HNN Extensions of von Neumann Algebras

Yoshimichi UEDA
Graduate School of Mathematics
Kyushu University

1. HNN Extensions in Group Theory

HNN extensions were first appeared in:

H-N-N's Thm. (1949)
$\forall$ countable discrete group G has
a two generators group $\widetilde{G}$ with with $G \hookrightarrow \widetilde{G}$.

H-N-N = G.Higman, H.Neumann & B.Neumann

Def. **(HNN Extensions)**

Given data:

- a countable group G;
- a subgroup $H(\subseteq G)$;
- an injective grp-homo. $\theta : H \to G$.

Then,

$$G^* = \left\langle G, t : t\theta(h)t^{-1} = h \ (h \in H) \right\rangle,$$

is called the **HNN extension** with **base group** G and **stable letter** t, and denoted by $G \bigstar_H \theta$.

Typical Examples:

- Baumslag-Solitar Group BS(n, m),
- Richard Thompson's Group F,

  etc...

A Group Theoretic Realization:

(cf. Serre's book [Tree, Springer])

$$\begin{aligned}\widehat{G} &:= \cdots \underset{H}{{}_{\mathrm{id}}\bigstar_{\theta}} \overset{-1\mathrm{th}}{G} \underset{H}{{}_{\mathrm{id}}\bigstar_{\theta}} \overset{0\mathrm{th}}{G} \underset{H}{{}_{\mathrm{id}}\bigstar_{\theta}} \overset{1\mathrm{th}}{G} \underset{H}{{}_{\mathrm{id}}\bigstar_{\theta}} \cdots \\ &= \left\langle \underset{\mathbb{Z}}{\bigstar} G \ : \ \iota_n(h) = \iota_{n+1}\left(\theta(h)\right), \ (h \in H) \right\rangle,\end{aligned}$$

with ι_n = the embedding map of G onto nth free component of $\underset{\mathbb{Z}}{\bigstar} G$ and

$$\underset{H}{{}_{\mathrm{id}}\bigstar_{\theta}} \quad = \quad \left[\begin{tikzcd} G & \bigstar & G \\ & H \arrow[ul, "\mathrm{id}"] \arrow[ur, "\theta"'] & \end{tikzcd}\right].$$

$\widehat{\theta}$:= Right Shift on $\widehat{G}$ i.e., $\iota_n(g) \mapsto \iota_{n+1}(g)$.

Then,

$$G \bigstar_H \theta = \langle \iota_0(G), t \rangle = \widehat{G} \rtimes_{\widehat{\theta}} \mathbb{Z}$$

with t = the generator of $\mathbb{Z} \left(\subseteq \widehat{G} \rtimes_{\widehat{\theta}} \mathbb{Z} \right)$.

NB ! $\widehat{\theta}$ is well-defined.

2. Reduced HNN Extensions

Given Data:

- v.N.algebra N;
- v.N.subalgebra $D\,(\subseteq N)$;
- family Θ of injective normal $*$-homo.s

$$\theta : D \to N \ (\theta \in \Theta);$$

- $\exists$ f.n.condi.exp.s

$$E_D^N : N \to D,$$
$$E_{\theta(D)}^N : N \to \theta(D) \ (\theta \in \Theta).$$

Def. (Reduced HNN Extension)

$$\left(M, E_N^M\right) := \left(N, E_D^N\right) \underset{D}{\bigstar} \left(\Theta, \left\{E_{\theta(D)}^N\right\}_{\theta\in\Theta}\right)$$

is a pair of

- v.N.algebra $M = W^*\left\langle N, u(\theta) \ (\theta \in \Theta)\right\rangle$;
- f.n.condi.exp. $E_N^M : M \to N$

with conditions (A), (M).

Condition (A) = Algebraic Property:

$$u(\theta) \cdot \theta(d) \cdot u(\theta)^* = d, \quad \forall d \in D,\ \forall \theta \in \boldsymbol{\Theta},$$

or other words,

$$\theta = \mathsf{Ad}u(\theta)^*\Big|_D : D \to \theta(D) \subseteq N.$$

Call $u(\theta)$ the **stable unitary** associated with $\theta \in \boldsymbol{\Theta}$.

Condition (M) = Property of Moments:

$$E_N^M(w) = 0.$$

for ∀ reduced word w in N and the $u(\theta)$'s.

Reduced Words:

$$u\left(\theta_0\right)^{\varepsilon_0}\cdot n_1 u\left(\theta_1\right)^{\varepsilon_1}\cdot n_2\cdots n_\ell\cdot u\left(\theta_\ell\right)^{\varepsilon_\ell}$$

with $\varepsilon_j \in \{\cdot, *\}$ is **reduced form** if

$$\begin{aligned}&\Big(\varepsilon_{j-1}=\cdot,\ \varepsilon_j=*\Big)\Rightarrow\ \left(n_j\in \mathsf{Ker}E^N_{\theta_j(D)}\right);\\&\Big(\varepsilon_{j-1}=*,\ \varepsilon_j=\cdot\Big)\Rightarrow\ \Big(n_j\in \mathsf{Ker}E^N_D\Big)\end{aligned}$$

whenever $\theta_{j-1}=\theta_j$.

Remark: Words starting and ending at $u(\theta)$'s are enough to formulate the property thanks to the bimodule property of E^M_N !

Example:

$$u\left(\theta\right)^{*}\cdot n_{1}\cdot u\left(\theta\right)\cdot n_{2}u\left(\theta'\right)\cdot n_{3}\cdot u\left(\theta'\right)^{*}$$

is of reduced form

$$\Longleftrightarrow\begin{cases}n_{1}\in N^{\circ}:=\mathsf{Ker}E_{D}^{N},\\ n_{3}\in N_{\theta'}^{\circ}:=\mathsf{Ker}E_{\theta'(D)}^{N}.\end{cases}$$

(no need to specify the position of n_2 !)

Voiculecu's theory of freeness suggests that **a "reduced word thm" can be formulated as some kind of "independence"**,
and more precisely

Freeness (with amalgamation)

$$\Updownarrow$$

Reduced Word Thm

for Amalgamated Free Products.

Analogously,

Condition (M)

$$\Updownarrow$$

Reduced Word Thm (Briton's lemma)

for HNN Extensions.

3. Construction

Assume $\Theta = \{\theta\}$.

- $\Delta_2 :=$ the diagonals in $M_2(\mathbf{C})$.
- The inclusion map

$$\iota_1 : D \otimes \Delta_2 \hookrightarrow N \otimes M_2(\mathbf{C});$$
$$\begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \mapsto \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix}.$$

- The f.n.condi.exp.:

$$E_1 : N \otimes M_2(\mathbf{C}) \to D \otimes \Delta_2;$$
$$E_1 := \begin{bmatrix} E_D^N & \\ & E_D^N \end{bmatrix}.$$

- The distinguished embedding map:

$$\iota_{\Theta} : D \otimes \Delta_2 \hookrightarrow N \otimes M_2(\mathbf{C});$$
$$\begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \mapsto \begin{bmatrix} d_1 & \\ & \theta(d_2) \end{bmatrix}.$$

- The f.n.condi.exp.:

$$E_{\Theta} : N \otimes M_2(\mathbf{C}) \to D \otimes \Delta_2;$$
$$E_{\Theta} := \begin{bmatrix} E_D^N & \\ & E_{\theta(D)}^N \end{bmatrix}.$$

Denote

$$N^{(2)} := N \otimes M_2(\mathbf{C})$$
$$\Delta_2^D := D \otimes \Delta_2$$

Consider

$$(\mathcal{N}, \mathcal{E}) = \left(N^{(2)}, E_\ominus : \iota_\ominus\right) \underset{\Delta_2^D}{\bigstar} \left(N^{(2)}, E_1 : \iota_1\right).$$

$\lambda_\ominus :=$ Embedding of $N^{(2)}$ onto 1st free-part;
$\lambda_1 :=$ Embedding $N^{(2)}$ onto 2nd free-part;
$\lambda :=$ Embedding of Δ_2^D into $\mathcal{N}$.

Fact.

$$\begin{aligned}\lambda_{\ominus}\left(\begin{bmatrix} d_1 & \\ & \theta\left(d_2\right)\end{bmatrix}\right) &= \lambda_{\ominus} \circ \iota_{\ominus}\left(\begin{bmatrix} d_1 & \\ & d_2\end{bmatrix}\right) \\ &= \lambda\left(\begin{bmatrix} d_1 & \\ & d_2\end{bmatrix}\right) \\ &= \lambda_1 \circ \iota_1\left(\begin{bmatrix} d_1 & \\ & d_2\end{bmatrix}\right) \\ &= \lambda_1\left(\begin{bmatrix} d_1 & \\ & d_2\end{bmatrix}\right).\end{aligned}$$

- $p := \lambda\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right) = \lambda_{\ominus}\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right)$.

- $\pi(n) := \lambda_{\ominus}\left(\begin{bmatrix} n & \\ & 0 \end{bmatrix}\right)$, $\forall n \in N$

$$\rightsquigarrow \ \pi : N \to p\left(\lambda_{\ominus}\left(N^{(2)}\right)\right)p \subseteq p\mathcal{N}p.$$

- $u(\theta) := \lambda_1\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\lambda_{\ominus}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)$

$$\rightsquigarrow \ \begin{cases} \text{(i) a unitary in } p\mathcal{N}p, \\ \text{(ii) a reduced word of length 2.} \end{cases}$$

Why does Condition (A) hold ?

$$
\begin{aligned}
& u(\theta) \cdot \pi\left(\theta(d)\right) \cdot u(\theta)^* \\
&= \lambda_1\left(\begin{bmatrix} & 1 \\ & \end{bmatrix}\right) \lambda_\ominus\left(\begin{bmatrix} & \\ 1 & \end{bmatrix} \begin{bmatrix} \theta(d) & \\ & \end{bmatrix} \begin{bmatrix} & 1 \\ & \end{bmatrix}\right) \lambda_1\left(\begin{bmatrix} & \\ 1 & \end{bmatrix}\right) \\
&= \lambda_1\left(\begin{bmatrix} & 1 \\ & \end{bmatrix}\right) \lambda_\ominus\left(\begin{bmatrix} & \\ & \theta(d) \end{bmatrix}\right) \lambda_1\left(\begin{bmatrix} & \\ 1 & \end{bmatrix}\right) \\
&= \lambda_1\left(\begin{bmatrix} & 1 \\ & \end{bmatrix}\right) \lambda_1\left(\begin{bmatrix} & \\ & d \end{bmatrix}\right) \lambda_1\left(\begin{bmatrix} & \\ 1 & \end{bmatrix}\right) \\
&= \lambda_1\left(\begin{bmatrix} d & \\ & \end{bmatrix}\right) \\
&= \lambda_\ominus\left(\begin{bmatrix} d & \\ & \end{bmatrix}\right) \\
&= \pi(d).
\end{aligned}
$$

- $M := \pi(N) \vee \{u(\theta)\}$.
- $E^{M}_{\pi(N)}$:= the restriction:

$$\left(\text{The Cond.Exp.: } \mathcal{N} \to \lambda_{\Theta}\left(N^{(2)}\right)\right)\Big|_{M},$$

giving a condi.exp.: $M \to \pi(N)$ since

$$p \in \lambda_{\Theta}\left(N^{(2)}\right), \quad p\lambda_{\Theta}\left(N^{(2)}\right)p = \pi(N).$$

$$\rightsquigarrow \left(M, E^{M}_{\pi(N)}\right) = \left(N, E^{N}_{D}\right) \underset{D}{\bigstar} \left(\theta, E^{N}_{\theta(D)}\right).$$

Condition (M) follows from that

$u(\theta)$ is a (special) reduced word in $\mathcal{N}$ of length 2.

4. Modular Theory

Thm. $\forall$ f.n.s.weight ψ on D,

$$\sigma_t^{\psi\circ E_D^N \circ E_N^M}(u(\theta)) \\ = u(\theta)\left[D\psi\circ\theta^{-1}\circ E_{\theta(D)}^N : D\psi\circ E_D^N\right]_t .$$

Cor. If τ is a trace on N s.t.

$$\tau\circ E_D^N = \tau, \quad \tau\circ E_{\theta(D)}^N = \tau, \\ \left(\tau\Big|_{\theta(D)}\right) = \left(\tau\Big|_D\right)\circ\theta,$$

then $\tau\circ E_N^M$ becomes again a trace.

Continuous Cores:

$$M \overset{E_N^M}{\supseteq} N \overset{E_D^N}{\supseteq} D \rightsquigarrow \widetilde{M} \overset{\widehat{E_N^M}}{\supseteq} \widetilde{N} \overset{\widehat{E_D^N}}{\supseteq} \widetilde{D}.$$

$$\theta : D \to N \rightsquigarrow \tilde{\theta} : \widetilde{D} \to \widetilde{N},$$

$$E^N_{\theta(D)} : N \to \theta(D) \rightsquigarrow \widehat{E^N_{\theta(D)}} : \widetilde{D} \to \theta\widetilde{(D)} = \tilde{\theta}\left(\widetilde{D}\right).$$

Thm.

$$\left(\widetilde{M}, \widehat{E_N^M}\right) \cong \left(\widetilde{N}, \widehat{E_D^N}\right) \mathop{\bigstar}_{\widetilde{D}} \left(\widetilde{\Theta}, \left\{\widehat{E^N_{\theta(D)}}\right\}_{\theta\in\Theta}\right),$$

with $\widetilde{\Theta} := \left\{\tilde{\theta} : \theta \in \Theta\right\}$.

5. Factoriality

Let

$$\left(M, E_N^M\right) := \left(N, E_D^N\right) \underset{D}{\bigstar} \left(\theta, E_{\theta(D)}^N\right).$$

Prop.

Suppose

- $\exists$ f.n.states φ, φ_θ on D;
- $\exists$ unitaries $v \in N_{\varphi\circ E_D^N}$, $v_\theta \in N_{\varphi_\theta\circ\theta^{-1}\circ E_{\theta(D)}^N}$

s.t.

$$E_D^N\left(v^n\right) = E_{\theta(D)}^N\left(v^n\right) = 0, \quad E_{\theta(D)}^N\left(v_\theta^n\right) = 0$$

as long as $n \neq 0$.

Then,

$$\{v, v_\theta\}' \cap M^\omega \subseteq N^\omega,$$

and in particular,

$$\{v, v_\theta\}' \cap M \subseteq N.$$

Thm.

Suppose

- $\exists$ f.n.states $\varphi,\ \varphi_\theta$ on D;
- $\exists$ unitaries $v \in N_{\varphi\circ E_D^N},\ v_\theta \in N_{\varphi_\theta\circ\theta^{-1}\circ E_{\theta(D)}^N}$

s.t.

$$E_D^N\left(v^n\right) = E_{\theta(D)}^N\left(v^n\right) = 0, \quad E_{\theta(D)}^N\left(v_\theta^n\right) = 0$$

as long as $n \neq 0$.

Then,

$$\mathcal{Z}(M) = \left\{x \in D \cap \theta(D) \cap N' : \theta(x) = x\right\};$$
$$\mathcal{Z}\left(\widetilde{M}\right) = \left\{x \in \widetilde{D} \cap \tilde{\theta}\left(\widetilde{D}\right) \cap \widetilde{N}' : \tilde{\theta}(x) = x\right\};$$
$$M' \cap M^\omega = \left\{x \in D^\omega \cap \theta^\omega\left(D^\omega\right) \cap N' : \theta^\omega(x) = x\right\}.$$

6. Analog of H-N-N's Thm

Prop.
$\forall$ finite v.N.algebra P has a type II_1 factor $\widetilde{P}$ such that

- it is full;
- it is generated by two **Haar** unitaries;
- $P \hookrightarrow \widetilde{P}$.

7. Ascending HNN Etensions

This part is part of a work still in progress.

Given Data:

- v.N.algebra N;
- endomorphism $\rho : N \to N$;
- f.n.condi.exp.

$$E_\rho : N \to \rho(N).$$

Def. (Ascending HNN Extensions)

$$\left(M, E_N^M\right) = (N, \mathrm{id}_N) \mathop{\bigstar}_{N} (\rho, E_\rho)$$

is called an ascending HNN extension.

For $M \overset{E}{\supseteq} N$, the dual op.-valued weight

$$\widehat{E} : \langle M, N \rangle \to M$$

is defined in such a way that

$$\widehat{E}\,(e_N) = 1$$

with the Jones proj e_N.

<u>Def.</u> (Relative Følner-type Condition)
$\forall \varepsilon > 0$, $\forall$ finite subset $\mathcal{F}\,(\subseteq \mathcal{U}(M))$,
$\exists$ proj. $f \in \langle M, N \rangle$ s.t.

- $\widehat{E}(f) \in M$,
- $\|ufu^* - f\|_{L^2(\langle M,N \rangle)} < \varepsilon \cdot \|f\|_{L^2(\langle M,N \rangle)}$
  for $\forall u \in \mathcal{F}$.

Prop. Let

$$\left(M, E_N^M\right) = (N, \mathrm{id}_N) \underset{N}{\bigstar} (\rho, E_\rho)$$

be an ascending HNN extension.
Suppose $\exists$ f.n.state φ on N s.t.

$$\varphi = \varphi \circ \rho^{-1} \circ E_\rho.$$

Then, $M \supseteq N$ satisfies relative Følner-type condition.

- $\widehat{\rho}$:= the (free) Bernoulli shift auto. on

$$\left(\widehat{N}, \widehat{\varphi}\right) := \bigotimes_{\mathbb{Z}} (P, \varphi_P) \text{ or } \mathop{\bigstar}_{\mathbb{Z}} (P, \varphi_P)\,.$$

- ρ := the restriction of $\widehat{\rho}$ to

$$(N, \varphi) := \bigotimes_{\mathbb{N}\sqcup\{0\}} (P, \varphi_P) \text{ or } \mathop{\bigstar}_{\mathbb{N}\sqcup\{0\}} (P, \varphi_P)\,.$$

- E_ρ := the $\widehat{\varphi}$-condi.exp. : $\widehat{N} \to N$.

Prop. Let

$$\left(M, E_N^M\right) = (N, \mathrm{id}_N) \mathop{\bigstar}_{N} (\rho, E_\rho)\,.$$

Then,

$$\left(M,\ \varphi \circ E_N^M\right) \cong \left(\widehat{N} \rtimes_{\widehat{\rho}} \mathbb{Z},\ \widehat{\varphi} \circ E_{\widehat{N}}^{\widehat{N} \rtimes_{\widehat{\rho}} \mathbb{Z}}\right).$$

Monod and Popa's type example:

$$\begin{array}{c}\mathbb{F}_2 = \mathbb{F}(\mathbb{Z}) \rtimes \mathbb{Z} \\ \cup \\ \mathbb{F}(\mathbb{Z}) \\ \cup \\ \mathbb{F}(\mathbb{Z}_{\geq 0}),\end{array}$$

$$\mathbb{F}(\mathbb{Z}) \rtimes \mathbb{Z} = \mathrm{HNN}\left(\mathbb{F}(\mathbb{Z}_{\geq 0}), \text{restricted free shift}\right).$$

Pestov's one:

$$\begin{array}{c}\mathbb{F}_2 = \mathbb{F}(\mathbb{Z}) \rtimes \mathbb{Z} \\ \cup \\ \mathbb{F}(\mathbb{Z}) = \Gamma_0 \rtimes \mathbb{F}(\mathbb{Z}_{\leq -1}) \\ \cup \\ \Gamma_0 := \bigvee_{g \in \mathbb{F}(\mathbb{Z}_{\leq -1})} g\mathbb{F}\left(\mathbb{Z}_{\geq 0}\right) g^{-1} \\ \cup \\ \mathbb{F}\left(\mathbb{Z}_{\geq 0}\right).\end{array}$$