I.

An overview of the theory of Zeta functions and L-series

K. Consani – Johns Hopkins University

Vanderbilt University, May 2006

(a) **Arithmetic L-functions**

(a1) **Riemann zeta function**: $\zeta(s)$, $s \in \mathbb{C}$

(a2) Dirichlet L-series: $L(\chi, s)$

$\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

(a3) **Dedekind zeta funct**: $\zeta_K(s)$, $[K : \mathbb{Q}] \leq \infty$

(a4) Hecke L-series: $L_K(\chi, s)$

(a5) **Artin L-function**: $L(\rho, s)$

$\rho : Gal(K/\mathbb{Q}) \to GL_n(\mathbb{C})$ Galois representation

(a6) **Motivic L-function**: $L(M, s)$

M pure or mixed motive

(b) **Automorphic L-functions**

(b1) Classical theory (before Tate's thesis 1950)

$L(f,s)$; $L(f,\chi,s)$ **modular L-function**

associated to a modular cusp form $f : \mathfrak{H} \to \mathbb{C}$

(b2) Modern adelic theory: $L(\pi, s)$

automorphic L-function

$\pi = \otimes'_v \pi_v$, (π_v, V_{π_v}) = irreducible (admissible) representation of $GL_n(\mathbb{Q}_v)$

(a1) **The Riemann zeta function**

$s \in \mathbb{C}$, $$\boxed{\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}}$$

Main Facts

- converges absolutely and uniformly on $Re(s) > 1$

$(Re(s) \geq 1 + \delta \ (\delta > 0), \quad \sum_{n=1}^{\infty} |\frac{1}{n^s}| \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}})$

$\Rightarrow \zeta(s)$ represents an analytic function in $Re(s) > 1$

- **Euler's identity**: $$\boxed{\zeta(s) = \prod_{\substack{p \\ \text{prime}}} (1 - p^{-s})^{-1}}$$

$$\left(\left|\prod_{p \leq N} (1 - p^{-s})^{-1} - \zeta(s)\right| \leq \sum_{n > N} \frac{1}{n^{1+\delta}}\right)$$

Number-theoretic significance of the zeta-function:

▶ Euler's identity expresses the law of unique prime factorization of natural numbers

$$\boxed{\Gamma(s) := \int_0^\infty e^{-y} y^s \ \frac{dy}{y}}$$ **Gamma-function**

$s \in \mathbb{C}$, $Re(s) > 0$; absolutely convergent

- $\Gamma(s)$ analytic, has meromorphic continuation to $\mathbb{C}$

- $\Gamma(s) \neq 0$, has <u>simple</u> poles at $s = -n$, $n \in \mathbb{Z}_{\geq 0}$

$$Res_{s=-n}\Gamma(s) = \frac{(-1)^n}{n!}$$

- functional equations

$$\Gamma(s+1) = s\Gamma(s), \quad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

- Legendre's duplication formula

$$\Gamma(s)\Gamma(s+\tfrac{1}{2}) = \frac{2\sqrt{\pi}}{2^{2s}}\Gamma(2s)$$

- special values

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \quad \Gamma(1) = 1, \quad \Gamma(k+1) = k!, \quad k \in \mathbb{Z}_{\geq 0}$$

The connection between $\Gamma(s)$ and $\zeta(s)$

$$y \mapsto \pi n^2 y \quad \Rightarrow \quad \pi^{-s}\Gamma(s)\tfrac{1}{n^{2s}} = \int_0^\infty e^{-\pi n^2 y} y^s \frac{dy}{y}$$

sum over $n \in \mathbb{N}$

$$\boxed{\pi^{-s}\Gamma(s)\zeta(2s) = \int_0^\infty \sum_{n\geq 1} e^{-\pi n^2 y} y^s \frac{dy}{y}} \qquad g(y) := \sum_{n\geq 1} e^{-\pi n^2 y}$$

$$\Theta(z) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} = 1 + 2\sum_{n=1}^{\infty} e^{\pi i n^2 z} \qquad \textbf{Jacobi's theta}$$

$$g(y) = \tfrac{1}{2}(\Theta(iy) - 1), \qquad \boxed{Z(s) := \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s)}$$

Main Facts

(1) $Z(s)$ admits the integral representation

$$Z(s) = \tfrac{1}{2}\int_0^\infty (\Theta(iy) - 1) y^{s/2} \frac{dy}{y} \qquad \overset{\text{Mellin Principle}}{\Rightarrow}$$

(2) $Z(s)$ admits an analytic continuation to $\mathbb{C} \setminus \{0, 1\}$, has simple poles at $s = 0$, $s = 1$

$$Res_{s=0} Z(s) = -1, \quad Res_{s=1} Z(s) = 1$$

(3) functional eq $\quad Z(s) = Z(1-s)$

Implications for the Riemann zeta $\zeta(s)$

(4) $\zeta(s)$ admits an analytic continuation to $\mathbb{C} \setminus \{1\}$

has simple pole at $s = 1$, $\quad Res_{s=1}\zeta(s) = 1$

(5) (functional eq) $\quad \zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos(\frac{\pi s}{2})\zeta(s)$

Moreover, from $Z(s) = Z(1-s) \quad \Rightarrow$

- the only zeroes of $\zeta(s)$ in $Re(s) < 0$ are the poles of $\Gamma(\frac{s}{2})$ ($s \in 2\mathbb{Z}_{<0}$, "trivial zeroes")
- other zeroes of $\zeta(s)$ (*i.e.* on $Re(s) > 0$) must lie on the **critical strip**: $\quad 0 \leq Re(s) \leq 1$

Riemann Hypothesis The "non-trivial" zeroes of $\zeta(s)$ lie on the line $\quad Re(s) = \frac{1}{2}$

(a2) **Dirichlet L-series**

$$m \in \mathbb{N}, \quad \chi : (\mathbb{Z}/m\mathbb{Z})^* \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

Dirichlet character mod.m

$$\chi : \mathbb{Z} \to \mathbb{C}, \quad \chi(n) = \begin{cases} \chi(n \text{ mod } m) & (n,m) = 1 \\ 0 & (n,m) \neq 1 \end{cases}$$

$$s \in \mathbb{C}, \quad \boxed{L(\chi, s) := \sum_{n \geq 1} \frac{\chi(n)}{n^s}} \qquad Re(s) > 1$$

for $\chi = 1$ (principal character): $L(1, s) = \zeta(s)$

Main Facts

(1) Euler's identity: $\boxed{L(\chi, s) = \prod_p (1 - \chi(p)p^{-s})^{-1}}$

(2) $L(\chi, s)$ converges absolutely and unif. on $Re(s) > 1$
(represents an analytic function)

$$\chi(-1) = (-1)^p \chi(1), \quad p \in \{0,1\} \textbf{ exponent}$$

$$\chi : \{(n) \subset \mathbb{Z} \mid (n,m) = 1\} \to S^1$$

$$\chi((n)) := \chi(n)(\tfrac{n}{|n|})^p$$

Grössencharacter mod.m   (multiplicative fct)

$$\Gamma(\chi, s) := \Gamma(\tfrac{s+p}{2}) = \int_0^\infty e^{-y} y^{(s+p)/2} \frac{dy}{y} \quad \textbf{Gamma integral}$$

$$y \mapsto \pi n^2 y/m, \quad \theta(\chi, iy) = \textstyle\sum_n \chi(n) n^p e^{-\pi n^2 y/m} \quad \ldots \Rightarrow$$

$$\boxed{L_\infty(\chi, s) := (\tfrac{m}{\pi})^{\frac{s}{2}} \Gamma(\chi, s)} \quad \textbf{archimedean Euler factor}$$

$$\Lambda(\chi, s) := L_\infty(\chi, s) L(\chi, s), \qquad Re(s) > 1$$

completed L-series of the character χ

$\Lambda(\chi, s)$ has integral representation $\stackrel{\text{Mellin principle}}{\Rightarrow}$

- Functional eq.: If $\chi \neq 1$ is a primitive character,

$\Lambda(\chi, s)$ admits an analytic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(\chi, s) = W(\chi) \Lambda(\bar{\chi}, 1-s), \quad |W(\chi)| = 1$$

($\bar{\chi}$ = complex conjugate character)

(a3) **Dedekind zeta function**

$K/\mathbb{Q}$ number field, $[K:\mathbb{Q}] = n$

$s \in \mathbb{C}$ $$\boxed{\zeta_K(s) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s}}$$

$\mathfrak{a}$ = integral ideal of K, $N(\mathfrak{a})$ = absolute norm

Main Facts

(1) $\zeta_K(s)$ converges absolutely and unif. on $Re(s) > 1$

(2) (Euler's identity)

$$\boxed{\zeta_K(s) = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1}} \quad Re(s) > 1$$

$Cl_K = J/P$ **ideal class group** of K

$$\zeta_K(s) = \sum_{[\mathfrak{b}] \in Cl_K} \zeta(\mathfrak{b}, s), \quad \zeta(\mathfrak{b}, s) := \sum_{\substack{\mathfrak{a} \in [\mathfrak{b}] \\ \text{integral}}} \frac{1}{N(\mathfrak{a})^s}$$

$\zeta(\mathfrak{b}, s)$ **partial zeta functions**

$$L_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$$

$$L_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$$

$r_1 :=$ number of real embeddings $\quad v = \bar{v} : K \to \mathbb{C}$

$r_2 :=$ number of pairs of complex embeddings $\{v, \bar{v}\} : K \to \mathbb{C}$

$d_K =$ discriminant of K

$$\boxed{Z_\infty(s) := |d_K|^{s/2} L_{\mathbb{R}}(s)^{r_1} L_{\mathbb{C}}(s)^{r_2}}$$

Euler's factor at infinity of $\zeta(\mathfrak{b}, s)$

► $Z(\mathfrak{b}, s) := Z_\infty(s)\zeta(\mathfrak{b}, s), \quad Re(s) > 1$

admits an analytic continuation to $\mathbb{C} \setminus \{0, 1\}$ and satisfies a functional equation

$$Z_K(s) := \sum_{\mathfrak{b}} Z(\mathfrak{b}, s) = Z_\infty(s)\zeta_K(s)$$

From the corresponding properties of $Z(\mathfrak{b}, s)$ one deduces

Main Facts

(1) $Z_K(s) = Z_\infty(s)\zeta_K(s)$ has analytic continuation to

$$\mathbb{C} \setminus \{0, 1\}$$

(2) (functional eq) $\quad Z_K(s) = Z_K(1-s)$

$Z_K(s)$ has simple poles at $s = 0,\ 1$

$$Res_{s=0} Z_K(s) = -\frac{2^r hR}{w}, \quad Res_{s=1} Z_K(s) = \frac{2^r hR}{w}$$

$r = r_1 + 2r_2$, $\quad h =$ class nb. of K, $\quad R =$ regulator of K
$w =$ number of roots of 1 in K

[Hecke] Subsequent results for $\zeta_K(s)$

(3) $\zeta_K(s)$ has analytic continuation to $\mathbb{C} \setminus \{1\}$

with a simple pole at $s = 1$

(4) Class number formula

$$Res_{s=1} \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d_K|}} \frac{hR}{w}$$

(5) (functional eq.) $\zeta_K(1-s) = A(s)\zeta_K(s)$

$$A(s) := |d_K|^{s-1/2}(\cos\tfrac{\pi s}{2})^{r_1+r_2}(\sin\tfrac{\pi s}{2})^{r_2} L_{\mathbb{C}}(s)^n$$

(6) $\zeta_K(s) \neq 0$ for $Re(s) > 1$ $\Rightarrow$

$$m \in \mathbb{Z}_{\geq 0}$$

$$ord_{s=-m}\zeta_K(s) = \begin{cases} r_1 + r_2 - 1 = rk(\mathcal{O}_K^*) & \text{if } m = 0 \\ r_1 + r_2 & \text{if } m > 0 \text{ even} \\ r_2 & \text{if } m > 0 \text{ odd} \end{cases}$$

The class number formula reads now as

$$\zeta_K^*(0) := \lim_{s\to 0} \frac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\frac{hR}{w}$$

(a5) **Artin L-functions**

$L/K =$ Galois extension of nb field K, $\quad G := Gal(L/K)$

Artin L-functions generalize the classical L-series

in the following way

$$L(\chi, s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1}, \ Re(s) > 1$$

$$\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*, \quad G := Gal(\mathbb{Q}(\mu_m)/\mathbb{Q}) \stackrel{\simeq}{\leftarrow} (\mathbb{Z}/m\mathbb{Z})^*$$

p mod $m \mapsto \varphi_p, \quad \varphi_p(\zeta_m) = \zeta_m^p$ **Frobenius**

$\chi : G \to GL_1(\mathbb{C})$ 1-dim Galois representation

$$\blacktriangleright\blacktriangleright \quad L(\chi, s) = \prod_{p \nmid m} (1 - \chi(\varphi_p)p^{-s})^{-1}$$

this is a description of the Dirichlet L-series in a purely Galois-theoretic fashion

More in general:

$V =$ finite dim $\mathbb{C}$-vector space

$$\rho : G = Gal(L/K) \to GL(V) = Aut_{\mathbb{C}}(V)$$

$\mathfrak{p}$ prime ideal in K, $\mathfrak{q}/\mathfrak{p}$ prime ideal of L above $\mathfrak{p}$

$$D_{\mathfrak{q}}/I_{\mathfrak{q}} \xrightarrow{\simeq} Gal(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})), \quad D_{\mathfrak{q}}/I_{\mathfrak{q}} =< \varphi_{\mathfrak{q}} >$$

$$\varphi_{\mathfrak{q}} \mapsto (x \mapsto x^q) \quad q = N(\mathfrak{p})$$

$\varphi_{\mathfrak{q}} \in End(V^{I_{\mathfrak{q}}})$ finite order endomorphism

$$\boxed{P_{\mathfrak{p}}(T) := \det(1 - \varphi_{\mathfrak{q}} T;\ V^{I_{\mathfrak{q}}})}, \quad \textbf{characteristic pol}$$

only depends on $\mathfrak{p}$ (not on $\mathfrak{q}/\mathfrak{p}$)

$$\boxed{\zeta_{L/K}(\rho, s) := \prod_{\substack{\mathfrak{p}\text{prime} \\ \text{in}K}} \det(1 - \varphi_{\mathfrak{q}} N(\mathfrak{p})^{-s};\ V^{I_{\mathfrak{q}}})^{-1}}$$

Artin L-series

$$\det(1 - \varphi_{\mathfrak{q}} N(\mathfrak{p})^{-s};\ \ V^{I_{\mathfrak{q}}}) = \prod_{i=1}^{d} (1 - \epsilon_i N(\mathfrak{p})^{-s})$$

$\epsilon_i =$ roots of 1: $\varphi_{\mathfrak{q}}$ has finite order

- $\zeta_{L/K}(\rho, s)$ converges absolutely and unif on $Re(s) > 1$

If $(\rho, \mathbb{C})$ is the trivial representation, then

$$\zeta_{L/K}(\rho, s) = \zeta_K(s) \quad \text{Dedekind zeta function}$$

- An additive expression analogous to $\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}$ does not exist for general Artin L-series.

- Artin L-series exhibit nice functorial behavior under change of extensions L/K and representations ρ

Character of (ρ, V) $\qquad \chi_\rho : Gal(L/K) \to \mathbb{C}$

$$\chi_\rho(\sigma) = tr(\rho(\sigma)), \quad \chi_\rho(1) = \dim V = \deg(\rho)$$

$$(\rho, V) \sim (\rho', V') \quad \Longleftrightarrow \quad \chi_\rho = \chi_{\rho'}$$

$\zeta_{L/K}(\rho, s) = \zeta_{L/K}(\chi_\rho, s)$; functorial behavior $\Rightarrow$

$$\zeta_L(s) = \zeta_K(s) \prod_{\substack{\chi \neq 1 \\ \chi \text{ irred of } G(L/K)}} \zeta_{L/K}(\chi, s)^{\chi(1)}$$

Artin conjecture $\forall \chi \neq 1$ irreducible, $\zeta_{L/K}(\chi, s)$ defines an entire function *i.e.* holom. function on $\mathbb{C}$

the conjecture has been proved for abelian extensions

For every <u>infinite</u> (archimedean) place $\mathfrak{p}$ of K

$$\zeta_{L/K,\mathfrak{p}}(\chi,s)=\begin{cases} L_{\mathbb{C}}(s)^{\chi(1)} & \mathfrak{p} \text{ complex} \\ L_{\mathbb{R}}(s)^{n_+}L_{\mathbb{R}}(s+1)^{n_-} & \mathfrak{p} \text{ real} \end{cases}$$

$$n_+=\tfrac{\chi(1)+\chi(\varphi_{\mathfrak{q}})}{2},\quad n_-=\tfrac{\chi(1)-\chi(\varphi_{\mathfrak{q}})}{2};\quad \varphi_{\mathfrak{q}}\in Gal(L_{\mathfrak{q}}/K_{\mathfrak{p}})$$

$\zeta_{L/K,\mathfrak{p}}(\chi,s)$ has also nice <u>functorial behavior</u>

<u>For $\mathfrak{p}$ real</u>, $\varphi_{\mathfrak{q}}$ induces decomp $\quad V=V^+\oplus V^-$

$$V^+=\{x\in V:\varphi_{\mathfrak{q}}x=x\},\ V^-=\{x\in V:\varphi_{\mathfrak{q}}x=-x\}$$

$$n_+=\dim V^+,\quad n_-=\dim V^-$$

$$\boxed{\zeta_{L/K,\infty}(\chi,s):=\prod_{\mathfrak{p}|\infty}\zeta_{L/K,\mathfrak{p}}(\chi,s)}$$

$$\Lambda_{L/K}(\chi,s):=c(L/K,\chi)^{\frac{s}{2}}\zeta_{L/K}(\chi,s)\zeta_{L/K,\infty}(\chi,s)$$

completed Artin series

$$c(L/K,\chi):=|d_K|^{\chi(1)}N(\mathfrak{f}(L/K,\chi))\in\mathbb{N}$$

$\mathfrak{f}(L/K,\chi)=\prod_{\mathfrak{p}\nmid\infty}\mathfrak{f}_{\mathfrak{p}}(\chi)$ Artin conductor of χ

$\mathfrak{f}_{\mathfrak{p}}(\chi)=\mathfrak{p}^{f(\chi)}$ local Artin conductor $\quad (f(\chi)\in\mathbb{Z})$

Main Facts

- $\Lambda_{L/K}(\chi, s)$ admits a meromorphic continuation to $\mathbb{C}$

- (Functional eq.) $\Lambda_{L/K}(\chi, s) = W(\chi)\Lambda_{L/K}(\bar{\chi}, 1-s)$

$$W(\chi) \in \mathbb{C}, \quad |W(\chi)| = 1$$

▶ the proof of the functional equation uses the fact that the Euler factors $\zeta_{L/K,\mathfrak{p}}(\chi, s)$ at the infinite places $\mathfrak{p}$ behave, under change of fields and characters, in exactly the same way as the Euler factors at the finite places:

$$\mathfrak{p} < \infty \qquad \zeta_{L/K,\mathfrak{p}}(\chi, s) := \det(1 - \varphi_{\mathfrak{q}} N(\mathfrak{q})^{-s};\ V^{I_{\mathfrak{q}}})^{-1}$$

this uniform behavior that might seem at first in striking contrast with the definition of the archimedean Euler factors has been motivated by a unified interpretation of the Euler's factors (archimedean and non)

▶▶ [Deninger 1991-92, Consani 1996]

$$\zeta_{L/K,\mathfrak{p}}(\chi, s) = \det{}_\infty\left(\tfrac{\log N(\mathfrak{p})}{2\pi i}(sid - \Theta_{\mathfrak{p}});\ H(X(\mathfrak{p})/\mathbb{L}_{\mathfrak{p}})\right)^{-1}$$

this result reaches far beyond Artin L-series and suggests a complete analogy with the theory of L-series of algebraic varieties over finite fields.

(b) Automorphic L-functions

(b1) Classical theory (before Tate's thesis)

$f : \mathfrak{H} \to \mathbb{C}$ modular form of weight k for $\Gamma \subset SL_2(\mathbb{Z})$

- f holomorphic, $\mathfrak{H} =$ upper-half complex plane
- $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$
- f is regular at the cusps z of Γ, $|SL_2(\mathbb{Z}) : \Gamma| < \infty$
  ($z \in \mathbb{Q} \cup \{i\infty\}$ fixed pts of parabolic elements of Γ)

Examples

- $\theta_q(z)$ theta series attached to a quadratic form $q(\underline{x})$

$$\theta_q(z) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n z}, \quad a(n) = \text{Card}\{\underline{v} : q(\underline{v}) = n\}$$

- $\Delta(z)$ discriminant function from the theory of elliptic modular functions

$$\Delta(z) = 2^{-4}(2\pi)^{12} \sum_{n=1}^{\infty} \tau(nz) e^{2\pi i n z}$$

For simplicity will assume: $\Gamma = SL_2(\mathbb{Z})$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma \quad \Rightarrow \quad f(z+1) = f(z)$$

$$\boxed{f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z}}$$ **Fourier expansion**

f is a **cusp form** if $a_0 = 0$ *i.e.*

$$f(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$$

the Fourier coefficients a_n often carry interesting arithmetical information:

- $f(z) = \theta_q(z)$, a_n counts the number of times n is represented by the quadratic form $q(\underline{x})$

- $f(z) = \Delta(z)$, $a_n = \tau(n)$ Ramanujan's τ-function

[Hecke 1936] Attached to each cusp form there is a complex analytic invariant function: its L-function

$$\boxed{L(f,s) = \sum_{n \geq 1} \frac{a_n}{n^s}}$$ **Dirichlet series**

$$L(f,s) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

This L-function is connected to f by an **integral representation**: its Mellin transform

$$\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s) = \int_0^\infty f(iy)y^s d^\times y$$

through this integral representation one gets

[Hecke] $L(f,s)$ is entire and satisfies the functional equation

$$\Lambda(f,s) = i^k \Lambda(f,k-s)$$

the functional equation is a consequence of having a modular transformation law under $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ sending $z \mapsto -\frac{1}{z}$

Since the Mellin transform has an inverse integral transform, one gets

Converse theorem [Hecke] If

$$D(s) = \sum_n \frac{a_n}{n^s}$$

has a "nice" behavior and satisfies the correct functional equation (as above) then

$$f(z) = \sum_n a_n e^{2\pi i n z}$$

is a cusp form (of weight k) for $SL_2(\mathbb{Z})$ and

$$D(s) = L(f, s)$$

in particular: the modularity of $f(z)$ is a consequence of the Fourier expansion and the functional equation

[Weil 1967] the Converse theorem for $\Gamma_0(N)$ holds, by using the functional equation not just for $L(f, s)$ but also for

$$L(f, \chi, s) = \sum_{n \geq 1} \frac{\chi(n) a_n}{n^s}$$

$\chi =$ Dirichlet character of conductor prime to the level $N \in \mathbb{N}$

$$\Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \; mod \; N \}$$

[Hecke 1936] An algebra of operators

$\mathcal{H} = \{T_n\}$ **Hecke operators**

acts on modular forms.

If $f(z)$ is a simultaneous eigen-function for the operators in $\mathcal{H}$, then $L(f,s)$ has an **Euler's product**

$$\boxed{L(f,s) = \prod_p (1 - a_p p^{-s} + p^{-2s})^{-1}}$$

Conclusion

Arithmetic L-functions

are described by Euler's products, analytic properties are conjectural, arithmetic meaning is clear

Automorphic L-functions

are defined by Dirichlet series, characterized by analytic properties, Euler's product and arithmetic meaning are more mysterious...

(b2) Modern adelic theory

The modular form $f(z)$ for $SL_2(\mathbb{Z})$ (or congruence subgroup) is replaced by an automorphic representation of $GL_2(\mathbb{A})$

(in general by an automorphic representation of $GL_n(\mathbb{A})$)

this construction is a generalization of Tate's thesis for $GL_1(\mathbb{A})$

$\boxed{\mathbb{A} := \prod_p{}' \mathbb{Q}_p \times \mathbb{R}}$ ring of **adèles of** $\mathbb{Q}$

locally compact topological ring ($\prod'$ = restricted product)

$\mathbb{Q} \subset \mathbb{A}$ diagonal discrete embedding, $\mathbb{A}/\mathbb{Q}$ compact

$$GL_n(\mathbb{A}) = \prod_p{}' GL_n(\mathbb{Q}_p) \times GL_n(\mathbb{R})$$

$GL_n(\mathbb{Q}) \subset GL_n(\mathbb{A})$ diagonal discrete embedding

$Z(\mathbb{A})GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})$ finite volume

$GL_n(\mathbb{A})$ acts on the space

$\mathcal{A}_0(Z(\mathbb{A})GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ **cusp automorphic forms**

producing a decomposition:

$$\mathcal{A}_0(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})) = \bigoplus_{\pi} m(\pi)V_\pi$$

the (infinite dimensional) factors are the **cuspidal automorphic representations**

The decomposition of $GL_n(\mathbb{A})$ as a restricted product corresponds to a decomposition of the representations

$$\pi \simeq \otimes'_v \pi_v = (\otimes'_p \pi_p) \otimes \pi_\infty$$

(π_v, V_{π_v}) = irreducible (admissible) representations of $GL_n(\mathbb{Q}_v)$

Main relations

$$\pi_\infty \longrightarrow L(\pi_\infty, s) \longleftrightarrow \Gamma(s)$$

$$\pi_p \longrightarrow L(\pi_p, s) = Q_p(p^{-s})^{-1}$$

$$\pi \longrightarrow \Lambda(\pi, s) = \prod_p L(\pi_p, s)L(\pi_\infty, s) = L(\pi, s)L(\pi_\infty, s)$$

for $Re(s) >> 0$

[Jacquet,P-S,Shalika] $L(\pi, s) := \prod_p L(\pi_p, s)$ is entire and satisfies a functional equation

$$\Lambda(\pi, s) = \epsilon(\pi, s)\Lambda(\tilde{\pi}, 1-s)$$

[Cogdell, P-S] A Converse theorem holds

$$\Rightarrow$$

- "nice" degree n automorphic L-functions are modular, *i.e.* they are associated to a cuspidal automorphic representation π of $GL_n(\mathbb{A})$

The theory or Artin L-functions $L(\rho, s)$ associated to degree n representations ρ of $G_{\mathbb{Q}} := Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ (and their conjectural theory) has suggested

<u>Langlands' Conjecture (1967)</u>

$$\{\rho : G_{\mathbb{Q}} \to GL_n(\mathbb{C})\} \subset \{\pi | \text{autom.rep. of } GL_n(\mathbb{A})\}$$

$$\text{s.t.} \quad L(\rho, s) = L(\pi, s)$$

modularity of Galois representations

there is a local version of this conjecture

In fact the local version (now a theorem!) can be stated very precisely, modulo replacing the local Galois group $G_{\mathbb{Q}_v}$ by the (local) Weil and Deligne groups

$$G_{\mathbb{Q}_v} \rightsquigarrow W_{\mathbb{Q}_v},\ W'_{\mathbb{Q}_v}$$

[Harris-Taylor, Henniart 1996-98] there is a 1-1 correspondence satisfying certain natural compatibilities (e.g. compatibility with local functional equations and preservation of L and epsilon factors of pairs)

$$\{\rho_v : W'_{\mathbb{Q}_v} \to GL_n(\mathbb{C}) : \text{admissible}\} \leftrightarrow$$

$$\leftrightarrow \{\pi \ : \ \text{irred.admiss rep of } GL_n(\mathbb{Q}_v)\}$$

Conclusion: local Galois representations are modular!

Global Modularity?

There is a global version of the Weil group $W_{\mathbb{Q}}$ but there is no definition for a global Weil-Deligne group (the conjectural Langlands group)
At the moment there is a conjectural re-interpretation of it: an "avatar" of this global modularity:

Global (local) functoriality...

BIBLIOGRAPHY

- J. Neukirch   Algebraic Number Theory,   Springer

- Manin, Panchishkin   Introduction to Modern Number Theory,   Springer