$$a = \alpha_{1_1} + \alpha_{1_2} = Tr(F^*; H^1_{et}(E_{\bar{k}}, \mathbb{Q}_\ell)) \in \mathbb{Z}$$

(2) <u>**Zeta functions of arithmetic schemes**</u>

$X \to Spec(\mathbb{Z})$ scheme separated and of finite type

$\bar{X}(=|X|) = \{x \in X : \ \kappa(x)$ finite$\}$, $\quad N(x) = |\kappa(x)|$

$s \in \mathbb{C}$, $$\boxed{\zeta_X(s) := \prod_{x \in \bar{X}} (1 - N(x)^{-s})^{-1}}$$

Hasse-Weil Zeta function of X

<u>Examples</u>

1) $X = Spec(\mathbb{Z})$, $\qquad \zeta_X(s) = \prod_p (1 - p^{-s})^{-1} = \zeta(s)$

2) $X = Spec(\mathbb{Z}[T_1, \ldots, T_n]) = \mathbb{A}^n_{\mathbb{Z}}$

$$\zeta_X(s) = \prod_p (1 - p^{-(s-n)})^{-1} = \zeta(s-n)$$

3) $X = \mathbb{P}^n_{\mathbb{Z}}$

$$\zeta_X(s) = \prod_p \prod_{m=0}^{n} (1 - p^{-(s-m)})^{-1} = \prod_{m=0}^{n} \zeta(s-m)$$

4) $X = Spec(\mathcal{O}_K)$, $\mathcal{O}_K$ = ring of integers of $K/\mathbb{Q}$ number field

$$\zeta_X(s) = \zeta_K(s) = \prod_{\mathfrak{p}\subset\mathcal{O}_K} (1 - N(\mathfrak{p})^{-s})^{-1} \text{ Dedekind zeta}$$

Question on the asymptotic distribution of closed points on X (*i.e.* $x \in \bar{X}$) can be translated into analytic questions about $\zeta_X(s)$

Fact $\zeta_X(s)$ is absolutely convergent (holomorphic) in $Re(s) > \dim X$

Expected: $\zeta_X(s)$ has a meromorphic continuation to $\mathbb{C}$ and a functional equation (once suitably completed)

More in general, consider

$$X \xrightarrow{\pi} Spec(\mathcal{O}_K), \quad \pi = \text{proper}$$

irreducible, arithmetic scheme, K = number field

$$|X| = \coprod_{\substack{\mathfrak{p}\subset\mathcal{O}_K \\ \mathfrak{p}\text{prime}}} |X_\mathfrak{p}|, \qquad X_\mathfrak{p} := X \otimes_{\mathcal{O}_K} (\mathcal{O}_K/\mathfrak{p})$$

$$\zeta_X(s) = \prod_{\substack{\mathfrak{p}\subset\mathcal{O}_K \\ \mathfrak{p}\text{prime}}} \zeta_{X_\mathfrak{p}}(s), \quad Re(s) > \dim X$$

Assume: $X_K := X \times_{Spec(\mathcal{O}_K)} Spec(K)$ (generic fiber)

is smooth and proper (dim $X_K = m$)

Known: $X_{\mathfrak{p}}$ is smooth and proper for almost all $\mathfrak{p}$ (*i.e.* all $\mathfrak{p}$ except a finite number)

$$\zeta_X(s) = \prod_{i=0}^{2m} L_i(X,s)^{(-1)^{i+1}}$$

$$\boxed{L_i(X,s) := \prod_{\substack{\mathfrak{p} \\ X_{\mathfrak{p}}\text{smooth}}} P_{i,\mathfrak{p}}(X, N(\mathfrak{p})^{-s})^{-1} \times L_i^{(\text{bad})}(X,s)}$$

FACT: $\prod_{\substack{\mathfrak{p} \\ X_{\mathfrak{p}}\text{smooth}}} P_{i,\mathfrak{p}}(X, N(\mathfrak{p})^{-s})^{-1}$ depends only on X_K

$L_i^{(\text{bad})}(X,s)$ depends also on X (the “geometric model” of X_K)

$$\boxed{P_{i,\mathfrak{p}}(X, N(\mathfrak{p})^{-s}) := \det(1 - F_{\mathfrak{p}}^* N(\mathfrak{p})^{-s}; H^i(X_{\bar{K}}, \mathbb{Q}_\ell))}$$

$X_{\bar{\mathfrak{p}}} := X_{\mathfrak{p}} \times_{\kappa(\mathfrak{p})} \overline{\kappa(\mathfrak{p})}$, $q = N(\mathfrak{p})$, $F_{\mathfrak{p}}^{-1} \in Gal(\overline{\kappa(\mathfrak{p})}/\kappa(\mathfrak{p}))$

- ▶ $P_{i,\mathfrak{p}}(X, N(\mathfrak{p})^{-s}) = \det(1 - F_{\mathfrak{p}}^* N(\mathfrak{p})^{-s}; H^i(X_{\bar{\mathfrak{p}}}, \mathbb{Q}_\ell))$

because of the base-change theorem in étale cohomology

▶ [Deligne] $$\prod_{\substack{\mathfrak{p} \\ X_{\mathfrak{p}} \text{smooth}}} P_{i,\mathfrak{p}}(X, N(\mathfrak{p})^{-s})^{-1} = L(\rho_{X,i}, s)$$

$$\rho_{X,i} : G_K \to Aut(H^i_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)) \quad G_K = Gal(\bar{K}/K)$$

$$L(\rho_{X,i}, s) := \prod_{v \notin S} P_{v,\rho}((Nv)^{-s})^{-1}$$ **Artin L-series**

$$P_{v,\rho}((Nv)^{-s}) := \det(1 - F^*_{v,\rho} N(v)^{-s}; H^i(X_{\bar{K}}, \mathbb{Q}_\ell))$$

$$F^{-1}_{v,\rho} \in G_{k(v)} \cong D_w/I_w, \quad w|v, \quad \mathfrak{p} = \mathfrak{p}_v$$

$v \in \Sigma_K$ classes of normalized valutations of K

$S \subset \Sigma_K$, $S = \{v : X_{\mathfrak{p}_v} \text{ not smooth}\} \cup \{v : \text{archim}\} \cup \{w|\ell\}$

$\rho_{X,i}$ factors through $G_{k(v)} =< F_{\mathfrak{p}_v} >$

▶ [Deligne] The conjugacy classes $\{F_{v,\rho}\}$ describe a system of (local) Galois reprentations which defines $\rho_{X,i}$

Because the infinite product

$$\prod_{\substack{\mathfrak{p}\subset\mathcal{O}_K \\ X_{\mathfrak{p}}\text{smooth}}} P_{i,\mathfrak{p}}(X, N(\mathfrak{p})^{-s})^{-1}$$

is known to have in some cases (e.g. abelian varieties with CM) meromorphic continuation to $\mathbb{C}$ and functional equation, <u>if completed</u> at the bad and at the archimedean primes

▶ One is led to study $L_i(X,s)$ “per se” as a function associated to $H^i(X_{\bar{K}}, \mathbb{Q}_\ell)$: the ℓ-adic realization of the (pure) <u>motive</u> $h^i(X_K)$

▶ The definition of the Euler's factors at the places $\mathfrak{p}$ of <u>bad reduction</u> for X (*i.e.* where $X_{\mathfrak{p}}$ is not smooth) is deduced by analogy with the case of a scheme defined over a global field of positive characteristic

<u>Main Point</u> (Analogy with the function field case)

$$Y_{/\mathbb{F}_q} \text{ smooth, projective curve,} \quad K(Y) = K$$

$$X \xrightarrow{\pi} Spec(K), \quad Spec(K) \overset{j}{\hookrightarrow} Y$$

$$\mathcal{F} := j_* R^i \pi_* \mathbb{Q}_\ell = j_* H^i(X_{\bar{K}}, \mathbb{Q}_\ell), \quad (\ell, q) = 1$$

$$y \in |Y|, \quad \mathcal{F}_{\bar{y}} = H^i(X_{\bar{K}}, \mathbb{Q}_\ell)^{I_y} \cong H^i(X_{\bar{K}_y}, \mathbb{Q}_\ell)^{I_y}$$

$$\bar{K}_y = \text{completion of } K \text{ at } y, \quad I_y \subset G_{K_y} \text{ inertia group}$$

$$L_i(X,s) = \prod_{y\in|Y|} \det(1 - F_y^* N(y)^{-s}; H^i(X_{\bar{K}_y}, \mathbb{Q}_\ell)^{I_y})^{-1}$$

$$\zeta_Y(\mathcal{F},s) = \prod_{i=0}^{2} \det(1 - F_y^* N(y)^{-s}; H^i(Y_{\bar{\mathbb{F}}_q}, \mathcal{F}))^{(-1)^{i+1}}$$

has functional equation (as $Y_{/\mathbb{F}_q}$ is smooth and proper)

This result suggests to define in the number-field case $L_i^{(bad)}(X,s)$ as a product of local factors such as

$$\boxed{P_{i,\mathfrak{p}}^{(bad)}(X, N(\mathfrak{p})^{-s}) := \det(1 - F_{\mathfrak{p}} N(\mathfrak{p})^{-s}; H^i(X_{\bar{K}}, \mathbb{Q}_\ell)^{I_{\mathfrak{p}}})^{-1}}$$

and assuming that the coefficients belong to $\mathbb{Q}$ and are independent of ℓ

Example $X_{/K}$ algebraic curve, $K =$ number-field, $g(X) = g$

$$H^1_{et}(X_{\bar{K}}, \mathbb{Q}_\ell) \simeq T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell =: V_\ell(J) \simeq \mathbb{Q}_\ell^{2g}$$

Tate's module of the Jacobian $J = Jac(X)$ of X

$$T_\ell(X) := \lim_m Ker(J \xrightarrow{\ell^m} J) \simeq \mathbb{Z}_\ell^{2g}$$

$$L_1(X,s) = \prod_{\mathfrak{p}} P_{1,\mathfrak{p}}(X, N(\mathfrak{p})^{-s})^{-1} \quad \text{L-function of } X$$

$$L_0(X,s) = \zeta_K(s), \quad L_2(X,s) = \zeta_K(s-1)$$

Cohomology classes are represented by cocycles (cells for CW complexes)

Grothendieck conjectured that an analogue of the CW-decomposition should exist for any algebraic scheme.

The factorization of the zeta-function

$$\zeta_X(s) = \prod_{i=0}^{2m} L_i(X,s)^{(-1)^{i+1}}$$

should then be interpreted as an arithmetic manifestation of a decomposition, holding at the level of the geometric space, into more general types of "cells":

the motives $h^i(X)$

$h^i(X)$ are no longer algebraic schemes but elements of a suitable **abelian category** constructed by enlarging the category of smooth, projective schemes over K

(3) Motivic L-functions

$K,\ E =$ number fields

$\mathcal{M}_K(E) =$ category of (pure, mixed) motives over K with coefficients in E, endowed with **realization functors**

$$H^*_{\mathcal{H}} : \mathcal{M}_K(E) \to Vect_E$$

these functors describe the realizations of a motive M in a (Weil) cohomology theory with coefficients in E: $H^*_{\mathcal{H}}(M, E)$

Example

$H^*_{et,\ell}(M) = H^*_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$, $X_{/K} =$ smooth, projective K-scheme

ℓ-adic realization, ℓ prime number

$\mathfrak{p}|p$ prime ideal in K, $[K_{\mathfrak{p}} : \mathbb{Q}_p] < \infty$

$$I_{\mathfrak{p}} \subset G_{K_{\mathfrak{p}}}, \quad \varphi_{\mathfrak{p}} \in G_{K_{\mathfrak{p}}}/I_{\mathfrak{p}}, \quad \varphi_{\mathfrak{p}}(x) = x^{N(\mathfrak{p})},\ F_{\mathfrak{p}} = \varphi_{\mathfrak{p}}^{-1}$$

Fix $\quad \ell \neq p, \quad \iota : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$

$E \otimes \mathbb{C} \simeq \mathbb{C}^{Hom(E,\mathbb{C})}$, consider the functor

$$\mathcal{M}_{K_\mathfrak{p}}(E) \to F_\mathfrak{p}Mod_{E\otimes\mathbb{C}}$$

$F_\mathfrak{p}Mod_{E\otimes\mathbb{C}}$ = category of $(E\otimes\mathbb{C})[F_\mathfrak{p}]$-modules of finite rank over $E\otimes\mathbb{C}$

$$M \mapsto M^I_{\ell,\iota} := (M^{I_\mathfrak{p}}_{\ell,\iota,\sigma})_{\sigma\in Hom(E,\mathbb{C})}$$

$$M^{I_\mathfrak{p}}_{\ell,\iota,\sigma} = M^{I_\mathfrak{p}}_{\ell,\iota} \otimes_{E\otimes\mathbb{C},\sigma} \mathbb{C}, \;\; M^{I_\mathfrak{p}}_{\ell,\iota} = H^*_{et}(M_{\bar{K}_\mathfrak{p}}, \mathbb{Q}_\ell)^{I_\mathfrak{p}} \otimes_{\mathbb{Q}_{\ell,\iota}} \mathbb{C}$$

Expected These functors are isomorphic for different choices of ℓ and ι

- ▶ This is in fact the case if $M = h(X_{K_\mathfrak{p}})$, and $X_{K_\mathfrak{p}}$ is smooth, projective with good reduction (at $\mathfrak{p}$):

$$H^*_{et}(M_{\bar{K}_\mathfrak{p}}, \mathbb{Q}_\ell)^{I_\mathfrak{p}} = H^*_{et}(X_{\bar{K}_\mathfrak{p}}, \mathbb{Q}_\ell) \qquad E = \mathbb{Q}$$

In general

$$L_{\mathfrak{p}}(M,s) := (\det_{\mathbb{C}}(1 - F_{\mathfrak{p}} N(\mathfrak{p})^{-s}; M_{\ell,\iota,\sigma}^{I_{\mathfrak{p}}})^{-1})_{\sigma \in Hom(E,\mathbb{C})}$$

$$L_{\mathfrak{p}}(M,s) = (L_{\mathfrak{p}}(M,\sigma,s))_{\sigma \in Hom(E,\mathbb{C})}$$

Expected to be independent of ℓ and ι

If K is a number field, M_K a motive over K (with coefficients in E)

$M_{K_{\mathfrak{p}}} := M \otimes_K K_{\mathfrak{p}}$ is a motive over the local field $K_{\mathfrak{p}}$

$$L(M,s) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(M_{\mathfrak{p}},s)$$

expected to be independent of ℓ, ι

To state the convergency properties of the motivic L-function

consider the integer $w_m :=$ largest weight of M

Example

$w_m = 2n$, $X_{/K}$ = smooth projective algebraic variety,
$\dim X = n$, $M = h(X)$

$$\prod_{\mathfrak{p}} L_{\mathfrak{p}}(M_{\mathfrak{p}}, s)$$

FACT: this function converges absolutely in
$Re(s) > \frac{w_m}{2} + 1 = n + 1$

Expected $L(M, s)$ has meromorphic continuation to $\mathbb{C}$
with functional equation holding for the complete
L-function

$$\widehat{L}(M, s) := L(M, s) \cdot L_\infty(M, s)$$

The Archimedean factors $L_\infty(M,s)$

$$\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s), \quad \Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})$$

$L_\infty(M,s)$ depends on the isomorphic class of the Betti realization

$$H_B^m(M) \otimes \mathbb{C}$$

of the motive, endowed with the Hodge decomposition and an involution F_∞

Conjecture the completed motivic L-function $\widehat{L}(M,s)$ has a meromorphic continuation to $\mathbb{C}$ and

(functional eq) $\widehat{L}(M,s) = \epsilon(M,s)\widehat{L}(M^*,1-s)$

$M^* =$ dual motive, $\epsilon(M,s) =$ epsilon factor

▶ In all cases where the conjecture has been verified, the proof runs through the identification of $\widehat{L}(M,s)$ with an automorphic L-series!

If M is a pure, geometric motive of weight i, then $M^* \simeq M(i)$ and the (expected) functional equation is

$$\widehat{L}(M,s) = \epsilon(M,s)\widehat{L}(M,i+1-s)$$

Main Conjecture the zeroes of $\widehat{L}(M,s)$ lie on the line

$$Re(s) = \frac{i+1}{2}$$

BIBLIOGRAPHY

- Manin, Panchishkin Introduction to Modern Number Theory, Springer

- Deninger L-functions of mixed motives, in Motives I, Proceedings of Symposia in Pure Math. 55

- Deligne, Valeurs de fonctions L et périodes d'intégrales, Proc. Symposia Pure Math 33 (2)