Brown measures of sets of commuting operators in a II_1-factor.

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1 Idempotents in $\tilde{\mathcal{M}}$.

Consider a II_1-factor $\mathcal{M}$ with faithful, tracial state τ. Let $\tilde{\mathcal{M}}$ denote the set of closed, densely defined operators affiliated with $\mathcal{M}$, and let $\mathcal{I}(\tilde{\mathcal{M}})$ denote the set of idempotents in $\tilde{\mathcal{M}}$, i.e.

$$\mathcal{I}(\tilde{\mathcal{M}}) = \{e \in \tilde{\mathcal{M}} \,|\, e^2 = e\}. \tag{1.1}$$

Recall from [Ne] that with

$$\mathrm{N}(\varepsilon, \delta) := \{x \in \tilde{\mathcal{M}} \,|\, \exists P \in P(\mathcal{M}) : \|xP\| < \varepsilon,\ \tau(P^\perp) < \delta\}, \quad (\varepsilon, \delta > 0), \tag{1.2}$$

the family $(\mathrm{N}(\varepsilon, \delta))_{\varepsilon,\delta>0}$ defines a translation-invariant topology on $\tilde{\mathcal{M}}$ (the *measure topology*), in which the $\mathrm{N}(\varepsilon, \delta)$ form a fundamental system of neighbourhoods of 0. $\tilde{\mathcal{M}}$ is actually the completion of $\mathcal{M}$ w.r.t. this topology. Also recall that closed, densely defined operators x and y affiliated with $\mathcal{M}$ represent the same element in $\tilde{\mathcal{M}}$, if and only if they agree on a dense subspace of $\mathcal{H}$.

In this section we shall study the set $\mathcal{I}(\tilde{\mathcal{M}})$ in some detail. We shall repeatedly (and without further mention) make use of the following theorem (cf. the appendix of [Aa] for a (published) proof):

1.1 Theorem. *[A] Let $\mathcal{A}$ be a finite von Neumann algebra represented on the Hilbert space $\mathcal{H}$, and let E and F be (not necessarily closed) subspaces of $\mathcal{H}$ which are affiliated with $\mathcal{A}$.*[1] *Then $E \cap F$ is affiliated with $\mathcal{A}$, and*

$$\overline{E \cap F} = \overline{E} \cap \overline{F}. \tag{1.3}$$

The following proposition shows that $\mathcal{I}(\tilde{\mathcal{M}})$ is in one-to-one correspondence with the set of pairs of projections $P, Q \in \mathcal{M}$ such that $P \wedge Q = 0$ and $P \vee Q = \mathbf{1}$.

1.2 Proposition. *Let $e \in \mathcal{I}(\tilde{\mathcal{M}})$. Then the range of e, $\mathrm{range}(e)$, and the kernel of e, $\ker(e)$ (which is the range of $\mathbf{1} - e$), are closed subspaces of $\mathcal{H}$, with*

$$\mathrm{range}(e) \cap \ker(e) = \{0\} \tag{1.4}$$

and

$$\overline{\mathrm{range}(e) + \ker(e)} = \mathcal{H}. \tag{1.5}$$

[1] A subspace E of $\mathcal{H}$ is said to be *affiliated with* $\mathcal{A}$ if for all $x \in \mathcal{A}'$, $x(E) \subseteq E$. Note that if E is affiliated with $\mathcal{A}$, then the projection onto $\overline{E}$ belongs to $\mathcal{A}$, and if E is closed, then this is a necessary and sufficient condition for E to be affiliated with $\mathcal{A}$.

Conversely, if P, $Q \in \mathcal{M}$ are projections with $P \wedge Q = 0$ and $P \vee Q = \mathbf{1}$, then there is a unique idempotent $e \in \tilde{\mathcal{M}}$ with $\mathrm{range}(e) = P(\mathcal{H})$ *and* $\ker(e) = Q(\mathcal{H})$, *and e is given by*

$$e\xi = \begin{cases} \xi, & \xi \in P(\mathcal{H}) \\ 0, & \xi \in Q(\mathcal{H}) \end{cases} \tag{1.6}$$

Proof. According to [KR, Exercise 2.8.44], when $x \in \tilde{\mathcal{M}}$, then $\ker(x)$ is a closed subspace of $\mathcal{H}$. Hence, $\mathrm{range}(e) = \ker(\mathbf{1} - e)$ is closed. The rest of the proof is standard, and we leave it to the reader. ∎

1.3 Definition. We define $\mathrm{tr} : \mathcal{I}(\tilde{\mathcal{M}}) \to [0, 1]$ by

$$\mathrm{tr}(e) = \tau(\mathrm{supp}(e)), \qquad (e \in \mathcal{I}(\tilde{\mathcal{M}})), \tag{1.7}$$

where $\mathrm{supp}(e) \in P(\mathcal{M})$ denotes the support projection of e.

1.4 Remark. For arbitrary projections $P, Q \in \mathcal{M}$,

$$P \vee Q - P \sim Q - P \wedge Q. \tag{1.8}$$

Hence, Proposition 1.2 implies that for every $e \in \mathcal{I}(\tilde{\mathcal{M}})$,

$$\mathrm{supp}(e) = \mathbf{1} - P_{\ker(e)} \sim P_{\mathrm{range}(e)}, \tag{1.9}$$

where $P_{\ker(e)}$ and $P_{\mathrm{range}(e)}$ denote the projections onto $\ker(e)$ and $\mathrm{range}(e)$, respectively. Hence

$$\mathrm{tr}(e) = \tau(P_{\mathrm{range}(e)}). \tag{1.10}$$

1.5 Proposition. *Let $e_1, \ldots, e_n \in \mathcal{I}(\tilde{\mathcal{M}})$ with $e_i e_j = 0$ when $i \neq j$. Then $e_1 + \cdots + e_n \in \mathcal{I}(\tilde{\mathcal{M}})$, and*

- *(a)* $\ker(e_1 + \cdots + e_n) = \bigcap_{i=1}^n \ker(e_i)$,
- *(b)* $\mathrm{supp}(e_1 + \cdots + e_n) = \bigvee_{i=1}^n \mathrm{supp}(e_i)$,
- *(c)* $\mathrm{tr}(e_1 + \cdots + e_n) = \sum_{i=1}^n \mathrm{tr}(e_i)$.

Proof. It suffices to consider the case $n = 2$. The general case follows by induction over $n \in \mathbb{N}$. If $e_1, e_2 \in \mathcal{I}(\tilde{\mathcal{M}})$ with $e_1 e_2 = e_2 e_1 = 0$, then clearly, $e_1 + e_2 \in \mathcal{I}(\tilde{\mathcal{M}})$. Let $P_i = \mathrm{supp}(e_i)$, $(i = 1, 2)$.

(a) Clearly, $\ker(e_1) \cap \ker(e_2) \subseteq \ker(e_1 + e_2)$. On the other hand, if $\xi \in \ker(e_1 + e_2)$, then $e_i\xi = e_i(e_1 + e_2)\xi = 0$, $(i = 1, 2)$, whence $\ker(e_1 + e_2) \subseteq \ker(e_1) \cap \ker(e_2)$.

(b) Since $\mathrm{supp}(e_1 + e_2) = [\ker(e_1 + e_2)]^\perp$, (b) follows from (a).

(c) $P_1 \wedge P_2 = 0$ because of $e_1 e_2 = e_2 e_1 = 0$. Then, since

$$P_1 \vee P_2 - P_1 \sim P_2 - P_1 \wedge P_2,$$

we get by application of (b) that

$$\begin{aligned} \mathrm{tr}(e_1 + e_2) &= \tau(P_1 \vee P_2) \\ &= \tau(P_1) + \tau(P_2) - \tau(P_1 \wedge P_2) \\ &= \mathrm{tr}(e_1) + \mathrm{tr}(e_2). \end{aligned}$$ ∎

1.6 Proposition. *Let $(e_n)_{n=1}^{\infty}$ be a sequence of idempotents in $\tilde{\mathcal{M}}$ with $e_n e_m = e_m e_n = 0$ when $n \neq m$. Then there is an idempotent in $\tilde{\mathcal{M}}$, which we denote by $\sum_{n=1}^{\infty} e_n$, such that $\sum_{n=1}^{N} e_n \to \sum_{n=1}^{\infty} e_n$ in measure as $N \to \infty$. Moreover, $\operatorname{supp}(\sum_{n=1}^{N} e_n) \nearrow \operatorname{supp}(\sum_{n=1}^{\infty} e_n)$, whence*

$$\operatorname{supp}\Big(\sum_{n=1}^{\infty} e_n\Big) = \bigvee_{n=1}^{\infty} \operatorname{supp}(e_n), \tag{1.11}$$

and

$$\operatorname{tr}\Big(\sum_{n=1}^{\infty} e_n\Big) = \sum_{n=1}^{\infty} \operatorname{tr}(e_n). \tag{1.12}$$

Proof. Let

$$f_n = \sum_{k=1}^{n} e_k, \qquad (n \in \mathbb{N}). \tag{1.13}$$

Then, according to Proposition 1.5, $\operatorname{supp}(f_n) = \bigvee_{k=1}^{n} \operatorname{supp}(e_k)$, and $\operatorname{tr}(f_n) = \sum_{k=1}^{n} \operatorname{tr}(e_k)$. We prove that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\tilde{\mathcal{M}}$. For $n, k \in \mathbb{N}$, let

$$P_{n,k} = \operatorname{supp}(f_{n+k} - f_n) = \bigvee_{l=1}^{k} \operatorname{supp}(e_{n+l}). \tag{1.14}$$

Then

$$(f_{n+k} - f_n) P_{n,k}^{\perp} = 0, \tag{1.15}$$

so for every $\varepsilon > 0$,

$$f_{n+k} - f_n \in \mathrm{N}(\varepsilon, \tau(P_{n,k})) = \mathrm{N}\Big(\varepsilon, \sum_{l=1}^{k} \operatorname{tr}(e_{n+l})\Big). \tag{1.16}$$

Now, $\sum_{k=1}^{n} \operatorname{tr}(e_k) = \operatorname{tr}(f_n) \leq 1$, so for arbitrary $\delta > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$\sum_{k=n_0}^{\infty} \operatorname{tr}(e_k) \leq \delta. \tag{1.17}$$

It follows from (1.16) and (1.17) that when $n \geq n_0$ and $k \geq 1$, then $f_{n+k} - f_n \in \mathrm{N}(\varepsilon, \delta)$. Thus, $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\tilde{\mathcal{M}}$. Put

$$e = \lim_{n \to \infty} f_n \in \tilde{\mathcal{M}}. \tag{1.18}$$

Then for every $n \in \mathbb{N}$,

$$e f_n = f_n e = f_n, \tag{1.19}$$

and therefore, $e^2 = e$.

Let $P_n = \operatorname{supp}(f_n) = \bigvee_{k=1}^{n} \operatorname{supp}(e_k)$. Then $P_n \nearrow P := \bigvee_{k=1}^{\infty} \operatorname{supp}(e_k)$ as $n \to \infty$, and

$$\begin{aligned} \sum_{k=1}^{\infty} \operatorname{tr}(e_k) &= \lim_{n \to \infty} \sum_{k=1}^{n} \operatorname{tr}(e_k) \\ &= \lim_{n \to \infty} \operatorname{tr}(f_n) \\ &= \lim_{n \to \infty} \tau(P_n) \\ &= \operatorname{tr}(P). \end{aligned}$$

It follows from (1.19) that for every $n \in \mathbb{N}$, $P_n \leq \operatorname{supp}(e)$. Hence $P \leq \operatorname{supp}(e)$. On the other hand, for every $n \in \mathbb{N}$, $f_n(\mathbf{1} - P_n) = 0$, so

$$e(\mathbf{1} - P) = \lim_{n\to\infty} [f_n(\mathbf{1} - P_n)] = 0 \tag{1.20}$$

(the limit refers to the measure topology). Thus, $\operatorname{supp}(e) \leq P$, and we have proved that $\operatorname{supp}(e) = P = \bigvee_{k=1}^{\infty} \operatorname{supp}(e_k)$. ■

1.7 Lemma. *Consider idempotents $e, f \in \tilde{\mathcal{M}}$. Let $P = \operatorname{supp}(e)$, $Q = \operatorname{supp}(\mathbf{1}-e)$, $R = \operatorname{supp}(f)$ and $S = \operatorname{supp}(\mathbf{1} - f)$. Then $ef = fe$ in $\tilde{\mathcal{M}}$ if and only if*

$$(P \wedge R) \vee (P \wedge S) \vee (Q \wedge R) \vee (Q \wedge S) = \mathbf{1}. \tag{1.21}$$

Proof. If $ef = fe$, then

$$\mathbf{1} = ef + e(\mathbf{1} - f) + (\mathbf{1} - e)f + (\mathbf{1} - e)(\mathbf{1} - f). \tag{1.22}$$

Let $g_1 = ef$, $g_2 = e(\mathbf{1}-f)$, $g_3 = (\mathbf{1}-e)f$ and $g_4 = (\mathbf{1}-e)(\mathbf{1}-f)$. Then $g_1, \dots, g_4$ are idempotents with ranges $E_1, \dots, E_4$, respectively, such that

$$\overline{E_1 + E_2 + E_3 + E_4} = \mathcal{H}. \tag{1.23}$$

Moreover, $P \wedge R = P_{E_1}$, $P \wedge S = P_{E_2}$, $Q \wedge R = P_{E_3}$, and $Q \wedge S = P_{E_4}$. Hence (1.21) holds.

On the other hand, assume that (1.21) holds. Let $\mathcal{H}_0 = \mathcal{D}(ef) \cap \mathcal{D}(fe)$. Then for every $\xi \in \mathcal{H}_0 \cap (P \wedge R)(\mathcal{H})$,

$$ef\xi = e\xi = \xi = f\xi = fe\xi, \tag{1.24}$$

and similarly, when $\xi \in \mathcal{H}_0 \cap (P \wedge S)(\mathcal{H})$, $\xi \in \mathcal{H}_0 \cap (Q \wedge R)(\mathcal{H})$ or $\xi \in \mathcal{H}_0 \cap (Q \wedge S)(\mathcal{H})$. Thus, ef agrees with fe on $\mathcal{H}_0 \cap [(P \wedge R)(\mathcal{H}) + (P \wedge S)(\mathcal{H}) + (Q \wedge R)(\mathcal{H}) + (Q \wedge S)(\mathcal{H})]$ which is dense in $\mathcal{H}$, and therefore $ef = fe$ in $\tilde{\mathcal{M}}$. ■

Inspired by the notion of a spectral measure we make the following defintion:

1.8 Definition. Let $(X, \mathcal{F})$ denote a measurable space. An *idempotent valued measure* on $(X, \mathcal{F})$ is a map e from $\mathcal{F}$ into $\mathcal{I}(\tilde{\mathcal{M}})$ such that

(i) $e(X) = \mathbf{1}$,

(ii) $e(F_1)e(F_2) = e(F_2)e(F_1) = 0$ when $F_1, F_2 \in \mathcal{F}$ with $F_1 \cap F_2 = \emptyset$,

(iii) when $(F_n)_{n=1}^{\infty}$ is a sequence of mutually disjoint sets from $\mathcal{F}$, then $\sum_{n=1}^{N} e(F_n)$ converges in measure as $N \to \infty$ to $e\Big(\bigcup_{n=1}^{\infty} F_n\Big)$, i.e.

$$e\Big(\bigcup_{n=1}^{\infty} F_n\Big) = \sum_{n=1}^{\infty} e(F_n).$$

Note that because of (ii) and Proposition 1.6, the limit in (iii) actually exists.

2 An idempotent valued measure associated with $T \in \mathcal{M}$.

As in the previous section, consider a II_1-factor $\mathcal{M}$ with faithful, tracial state τ. Recall from [HS] that for $T \in \mathcal{M}$ and $B \subseteq \mathbb{C}$ a Borel set there is a maximal T-invariant projection $P = P_T(B) \in \mathcal{M}$, such that the Brown measure of PTP (considered as an element of $P\mathcal{M}P$) is concentrated on B. Moreover, $P_T(B)$ is hyperinvariant for T,

$$\tau(P_T(B)) = \mu_T(B), \tag{2.1}$$

and the Brown measure of $P^\perp TP^\perp$ (considered as an element of $P^\perp \mathcal{M} P^\perp$) is concentrated on B^c. Let $\mathcal{K}_T(B)$ denote the range of $P_T(B)$. The aim of this section is to prove:

2.1 Theorem. *Let $T \in \mathcal{M}$, and for $B \in \mathbb{B}(\mathbb{C})$, let $e_T(B)$ be given by*

$$e_T(B)\xi = \begin{cases} \xi, & \xi \in \mathcal{K}_T(B) \\ 0, & \xi \in \mathcal{K}_T(B^c) \end{cases} \tag{2.2}$$

Then $e_T(B) \in \mathcal{I}(\tilde{\mathcal{M}})$, and $B \mapsto e_T(B)$ is an idempotent valued measure.

The proof of this theorem uses various results which we state and prove below. The first one of these is a lemma which we proved in [HS], but for the sake of completeness we give the proof here as well.

2.2 Lemma. *Let $T \in \mathcal{M}$, and let $P \in \mathcal{M}$ be a T-invariant projection. Then for every $B \in \mathbb{B}(\mathbb{C})$,*

$$\mathcal{K}_{T|_{P(\mathcal{H})}}(B) = \mathcal{K}_T(B) \cap P(\mathcal{H}). \tag{2.3}$$

Proof. Let $Q \in P\mathcal{M}P$ denote the projection onto $\mathcal{K}_{T|_{P(\mathcal{H})}}(B)$, and let $R = P_T(B) \wedge P$. We will prove that $Q \leq R$ and $R \leq Q$.

Clearly, $Q \leq P$. In order to see that $Q \leq P_T(B)$, recall that $P_T(B)$ is maximal w.r.t. the properties

(i) $P_T(B)TP_T(B) = TP_T(B)$,

(ii) $\mu_{P_T(B)TP_T(B)}$ (computed relative to $P_T(B)\mathcal{M}P_T(B)$) is concentrated on B.

Since

$$QTQ = QTPQ = TPQ = TQ, \tag{2.4}$$

and $\mu_{QTQ} = \mu_{QTPQ}$ (computed relative to $Q\mathcal{M}Q$) is concentrated on B, we get that $Q \leq P_T(B)$, and hence $Q \leq R$.

Similarly, to prove that $R \leq Q$, prove that

(i') $RTPR = TPR$, i.e. $RTR = TR$,

(ii') $\mu_{RTPR} = \mu_{RTR}$ (computed relative to $R\mathcal{M}R$) is concentrated on B.

Note that if $P_T(B) = 0$, then $R \leq Q$, so we may assume that $P_T(B) \neq 0$. (i') holds, because $R(\mathcal{H}) = P(\mathcal{H}) \cap P_T(B)(\mathcal{H})$ is T-invariant when $P(\mathcal{H})$ and $P_T(B)(\mathcal{H})$ are T-invariant. In order to prove (ii'), at first note that $R(\mathcal{H})$ is $TP_T(B)$-invariant. Hence

$$\mu_{TP_T(B)} = \tau_1(R) \cdot \mu_{RTR} + \tau_1(R^\perp) \cdot \mu_{R^\perp TR^\perp}, \tag{2.5}$$

where $\tau_1 = \frac{1}{\tau(P_T(B))} \cdot \tau|_{P_T(B)\mathcal{M}P_T(B)}$. It follows that

$$\tau_1(R) \cdot \mu_{RTR}(B^c) \leq \mu_{TP_T(B)}(B^c) = 0, \tag{2.6}$$

and thus, if $R \neq 0$, then $\mu_{RTR}(B^c) = 0$, and (ii') holds. If $R = 0$, then $R \leq Q$ is trivially fulfilled. ■

2.3 Proposition. *For every Borel set $B \subseteq \mathbb{C}$,*

$$\mathcal{K}_T(B) = \mathcal{K}_{T^*}((B^c)^*)^\perp, \tag{2.7}$$

where $A^* := \{\overline{z} \mid z \in A\}$ for $A \subseteq \mathbb{C}$. Moreover, for all Borel sets $A, B \subseteq \mathbb{C}$,

$$\mathcal{K}_T(A) \cap \mathcal{K}_T(B) = \mathcal{K}_T(A \cap B), \tag{2.8}$$

and

$$\mathcal{K}_T(A \cup B) = \overline{\mathcal{K}_T(A) + \mathcal{K}_T(B)}. \tag{2.9}$$

Proof. Let $B \in \mathbb{B}(\mathbb{C})$, and let $P = P_T(B)$. Then $P^\perp$ is T^*-invariant, and

$$\mu_{P^\perp T^* P^\perp}(B^*) = \mu_{(P^\perp T^* P^\perp)^*}(B) = \mu_{P^\perp TP^\perp}(B) = 0 \tag{2.10}$$

(recall that $\mu_{P^\perp TP^\perp}$ is concentrated on B^c). Thus, $\mu_{P^\perp T^* P^\perp}$ is concentrated on $\mathbb{C} \setminus B^*$, and maximality of $P_{T^*}(\mathbb{C} \setminus B^*)$ implies that

$$P_T(B)^\perp = P^\perp \leq P_{T^*}(\mathbb{C} \setminus B^*). \tag{2.11}$$

Since

$$\begin{aligned}
\tau(P_{T^*}(\mathbb{C} \setminus B^*)) &= \mu_{T^*}(\mathbb{C} \setminus B^*) \\
&= 1 - \mu_{T^*}(B^*) \\
&= 1 - \mu_T(B) \\
&= \tau(P_T(B)^\perp),
\end{aligned}$$

we get from (2.11) that $P_T(B)^\perp = P_{T^*}(\mathbb{C} \setminus B^*)$.

Next, let $A, B \in \mathbb{B}(\mathbb{C})$. By maximality of $P_T(A)$ and $P_T(B)$, $P_T(A \cap B) \leq P_T(A) \wedge P_T(B)$, so $\supseteq$ holds in (2.8). We let $\mathcal{K} := \mathcal{K}_T(A) \cap \mathcal{K}_T(B)$, and we let Q denote the projection onto $\mathcal{K}$. Then, according to Lemma 2.2,

$$\begin{aligned}
\mathcal{K} &= \mathcal{K}_{T|_{\mathcal{K}_T(A)}}(B) \\
&= \mathcal{K}_{T|_{\mathcal{K}_T(B)}}(A),
\end{aligned}$$

proving that μ_{QTQ} is concentrated on A and on B, hence on $A \cap B$. Consequently, $Q \leq P_T(A \cap B)$, so $\subseteq$ also holds in (2.8).

Finally, we infer from (2.7) and (2.8) that

$$\begin{aligned}
\mathcal{K}_T(A\cup B) &= \mathcal{K}_T((A^c\cap B^c)^c)\\
&= \mathcal{K}_{T^*}((A^c\cap B^c)^*)^{\perp}\\
&= \mathcal{K}_{T^*}((A^c)^*\cap (B^c)^*)^{\perp}\\
&= [\mathcal{K}_{T^*}((A^c)^*)\cap \mathcal{K}_{T^*}((B^c)^*)]^{\perp}\\
&= \overline{\mathcal{K}_{T^*}((A^c)^*)^{\perp}+\mathcal{K}_{T^*}((B^c)^*)^{\perp}}\\
&= \overline{\mathcal{K}_T(A)+\mathcal{K}_T(B)}. \qquad \blacksquare
\end{aligned}$$

It follows from Proposition 2.3 that for $B\in\mathbb{B}(\mathbb{C})$, $e_T(B)$ given by (2.2) belongs to $\mathcal{I}(\tilde{\mathcal{M}})$, as stated in Theorem 2.1.

2.4 Lemma. *Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\tilde{\mathcal{M}}$, and suppose $\tau(\mathrm{supp}(x_n))\to 0$ as $n\to\infty$. Then $x_n\to 0$ in the measure topology.*

Proof. This is standard. $\blacksquare$

Proof of Theorem 2.1. $e_T(\emptyset)=0$, because $P_T(\emptyset)=0$. If B_1, $B_2\in\mathbb{B}(\mathbb{C})$ with $B_1\cap B_2=\emptyset$, then $P_T(B_1)\cap P_T(B_2)=0$, and hence $e_T(B_1)e_T(B_2)=e_T(B_2)e_T(B_1)=0$.

Now, let $(B_n)_{n=1}^{\infty}$ be a sequence of mutually disjoint Borel sets. Then for each $N\in\mathbb{N}$ we get from Proposition 2.3 and Lemma 1.5 that

$$\begin{aligned}
\mathrm{supp}\Big(e_T\Big(\bigcup_{n=1}^{N}B_n\Big)\Big) &= \mathcal{K}_T\Big(\bigcup_{n=1}^{N}B_n\Big)\\
&= \overline{\mathcal{K}_T(B_1)+\cdots+\mathcal{K}_T(B_N)}\\
&= \mathrm{supp}(e_T(B_1)+\cdots+e_T(B_N))
\end{aligned}$$

and

$$\begin{aligned}
\ker\Big(e_T\Big(\bigcup_{n=1}^{N}B_n\Big)\Big) &= \mathcal{K}_T\Big(\Big(\bigcup_{n=1}^{N}B_n\Big)^c\Big)\\
&= \mathcal{K}_T\Big(\bigcap_{n=1}^{N}B_n^c\Big)\\
&= \bigcap_{n=1}^{N}\mathcal{K}_T(B_n^c)\\
&= \bigcap_{n=1}^{N}\ker(e_T(B_n))\\
&= \ker(e_T(B_1)+\cdots+e_T(B_N)).
\end{aligned}$$

Hence e_T is additive, i.e.

$$e_T\Big(\bigcup_{n=1}^{N}B_n\Big)=e_T(B_1)+\cdots+e_T(B_N),\qquad (N\in\mathbb{N}). \tag{2.12}$$

It follows from additivity of e_T that

$$
\begin{aligned}
e_T\Big(\bigcup_{n=1}^{\infty} B_n\Big) - \sum_{n=1}^{\infty} e_T(B_n) &= \lim_{N\to\infty} \Big(e_T\Big(\bigcup_{n=1}^{\infty} B_n\Big) - \sum_{n=1}^{N} e_T(B_n))\Big) \\
&= \lim_{N\to\infty} e_T\Big(\bigcup_{n=N+1}^{\infty} B_n\Big) \qquad (2.13)
\end{aligned}
$$

(the limits refer to the measure topology), where

$$
\begin{aligned}
\tau\Big(\operatorname{supp}\Big(e_T\Big(\bigcup_{n=N+1}^{\infty} B_n\Big)\Big)\Big) &= \tau\Big(P_T\Big(\bigcup_{n=N+1}^{\infty} B_n\Big)\Big) \\
&= \mu_T\Big(\bigcup_{n=N+1}^{\infty} B_n\Big) \\
&\to 0, \quad \text{as } N \to \infty.
\end{aligned}
$$

Combining this with (2.13) and Lemma 2.4, we find that e_T is σ-additive as well. ■

Note that in the case where T is a *normal* operator, $B \mapsto P_T(B)$ is just the spectral measure of T, and $e_T(B) = P_T(B)$.

3 The Brown measure of a set of commuting operators in $\mathcal{M}$.

3.1 Theorem. *Let $n \in \mathbb{N}$, and let $T_1, \dots, T_n \in \mathcal{M}$ be commuting operators. Then there is a probability measure $\mu_{T_1,\dots,T_n}$ on $(\mathbb{C}^n, \mathbb{B}(\mathbb{C}^n))$, which is uniquely determined by*

$$
\mu_{T_1,\dots,T_n}(B_1 \times \cdots \times B_n) = \tau\Big(\bigwedge_{i=1}^{n} P_{T_i}(B_i)\Big), \qquad (B_1, \dots, B_n \in \mathbb{B}(\mathbb{C})), \tag{3.1}
$$

where $P_{T_i}(B_i) \in \mathcal{M}$ is the projection onto $\mathcal{K}_{T_i}(B_i)$ (cf. Section 2).

The idea of proof is as follows:

If $S \in \mathcal{M}$ commutes with $T \in \mathcal{M}$, then for every $B \in \mathbb{B}(\mathbb{C})$, $\mathcal{K}_T(B)$ and $\mathcal{K}_T(B^c)$ are S-invariant, and therefore S commutes with $e_T(B)$ as well. We prove that, as a consequence of this, $[e_S(A), e_T(B)] = 0$ for every $A \in \mathbb{B}(\mathbb{C})$, and we may therefore define a map $e_{T_1,\dots,T_n}$ from $\mathbb{B}(\mathbb{C})^n$ into $\mathcal{I}(\tilde{\mathcal{M}})$ by

$$
e_{T_1,\dots,T_n}(B_1, \dots, B_n) = e_{T_1}(B_1)e_{T_2}(B_2)\cdots e_{T_n}(B_n), \qquad (B_1, \dots, B_n \in \mathbb{B}(\mathbb{C})). \tag{3.2}
$$

We will then define ν on $\mathbb{B}(\mathbb{C})^n$ by

$$
\nu(B_1, \dots, B_n) = \tau(\operatorname{supp}[e_{T_1,\dots,T_n}(B_1, \dots, B_n)]) = \tau\Big(\bigwedge_{i=1}^{n} P_{T_i}(B_i)\Big), \qquad (B_1, \dots, B_n \in \mathbb{B}(\mathbb{C})), \tag{3.3}
$$

and we will prove that ν extends (uniquely) to a probability measure, $\mu_{T_1,\dots,T_n}$, on $(\mathbb{C}^n, \mathbb{B}(\mathbb{C}^n))$.

3.2 Lemma. *Let $T \in \mathcal{M}$, and let $e \in \mathcal{I}(\tilde{\mathcal{M}})$ with $[e,T] = 0$. Then for every $B \in \mathbb{B}(\mathbb{C})$, $[e, e_T(B)] = 0$. In particular, if $S \in \mathcal{M}$ commutes with T, then $[e_S(\cdot), e_T(\cdot)] = 0$.*

Proof. Let $P = \operatorname{supp}(e)$, $Q = \operatorname{supp}(\mathbf{1} - e)$, $R = \operatorname{supp}(e_T(B))$ and $S = \operatorname{supp}(\mathbf{1} - e_T(B)$. We prove that (1.21) holds. Since $eT = Te$, $P(\mathcal{H})$ and $Q(\mathcal{H})$ are T-invariant. Then by Lemma 2.2,

$$\begin{aligned} \mathcal{K}_{T|_{P(\mathcal{H})}}(B) &= \mathcal{K}_T(B) \cap P(\mathcal{H}) \\ &= R(\mathcal{H}) \cap P(\mathcal{H}), \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}_{T|_{P(\mathcal{H})}}(B^c) &= \mathcal{K}_T(B^c) \cap P(\mathcal{H}) \\ &= S(\mathcal{H}) \cap P(\mathcal{H}). \end{aligned}$$

Hence $(R \wedge P) \vee (S \wedge P) = P$, and similarly, $(R \wedge Q) \vee (S \wedge Q) = Q$. It follows that

$$\mathbf{1} = P \vee Q = (R \wedge P) \vee (S \wedge P) \vee (R \wedge Q) \vee (S \wedge Q),$$

as desired. ■

3.3 Theorem. *Consider complete, seperable metric spaces $(X_1, d_1), \ldots, (X_n, d_n)$. Suppose $\nu : \mathbb{B}(X_1) \times \cdots \times \mathbb{B}(X_n) \to [0, \infty[$ is a map satisfying*

(1) for all $B_2 \in \mathbb{B}(X_2), B_3 \in \mathbb{B}(X_3), \ldots, B_n \in \mathbb{B}(X_n)$, $B \mapsto \nu(B, B_2, \ldots, B_n)$ is a measure on $(X_1, \mathbb{B}(X_1))$,

(2) for all , $B_1 \in \mathbb{B}(X_1), B_3 \in \mathbb{B}(X_3), \ldots, B_n \in \mathbb{B}(X_n)$, $B \mapsto \nu(B_1, B, B_3, \ldots, B_n)$ is a measure on $(X_2, \mathbb{B}(X_2))$,

$\vdots$

(n) for all , $B_1 \in \mathbb{B}(X_1), B_2 \in \mathbb{B}(X_2), \ldots, B_{n-1} \in \mathbb{B}(X_{n-1})$, $B \mapsto \nu(B_1, B_2, \ldots, B_{n-1}, B)$ is a measure on $(X_n, \mathbb{B}(X_n))$.

Then there is a unique measure μ on $\otimes_{i=1}^n \mathbb{B}(X_i)$, such that for all $B_1 \in \mathbb{B}(X_1), B_2 \in \mathbb{B}(X_2), \ldots, B_n \in \mathbb{B}(X_n)$,

$$\mu(B_1 \times B_2 \times \cdots \times B_n) = \nu(B_1, B_2, \ldots, B_n). \tag{3.4}$$

Proof. According to [Ku, p. 227], $(X_i, \mathbb{B}(X_i))$ is Borel equivalent to $([0,1], \mathbb{B}([0,1])$, i.e. there is a bijective bimeasurable map $\phi_i : (X_i, \mathbb{B}(X_i)) \to ([0,1], \mathbb{B}([0,1]))$. Therefore we may as well assume that $X_i = \mathbb{R}$, $(i = 1, \ldots, n)$. We may also assume that $\nu(\mathbb{R}, \mathbb{R}, \ldots, \mathbb{R}) = 1$. We define $F : \mathbb{R}^n \to [0,1]$ by

$$F(x_1, \ldots, x_n) = \nu(]-\infty, x_1], \ldots,]-\infty, x_n]), \qquad (x_1, \ldots, x_n \in \mathbb{R}). \tag{3.5}$$

Because of (1)–(n), F satisfies

(a) if $x_i^{(k)} \searrow x_i$, $(i = 1, \ldots, n)$, then $F(x_1^{(k)}, \ldots, x_n^{(k)}) \searrow F(x_1, \ldots, x_n)$,

(b) if $x_i \searrow -\infty$ for some $i \in \{1, \ldots, n\}$, then $F(x_1, \ldots, x_n) \searrow 0$,

(c) if $x_i \nearrow \infty$ for all $i \in \{1, \ldots, n\}$, then $F(x_1, \ldots, x_n) \nearrow 1$.

Then, according to [Br, Corollary 2.27], there is a (unique) probability measure μ on $(\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n))$ such that for all $x_1, \ldots, x_n \in \mathbb{R}$,

$$\mu(]-\infty, x_1] \times \cdots \times]-\infty, x_n]) = F(x_1, \ldots, x_n). \tag{3.6}$$

Let $x_2, \ldots, x_n \in \mathbb{R}$ be fixed but arbitrary. Then the (finite) measures

$$B \mapsto \mu(B \times]-\infty, x_2] \times \cdots \times]-\infty, x_n])$$

and

$$A \mapsto \nu(B,]-\infty, x_2], \ldots,]-\infty, x_n])$$

have the same distribution functions. Hence they must be identical. That is, for all $B \in \mathbb{B}(\mathbb{R})$,

$$\mu(B \times]-\infty, x_2] \times \cdots \times]-\infty, x_n]) = \nu(B,]-\infty, x_2], \ldots,]-\infty, x_n]). \tag{3.7}$$

Now, let $B_1 \in \mathbb{B}(\mathbb{R})$ and $x_3, \ldots, x_n \in \mathbb{R}$ be fixed but arbitrary. Then (3.7) shows that the (finite) measures

$$B \mapsto \mu(B_1 \times B \times]-\infty, x_3] \times \cdots \times]-\infty, x_n])$$

and

$$B \mapsto \nu(B_1, B,]-\infty, x_3], \ldots,]-\infty, x_n])$$

have the same distribution functions, so they must be identical as well. That is, for all $B \in \mathbb{B}(\mathbb{R})$,

$$\mu(B_1 \times B \times]-\infty, x_3] \times \cdots \times]-\infty, x_n]) = \nu(B_1, B,]-\infty, x_3], \ldots,]-\infty, x_n]). \tag{3.8}$$

Continuing like this we find that (3.4) holds. ■

It follows from Theorem 3.3 that in order to show that $\mu_{T_1,\ldots,T_n}$ exists, we must prove that (1)–(n) of Theorem 3.3 hold in the case where $X_1 = \cdots = X_n = \mathbb{C}$, and where ν is given by (3.3).

From now on we will, in order to simplify notation a little, consider the case $n = 2$, and we will assume that $S, T \in \mathcal{M}$ are commuting operators. It should be clear that the proof given below may be generalized to the case of arbitrary $n \in \mathbb{N}$.

3.4 Lemma. *For fixed $A \in \mathbb{B}(\mathbb{C})$, $\nu_A : \mathbb{B}(\mathbb{C}) \to [0, 1]$ given by*

$$\nu_A(B) = \nu(A, B) = \tau(P_S(A) \wedge P_T(B)), \qquad (B \in \mathbb{B}(\mathbb{C})) \tag{3.9}$$

(cf. (3.3)) is a measure on $(\mathbb{C}, \mathbb{B}(\mathbb{C}))$.

Proof. According to Theorem 2.1 and Definition 1.8, $e_T(\emptyset) = 0$, so

$$\nu_A(\emptyset) = \tau(\mathrm{supp}[e_S(A)e_T(\emptyset)]) = 0.$$

Let $(B_n)_{n=1}^{\infty}$ be a sequence of mutually disjoint sets from $\mathbb{B}(\mathbb{C})$. Then $e_T\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} e_T(B_n)$, so

$$e_S(A)e_T\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} e_S(A)e_T(B_n) \tag{3.10}$$

with $e_S(A)e_T(B_n)e_S(A)e_T(B_m) = e_S(A)e_T(B_n)e_T(B_m) = 0$ when $n \neq m$. Hence by Proposition 1.6,

$$\mathrm{tr}\Big(e_S(A)e_T\Big(\bigcup_{n=1}^{\infty} B_n\Big)\Big) = \sum_{n=1}^{\infty} \mathrm{tr}(e_S(A)e_T(B_n)). \tag{3.11}$$

This shows that ν_A is a measure. ■

It now follows from Lemma 3.4 and Theorem 3.3 that there is one and only one (probability) measure $\mu_{S,T}$ on $\mathbb{B}(\mathbb{C}^2)$ such that for all A, $B \in \mathbb{C}(\mathbb{B})$,

$$\mu_{S,T}(A \times B) = \tau(\mathrm{supp}[e_{S,T}(A,B)]) = \tau(P_S(A) \wedge P_T(B)), \tag{3.12}$$

and this proves Theorem 3.1 in the case $n = 2$.

4 Spectral subspaces for commuting operators $S, T \in \mathcal{M}$.

4.1 Theorem. *Let $S, T \in \mathcal{M}$ be commuting operators, and let $B \subseteq \mathbb{C}^2$ be any Borel set. Then there is a maximal, closed, S- and T-invariant subspace $\mathcal{K} = \mathcal{K}_{S,T}(B)$ affiliated with $\mathcal{M}$, such that the Brown measure $\mu_{S|_{\mathcal{K}},T|_{\mathcal{K}}}$ is concentrated on B. Let $P_{S,T}(B) \in \mathcal{M}$ denote the projection onto $\mathcal{K}_{S,T}(B)$. Then more precisely,*

(i) if $B = B_1 \times B_2$ with $B_1, B_2 \in \mathbb{B}(\mathbb{C})$, then

$$P_{S,T}(B) = P_S(B_1) \wedge P_T(B_2), \tag{4.1}$$

(ii) if B is a disjoint union of rectangles $(B_1^{(k)} \times B_2^{(k)})_{k=1}^{\infty}$, where $B_i^{(k)} \in \mathbb{B}(\mathbb{C})$, $k \in \mathbb{N}$, $i = 1, 2$, then

$$P_{S,T}(B) = \bigvee_{k=1}^{\infty} [P_S(B_1^{(k)}) \wedge P_T(B_2^{(k)})], \tag{4.2}$$

(iii) and for general $B \in \mathbb{B}(\mathbb{C}^2)$,

$$P_{S,T}(B) = \bigwedge_{B \subseteq U,\ U \subseteq \mathbb{C}^2 \text{ open}} P_{S,T}(U). \tag{4.3}$$

Moreover,

$$\mu_{S,T}(B) = \tau(P_{S,T}(B)), \quad (B \in \mathbb{B}(\mathbb{C}^2)). \tag{4.4}$$

4.2 Remark. Every non-empty, open subset of $\mathbb{C}^2 \cong \mathbb{R}^4$ is a disjoint union of countably many *standard intervals*, i.e. sets of the form $\prod_{i=1}^{4}]a_i, b_i]$, where $-\infty < a_i < b_i < \infty$, $1 \leq i \leq 4$. Hence, it follows from Theorem 4.1 that the map $\mathbb{B}(\mathbb{C}^2) \to P(\mathcal{M}) : B \mapsto P_{S,T}(B)$ is uniquely determined by its values on such standard intervals.

Proof of Theorem 4.1. We let $\mathbb{K}$ denote the set of rectangles $B_1 \times B_2$ with $B_1, B_2 \in \mathbb{B}(\mathbb{C})$.

Consider an arbitrary sequence of mutually disjoint sets from $\mathbb{K}$, $(B_1^{(k)} \times B_2^{(k)})_{k=1}^{\infty}$, and define

$$P_{S,T}\Big(\bigcup_{k=1}^{\infty} (B_1^{(k)} \times B_2^{(k)})\Big) := \bigvee_{k=1}^{\infty} [P_S(B_1^{(k)}) \wedge P_T(B_2^{(k)})]. \tag{4.5}$$

Clearly, $P := P_{S,T}\Big(\bigcup_{k=1}^{\infty}(B_1^{(k)} \times B_2^{(k)})\Big)$ satisfies that

(a) P is S- and T-invariant.

In addition, we prove that with $\mathcal{K} = P(\mathcal{H})$,

(b) $\mu_{S|_\mathcal{K},T|_\mathcal{K}}$ is concentrated on $B := \bigcup_{k=1}^{\infty}(B_1^{(k)} \times B_2^{(k)})$,

and

(c) P is maximal w.r.t. the properties (a) and (b).

(c) will entail that the right-hand side of (4.5) is independent of the way in which we write B as a disjoint union of countably many rectangles, and hence, that $P_{S,T}(B)$ *does*, as indicated by the notation, only depend on the set B.

To see that (b) holds, note that if $Q \in \mathcal{M}$ is *any* S- and T-invariant projection, and if we let $\mathcal{L} = Q(\mathcal{H})$, then by (3.1) and Lemma 2.2,

$$\begin{aligned}
\mu_{S|_\mathcal{L},T|_\mathcal{L}}(B) &= \sum_{k=1}^{\infty} \mu_{S|_\mathcal{L},T|_\mathcal{L}}(B_1^{(k)} \times B_2^{(k)}) \\
&= \sum_{k=1}^{\infty} \tau_{Q\mathcal{M}Q}(P_{S|_\mathcal{L}}(B_1^{(k)}) \wedge P_{T|_\mathcal{L}}(B_2^{(k)})) \\
&= \sum_{k=1}^{\infty} \tau_{Q\mathcal{M}Q}(P_S(B_1^{(k)}) \wedge P_T(B_2^{(k)}) \wedge Q).
\end{aligned} \tag{4.6}$$

Then, using Proposition 1.6, we get that

$$\begin{aligned}
\mu_{S|_\mathcal{K},T|_\mathcal{K}}(B) &= \sum_{k=1}^{\infty} \tau_{P\mathcal{M}P}(P_S(B_1^{(k)}) \wedge P_T(B_2^{(k)}) \wedge P) \\
&= \sum_{k=1}^{\infty} \tau_{P\mathcal{M}P}(P_S(B_1^{(k)}) \wedge P_T(B_2^{(k)})) \\
&= \frac{1}{\tau(P)} \sum_{k=1}^{\infty} \mathrm{tr}(e_S(B_1^{(k)}) e_T(B_2^{(k)})) \\
&= \frac{1}{\tau(P)} \mathrm{tr}\Big(\sum_{k=1}^{\infty} e_S(B_1^{(k)}) e_T(B_2^{(k)})\Big) \\
&= \frac{1}{\tau(P)} \tau\Big(\bigvee_{k=1}^{\infty} [P_S(B_1^{(k)}) \wedge P_T(B_2^{(k)})]\Big) \\
&= 1.
\end{aligned}$$

Thus, (b) holds.

Now, suppose that $Q \in \mathcal{M}$ is an S- and T-invariant projection, and that $\mu_{S|_{\mathcal{L}},T|_{\mathcal{L}}}$ is concentrated on B, where $\mathcal{L} = Q(\mathcal{H})$. Then by Lemma 2.2 and Proposition 1.6,

$$\begin{aligned} P \wedge Q &= \Big(\bigvee_{k=1}^{\infty} [P_S(B_1^{(k)}) \wedge P_T(B_2^{(k)})] \Big) \wedge Q \\ &= \bigvee_{k=1}^{\infty} [P_{S|_{\mathcal{L}}}(B_1^{(k)}) \wedge P_{T|_{\mathcal{L}}}(B_2^{(k)})] \\ &= \operatorname{supp}\Big(\sum_{k=1}^{\infty} e_{S|_{\mathcal{L}}}(B_1^{(k)}) e_{T|_{\mathcal{L}}}(B_2^{(k)}) \Big). \end{aligned}$$

Hence, Proposition 1.6 and (4.6) imply that

$$\begin{aligned} \tau_{Q\mathcal{M}Q}(P \wedge Q) &= \operatorname{tr}_{Q\mathcal{M}Q}\Big(\sum_{k=1}^{\infty} e_{S|_{\mathcal{L}}}(B_1^{(k)}) e_{T|_{\mathcal{L}}}(B_2^{(k)}) \Big) \\ &= \sum_{k=1}^{\infty} \operatorname{tr}_{Q\mathcal{M}Q}(e_{S|_{\mathcal{L}}}(B_1^{(k)}) e_{T|_{\mathcal{L}}}(B_2^{(k)})) \\ &= \sum_{k=1}^{\infty} \tau_{Q\mathcal{M}Q}(P_{S|_{\mathcal{L}}}(B_1^{(k)}) \wedge P_{T|_{\mathcal{L}}}(B_2^{(k)})) \\ &= \mu_{S|_{\mathcal{L}},T|_{\mathcal{L}}}(B) \\ &= 1. \end{aligned}$$

Thus, $P \wedge Q = Q$, and this shows that (c) holds.

As mentioned in Remark 4.2, every open set $U \subseteq \mathbb{C}^2$ may be written as a union of countably many mutually disjoint sets from $\mathbb{K}$. Thus, we have now proved existence of $P_{S,T}(U)$ for every such U, and for general $B \in \mathbb{B}(\mathbb{C}^2)$ we will define

$$P_{S,T}(B) := \bigwedge_{B \subseteq U,\ U \subseteq \mathbb{C}^2 \text{ open}} P_{S,T}(U). \tag{4.7}$$

Then again, $P := P_{S,T}(B)$ satisfies that

(a) P is S- and T-invariant.

Moreover, we prove that with $\mathcal{K} = P(\mathcal{H})$,

(b) $\mu_{S|_{\mathcal{K}},T|_{\mathcal{K}}}$ is concentrated on B,

and

(c) P is maximal w.r.t. the properties (a) and (b).

These propterties will entail that when B happens to be a union of countably many mutually disjoint sets from $\mathbb{K}$, then (4.7) agrees with the previous definition of $P_{S,T}(B)$ (cf. (4.5)).

Now, to see that (b) holds, note that $\mu_{S|_{\mathcal{K}},T|_{\mathcal{K}}}$ is regular (cf. [Fo, Theorem 7.8]), and hence

$$\mu_{S|_{\mathcal{K}},T|_{\mathcal{K}}}(B) = \inf\{\mu_{S|_{\mathcal{K}},T|_{\mathcal{K}}}(U) \mid B \subseteq U,\ U \subseteq \mathbb{C}^2 \text{ open}\}. \tag{4.8}$$

Let U be any open subset of $\mathbb{C}^2$ containing B. Write U as a union of countably many mutually disjoint sets from $\mathbb{K}$:

$$U = \bigcup_{k=1}^{\infty} (B_1^{(k)} \times B_2^{(k)}).$$

Then, according to (4.6),

$$\mu_{S|_{\mathcal{K}},T|_{\mathcal{K}}}(U) = \sum_{k=1}^{\infty} \tau_{P\mathcal{M}P}(P_S(B_1^{(k)}) \wedge P_T(B_2^{(k)}) \wedge P),$$

and using Proposition 1.6 and Lemma 2.2 we find that

$$\begin{aligned}
\mu_{S|_{\mathcal{K}},T|_{\mathcal{K}}}(U) &= \mathrm{tr}_{P\mathcal{M}P}\Big(\sum_{k=1}^{\infty} e_{S|_{\mathcal{K}}}(B_1^{(k)}) e_{T|_{\mathcal{K}}}(B_2^{(k)})\Big) \\
&= \tau_{P\mathcal{M}P}\Big(\bigvee_{k=1}^{\infty} [P_{S|_{\mathcal{K}}}(B_1^{(k)}) \wedge P_{T|_{\mathcal{K}}}(B_2^{(k)})]\Big) \\
&= \tau_{P\mathcal{M}P}(P_{S,T}(U) \wedge P) \\
&= \tau_{P\mathcal{M}P}(P) \\
&= 1,
\end{aligned}$$

where $P_{S,T}(U)$ is given by (4.5). Hence by (4.8), $\mu_{S|_{\mathcal{K}},T|_{\mathcal{K}}}$ is concentrated on B.

Finally, if $Q \in \mathcal{M}$ is any S- and T-invariant projection, and if $\mu_{S|_{\mathcal{L}},T|_{\mathcal{L}}}$ is concentrated on B, where $\mathcal{L} = Q(\mathcal{H})$, then $\mu_{S|_{\mathcal{L}},T|_{\mathcal{L}}}$ is concentrated on U for every open set U containing B. Hence, by the first part of the proof, $Q \leq P_{S,T}(U)$ for every such U, and it follows from the definition of $P_{S,T}(B)$ that $Q \leq P$.

Concerning (4.4), note that if $B = B_1 \times B_2$, where $B_1, B_2 \in \mathbb{B}(\mathbb{C})$, then, by the definitions of $\mu_{S,T}$ and $P_{S,T}(B)$, (4.4) holds. If B is a disjoint union of rectangles $(B^{(k)})_{k=1}^{\infty} = (B_1^{(k)} \times B_2^{(k)})_{k=1}^{\infty}$, where $B_i^{(k)} \in \mathbb{B}(\mathbb{C})$, $k \in \mathbb{N}$, $i = 1, 2$, then

$$\begin{aligned}
\mu_{S,T}(B) &= \sum_{k=1}^{\infty} \tau(P_{S,T}(B_1^{(k)} \times B_2^{(k)})) \\
&= \sum_{k=1}^{\infty} \tau(\mathrm{supp}(e_S(B_1^{(k)}) e_T(B_2^{(k)}))).
\end{aligned}$$

Applying Proposition 1.6 we thus find that

$$\begin{aligned}
\mu_{S,T}(B) &= \tau\Big(\mathrm{supp}\Big(\sum_{k=1}^{\infty} e_S(B_1^{(k)}) e_T(B_2^{(k)})\Big)\Big) \\
&= \tau\Big(\bigvee_{k=1}^{\infty} \mathrm{supp}(e_S(B_1^{(k)}) e_T(B_2^{(k)}))\Big) \\
&= \tau\Big(\bigvee_{k=1}^{\infty} P_{S,T}(B_1^{(k)} \times B_2^{(k)})\Big) \\
&= \tau(P_{S,T}(B)).
\end{aligned}$$

Finally, for general $B \in \mathbb{B}(\mathbb{C}^2)$, since $\mu_{S,T}$ is regular,

$$\begin{aligned}
\mu_{S,T}(B) &= \inf\{\mu_{S,T}(\mathcal{U}) \,|\, B \subseteq \mathcal{U} \subseteq \mathbb{C}^2,\, \mathcal{U}\,\text{open}\} \\
&= \inf\{\tau(P_{S,T}(\mathcal{U})) \,|\, B \subseteq \mathcal{U} \subseteq \mathbb{C}^2,\, \mathcal{U}\,\text{open}\} \\
&= \tau\Big(\bigwedge_{B\subseteq\mathcal{U}\subseteq\mathbb{C}^2,\,\mathcal{U}\,\text{open}} P_{S,T}(\mathcal{U})\Big) \\
&= \tau(P_{S,T}(B)). \qquad \blacksquare
\end{aligned}$$

The proof given above may clearly be generalized to the case of n commuting operators $T_1, \ldots, T_n \in \mathcal{M}$, meaning that Theorem 4.1 has a slightly more general version:

4.3 Theorem. *Let $n \in \mathbb{N}$, let $T_1, \ldots, T_n \in \mathcal{M}$ be commuting operators, and let $B \subseteq \mathbb{C}^n$ be any Borel set. Then there is a maximal closed subspace, $\mathcal{K} = \mathcal{K}_{T_1,\ldots,T_n}(B)$, affiliated with $\mathcal{M}$ which is T_i-invariant for every $i \in \{1, \ldots, n\}$, and such that the Brown measure $\mu_{T_1|_{\mathcal{K}},\ldots,T_n|_{\mathcal{K}}}$ is concentrated on B. Let $P_{T_1,\ldots,T_n}(B) \in \mathcal{M}$ denote the projection onto $\mathcal{K}_{T_1,\ldots,T_n}(B)$. Then more precisely,*

(i) if $B = B_1 \times \cdots \times B_n$ with $B_i \in \mathbb{B}(\mathbb{C})$, then

$$P_{T_1,\ldots,T_n}(B) = \bigwedge_{i=1}^{n} P_{T_i}(B_i), \tag{4.9}$$

(ii) if B is a disjoint union of 'boxes' $(B^{(k)})_{k=1}^{\infty} = (B_1^{(k)} \times \cdots \times B_n^{(k)})_{k=1}^{\infty}$, where $B_i^{(k)} \in \mathbb{B}(\mathbb{C})$, $k \in \mathbb{N}$, $i = 1, \ldots, n$, then

$$P_{T_1,\ldots,T_n}(B) = \bigvee_{k=1}^{\infty} P_{T_1,\ldots,T_n,}(B^{(k)}), \tag{4.10}$$

(iii) and for general $B \in \mathbb{B}(\mathbb{C}^n)$,

$$P_{T_1,\ldots,T_n}(B) = \bigwedge_{B\subseteq U,\ U\subseteq\mathbb{C}^n\ \text{open}} P_{T_1,\ldots,T_n}(U). \tag{4.11}$$

Moreover, for every $B \in \mathbb{B}(\mathbb{C}^n)$,

$$\mu_{T_1,\ldots,T_n}(B) = \tau(P_{T_1,\ldots,T_n}(B)). \tag{4.12}$$

5 An alternative characterization of $\mu_{S,T}$.

In this final section we are going to give a characterization of the Brown measure, $\mu_{S,T}$, of two commuting operators $S, T \in \mathcal{M}$, which is different from the one we gave in Theorem 3.1. Recall from [Br] that for $T \in \mathcal{M}$, the Brown measure, μ_T, is the unique compactly supported Borel probability measure on $\mathbb{C}$ which satisfies the identity

$$\tau(\log|T - \lambda\mathbf{1}|) = \int_{\mathbb{C}} \log|z - \lambda|\,\mathrm{d}\mu_T(z)$$

for almost every $\lambda \in \mathbb{C}$ w.r.t. Lebesgue measure.

We are going to prove that a similar property characterizes $\mu_{S,T}$:

5.1 Theorem. *Let $S, T \in \mathcal{M}$ be commuting operators. Then $\mu_{S,T}$ is the unique compactly supported Borel probability measure on $\mathbb{C}^2$ which satisfies the identity*

$$\tau(\log|\alpha S + \beta T - \lambda \mathbf{1}|) = \int_{\mathbb{C}^2} \log|\alpha z + \beta w - \lambda| \, \mathrm{d}\mu_{S,T}(z, w). \tag{5.1}$$

for all $\alpha, \beta \in \mathbb{C}$ and for almost every $\lambda \in \mathbb{C}$ w.r.t. Lebesgue measure.

5.2 Remark. Let $S, T \in \mathcal{M}$ be as in Theorem 5.1, and for $\alpha, \beta \in \mathbb{C}$, let $\nu_{\alpha,\beta}$ denote the push-forward measure of $\mu_{S,T}$ under the map $(z, w) \mapsto \alpha z + \beta w$. By Brown's characterization of $\mu_{\alpha S + \beta T}$, if (5.1) holds for almost every $\lambda \in \mathbb{C}$ w.r.t. Lebesgue measure, then $\nu_{\alpha,\beta} = \mu_{\alpha S+\beta T}$. On the other hand, if $\nu_{\alpha,\beta} = \mu_{\alpha S+\beta T}$, then (5.1) holds. Thus, (5.1) holds for all $\alpha, \beta \in \mathbb{C}$ and for almost every $\lambda \in \mathbb{C}$ w.r.t. Lebesgue measure iff for all $\alpha, \beta \in \mathbb{C}$, $\mu_{\alpha S+\beta T} = \nu_{\alpha,\beta}$.

In the proof of Theorem 5.1 we shall need the lemma below. Recall from [HS] that for $T \in \mathcal{M}$, the *modified spectral radius* of T, $r'(T)$, is given by

$$r'(T) := \max\{|z| \,|\, z \in \mathrm{supp}(\mu_T)\}.$$

Also recall from [HS, Corollary 2.6] that actually

$$r'(T) = \lim_{p \to \infty} \Big(\lim_{n \to \infty} \|T^n\|_{\frac{p}{n}}^{\frac{1}{n}} \Big). \tag{5.2}$$

5.3 Lemma. *Let $S, T \in \mathcal{M}$ be commuting operators. Then the modified spectral radii, $r'(S), r'(T), r'(ST)$ and $r'(S+T)$ satisfy the inequalities*

$$r'(ST) \le r'(S) \cdot r'(T), \tag{5.3}$$

and

$$r'(S+T) \le r'(S) + r'(T). \tag{5.4}$$

Proof. (5.3) follows from (5.2) and the generalized Hölder inequality (cf. [FK]): For $x, y \in \mathcal{M}$ and for $0 < p, q, r \le \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$,

$$\|xy\|_r \le \|x\|_p \|y\|_q.$$

To prove (5.4), note that

$$\mathrm{supp}(\mu_S) \subseteq \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z \le r'(S)\},$$

and

$$\mathrm{supp}(\mu_T) \subseteq \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z \le r'(T)\}.$$

According to [Br, Theorem 4.1], μ_{e^S} and μ_{e^T} are the push-forward measures of μ_S and μ_T, respectively, under the $z \mapsto \mathrm{e}^z$. Hence,

$$\mathrm{supp}(\mu_{\mathrm{e}^S}) \subseteq \{z \in \mathbb{C} \,|\, |z| \le \mathrm{e}^{r'(S)}\},$$

and

$$\mathrm{supp}(\mu_{\mathrm{e}^T}) \subseteq \{z \in \mathbb{C} \,|\, |z| \le \mathrm{e}^{r'(T)}\},$$

and it follows from (5.3) that

$$\begin{aligned} r'(\mathrm{e}^{S+T}) &= r'(\mathrm{e}^{S}\mathrm{e}^{T}) \\ &\leq \mathrm{e}^{r'(S)}\mathrm{e}^{r'(T)} \\ &= \mathrm{e}^{r'(S)+r'(T)}. \end{aligned}$$

Thus, $\mathrm{supp}(\mu_{\mathrm{e}^{S+T}}) \subseteq \overline{B(0, \mathrm{e}^{r'(S)+r'(T)})}$, and then, by one more application of [Br, Theorem 4.1],

$$\mathrm{supp}(\mu_{S+T}) \subseteq \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z \leq r'(S) + r'(T)\}.$$

Repeating this argument, we find that for arbitrary $\theta \in [0, 2\pi[$,

$$\mathrm{supp}(\mu_{\mathrm{e}^{\mathrm{i}\theta}(S+T)}) \subseteq \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z \leq r'(S) + r'(T)\},$$

i.e.

$$\mathrm{supp}(\mu_{S+T}) \subseteq \{z \in \mathbb{C} \,|\, \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\theta} z) \leq r'(S) + r'(T)\}.$$

We thus conclude that

$$\mathrm{supp}(\mu_{S+T}) \subseteq \overline{B(0, r'(S) + r'(T))},$$

and this proves (5.4). ■

5.4 Lemma. *Let $S, T \in \mathcal{M}$ be commuting operators, and let $\alpha, \beta \in \mathbb{C}$. Then $\mu_{\alpha S, \beta T}$ is the push-forward measure of $\mu_{S,T}$ under the map $h_{\alpha,\beta} : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ given by*

$$h_{\alpha,\beta}(z, w) = (\alpha z, \beta w).$$

Proof. Recall that $\mu_{\alpha S, \beta T}$ is uniquely determined by the property that for all $B_1, B_2 \in \mathbb{B}(\mathbb{C})$,

$$\mu_{\alpha S, \beta T}(B_1 \times B_2) = \tau(P_{\alpha S}(B_1) \wedge P_{\beta T}(B_2)). \tag{5.5}$$

Now, it is easily seen that for $\alpha \neq 0$ and $\beta \neq 0$, $P_{\alpha S}(B_1) = P_S(\frac{1}{\alpha}B_1)$ and $P_{\beta T}(B_2) = P_T(\frac{1}{\beta}B_2)$. Hence,

$$\begin{aligned} \mu_{\alpha S, \beta T}(B_1 \times B_2) &= \tau(P_S(\tfrac{1}{\alpha}B_1) \wedge P_T(\tfrac{1}{\beta}B_2)) \\ &= \mu_{S,T}(\tfrac{1}{\alpha}B_1 \times \tfrac{1}{\beta}B_2) \\ &= \mu_{S,T}(h_{\alpha,\beta}^{-1}(B_1 \times B_2)). \end{aligned} \tag{5.6}$$

If for instance $\alpha = 0$, then $P_{\alpha S}(B_1) = 0$ if $0 \notin B_1$ and $P_{\alpha S}(B_1) = \mathbf{1}$ if $0 \in B_1$. It then follows that (5.6) holds in this case as well. Similar arguments apply if $\beta = 0$. ■

Proof of Theorem 5.1. As noted in Remark 5.2, it suffices to prove that for all $\alpha, \beta \in \mathbb{C}$, $\mu_{\alpha S + \beta T}$ is the push-forward measure of $\mu_{S,T}$ under the map $(z, w) \mapsto \alpha z + \beta w$. At first we will consider the case $\alpha = \beta = 1$. Define $g : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$g(z, w) = z + w, \quad (z, w \in \mathbb{C}).$$

We are going to prove that for all $B \in \mathbb{B}(\mathbb{C})$,

$$\mu_{S+T}(B) = \mu_{S,T}(g^{-1}(B)). \tag{5.7}$$

We shall identify $\mathbb{C}$ with $\mathbb{R}^2$ and $\mathbb{B}(\mathbb{C})$ with $\mathbb{B}(\mathbb{R}^2)$. Note that under these identifications, $\mathbb{B}(\mathbb{C})$ is generated by the $\cap$-stable set

$$\mathbb{I} := \{]-\infty, a]\times]-\infty, b] \,|\, a, b \in \mathbb{R}\}.$$

Thus, it suffices to prove that for all $a, b \in \mathbb{R}$, (5.7) holds with $B =]-\infty, a]\times]-\infty, b]$. At first consider such a set B with $\mu_{S,T}(\partial g^{-1}(B)) = 0$. Note that

$$g^{-1}(B) = \{(z, w) \in \mathbb{C}^2 \,|\, \mathrm{Re}z + \mathrm{Re}w \leq a,\ \mathrm{Im}z + \mathrm{Im}w \leq b\}.$$

For each $n \in \mathbb{N}$ and for all $x_1, \dots, x_4 \in \mathbb{R}$ define

$$I^{(n)}_{x_1,\dots,x_4} = \Big(\big[\tfrac{x_1}{n}, \tfrac{x_1+1}{n}\big[\times \big[\tfrac{x_2}{n}, \tfrac{x_2+1}{n}\big[\Big) \times \Big(\big[\tfrac{x_3}{n}, \tfrac{x_3+1}{n}\big[\times \big[\tfrac{x_4}{n}, \tfrac{x_4+1}{n}\big[\Big) \subset \mathbb{C}\times\mathbb{C}.$$

Let

$$M_n = \{(m_1, \dots, m_4) \in \mathbb{Z}^4 \,|\, m_1 + m_3 \leq n\cdot a - 2,\ m_2 + m_4 \leq n \cdot b - 2\},$$

and note that $(I^{(n)}_{m_1,\dots,m_4})_{(m_1,\dots,m_4)\in M_n}$ is a family of disjoint sets contained in $g^{-1}(B)$ and that as $n \to \infty$, $\bigcup_{(m_1,\dots,m_4)\in M_n} I^{(n)}_{m_1,\dots,m_4} \nearrow g^{-1}(B) \setminus \partial g^{-1}(B)$. By definition, $\mathcal{K}_{S,T}(I^{(n)}_{m_1,\dots,m_4}) := P_{S,T}(I^{(n)}_{m_1,\dots,m_4})(\mathcal{H})$ is given by

$$\mathcal{K}_{S,T}(I^{(n)}_{m_1,\dots,m_4}) = \mathcal{K}_S(I^{(n)}_{m_1,m_2}) \cap \mathcal{K}_T(I^{(n)}_{m_3,m_4}),$$

where

$$I^{(n)}_{x_1,x_2} := \big[\tfrac{x_1}{n}, \tfrac{x_1+1}{n}\big[\times \big[\tfrac{x_2}{n}, \tfrac{x_2+1}{n}\big[, \qquad (x_1, x_2 \in \mathbb{R}).$$

In particular, $\mathcal{K}_{S,T}(I^{(n)}_{m_1,\dots,m_4}) \subseteq \mathcal{K}_S(I^{(n)}_{m_1,m_2})$, and therefore

$$\mathrm{supp}(\mu_{S|_{\mathcal{K}_{S,T}(I^{(n)}_{m_1,\dots,m_4})}}) \subseteq \overline{I^{(n)}_{m_1,m_2}}.$$

That is,

$$\mathrm{supp}(\mu_{(S-\frac{1}{n}(m_1+\frac{1}{2}+\mathrm{i}(m_2+\frac{1}{2}))\mathbf{1})|_{\mathcal{K}_{S,T}(I^{(n)}_{m_1,\dots,m_4})}}) \subseteq \big[-\tfrac{1}{2n}, \tfrac{1}{2n}\big] \times \big[-\tfrac{1}{2n}, \tfrac{1}{2n}\big]. \tag{5.8}$$

Similarly,

$$\mathrm{supp}(\mu_{(T-\frac{1}{n}(m_3+\frac{1}{2}+\mathrm{i}(m_4+\frac{1}{2}))\mathbf{1})|_{\mathcal{K}_{S,T}(I^{(n)}_{m_1,\dots,m_4})}}) \subseteq \big[-\tfrac{1}{2n}, \tfrac{1}{2n}\big] \times \big[-\tfrac{1}{2n}, \tfrac{1}{2n}\big]. \tag{5.9}$$

Then by Lemma 5.3,

$$r'((S + T - \tfrac{1}{n}(m_1 + m_3 + 1 + \mathrm{i}(m_2 + m_4 + 1))\mathbf{1})|_{\mathcal{K}_{S,T}(I^{(n)}_{m_1,\dots,m_4})}) \leq 2 \cdot \tfrac{1}{\sqrt{2}n} = \tfrac{\sqrt{2}}{n},$$

and thus,

$$\begin{aligned}\mathrm{supp}(\mu_{S+T|_{\mathcal{K}_{S,T}(I^{(n)}_{m_1,\dots,m_4})}}) &\subseteq \overline{B\Big(\tfrac{m_1+m_3+1}{n} + \mathrm{i}\tfrac{m_2+m_4+1}{n}, \tfrac{\sqrt{2}}{n}\Big)} \\ &\subseteq \big]-\infty, a + \tfrac{\sqrt{2}-1}{n}\big] \times \big]-\infty, b + \tfrac{\sqrt{2}-1}{n}\big].\end{aligned}$$

This implies that

$$\mathcal{K}_{S,T}(I^{(n)}_{m_1,\dots,m_4}) \subseteq \mathcal{K}_{S+T}\Big(\big]-\infty, a + \tfrac{\sqrt{2}-1}{n}\big] \times \big]-\infty, b + \tfrac{\sqrt{2}-1}{n}\big]\Big),$$

and therefore

$$\begin{aligned}\mathcal{K}_{S,T}\Big(\bigcup_{(m_1,\dots,m_4)\in M_n} I^{(n)}_{m_1,\dots,m_4}\Big) &= \overline{\operatorname{span}_{\mathbb{C}}}\Big\{\mathcal{K}_{S,T}(I^{(n)}_{m_1,\dots,m_4}) \,|\, (m_1,\dots,m_4)\in M_n\Big\} \\ &\subseteq \mathcal{K}_{S+T}\Big(\big]-\infty, a+\tfrac{\sqrt{2}-1}{n}\big]\times\big]-\infty, b+\tfrac{\sqrt{2}-1}{n}\big]\Big),\end{aligned}$$

so by Theorem 4.1

$$\begin{aligned}\mu_{S,T}\Big(\bigcup_{(m_1,\dots,m_4)\in M_n} I^{(n)}_{m_1,\dots,m_4}\Big) &= \tau\Big(P_{S,T}\Big(\bigcup_{(m_1,\dots,m_4)\in M_n} I^{(n)}_{m_1,\dots,m_4}\Big)\Big) \\ &\le \tau\Big(P_{S+T}\Big(\big]-\infty, a+\tfrac{\sqrt{2}-1}{n}\big]\times\big]-\infty, b+\tfrac{\sqrt{2}-1}{n}\big]\Big)\Big) \\ &= \mu_{S+T}\Big(\big]-\infty, a+\tfrac{\sqrt{2}-1}{n}\big]\times\big]-\infty, b+\tfrac{\sqrt{2}-1}{n}\big]\Big).\end{aligned}$$

Since $\bigcup_{(m_1,\dots,m_4)\in M_n} I^{(n)}_{m_1,\dots,m_4} \nearrow g^{-1}(B)\backslash\partial g^{-1}(B)$ and $]-\infty, a+\frac{\sqrt{2}-1}{n}]\times]-\infty, b+\frac{\sqrt{2}-1}{n}] \searrow B$ as $n\to\infty$, we conclude that

$$\mu_{S,T}(g^{-1}(B)) \le \mu_{S+T}(B). \tag{5.10}$$

To prove the reverse inequality, note that B^c is the disjoint union of the sets

$$\begin{aligned}B_1 &= \,]-\infty, a]\times[b,\infty[, \\ B_2 &= \,]a,\infty[\times]b,\infty[, \\ B_3 &= \,]a,\infty[\times]-\infty, b].\end{aligned}$$

Here, $\partial g^{-1}(B_i) = \partial g^{-1}(B)$, $i=1,2,3$, so by minor modification of the above arguments we find that

$$\mu_{S,T}(g^{-1}(B_i)) \le \mu_{S+T}(B_i), \qquad (i=1,2,3). \tag{5.11}$$

It now follows that

$$1 = \mu_{S,T}(g^{-1}(B)) + \mu_{S,T}(g^{-1}(B^c)) \le \mu_{S+T}(B) + \mu_{S+T}(B^c) = 1,$$

so equality must hold in (5.10) and (5.11). Hence, if $\mu_{S,T}(\partial g^{-1}(B)) = 0$, then

$$\mu_{S,T}(g^{-1}(B)) = \mu_{S+T}(B). \tag{5.12}$$

If $\mu_{S,T}(\partial g^{-1}(B)) \neq 0$, then consider the decreasing map $f :]0,\infty[\to[0,1]$ given by

$$f(t) = \mu_{S,T}\Big(g^{-1}\Big(\big]-\infty, a+\tfrac{1}{t}\big]\times\big]-\infty, b+\tfrac{1}{t}\big]\Big)\Big), \qquad (t>0).$$

It has at most countably many points of discontinuity, so we may choose $0<t_1<t_2<\cdots$, such that $t_n\nearrow\infty$, and for all $n\in\mathbb{N}$,

$$\mu_{S,T}\Big(\partial g^{-1}\Big(\big]-\infty, a+\tfrac{1}{t_n}\big]\times\big]-\infty, b+\tfrac{1}{t_n}\big]\Big)\Big) = 0.$$

By application of (5.12) and the fact that $]-\infty, a+\frac{1}{t_n}]\times]-\infty, b+\frac{1}{t_n}] \searrow B$, we then find that (5.12) holds for this B as well.

Now, if $\alpha, \beta \in \mathbb{C}$, then we conclude from the above and Lemma 5.4 that

$$\begin{aligned} \mu_{\alpha S+\beta T}(B) &= \mu_{\alpha S,\beta T}(g^{-1}(B)) \\ &= \mu_{S,T}(h_{\alpha,\beta}^{-1}(g^{-1}(B))) \\ &= \mu_{S,T}((g \circ h_{\alpha,\beta})^{-1}(B)), \end{aligned}$$

and since $g \circ h_{\alpha,\beta}(z,w) = \alpha z + \beta w$, this completes the proof of the identity (5.1).

To prove uniqueness of $\mu_{S,T}$, suppose that ν is a compactly supported Borel probability measure on $\mathbb{C}^2$, which satisfies the identity (5.1) for all $\alpha, \beta \in \mathbb{C}$ and for almost every $\lambda \in \mathbb{C}$ w.r.t Lebesgue measure. That is, for all $\alpha, \beta \in \mathbb{C}$, $\mu_{\alpha S+\beta T}$ is the push-forward measure of ν under the map $(z,w) \mapsto \alpha z + \beta w$. Then, to prove that $\nu = \mu_{S,T}$, it suffices to prove that for all $y = (y_1, \ldots, y_4) \in \mathbb{R}^4$,

$$\int_{\mathbb{R}^4} \mathrm{e}^{\mathrm{i}(y,x)}\, \mathrm{d}\mu_{S,T}(x) = \int_{\mathbb{R}^4} \mathrm{e}^{\mathrm{i}(y,x)}\, \mathrm{d}\nu(x) \tag{5.13}$$

(once more we identify $\mathbb{C}$ with $\mathbb{R}^2$). For $x = (x_1, \ldots, x_4) \in \mathbb{R}^4$ and $y = (y_1, \ldots, y_4) \in \mathbb{R}^4$, note that

$$(y,x) = \mathrm{Re}\Big((y_1 - \mathrm{i}y_2)(x_1 + \mathrm{i}x_2) + (y_3 - \mathrm{i}y_4)(x_3 + \mathrm{i}x_4)\Big),$$

and hence with $\alpha = y_1 - \mathrm{i}y_2$ and $\beta = y_3 - \mathrm{i}y_4$ we find that

$$\begin{aligned} \int_{\mathbb{R}^4} \mathrm{e}^{\mathrm{i}(y,x)}\, \mathrm{d}\mu_{S,T}(x) &= \int_{\mathbb{C}^2} \mathrm{e}^{\mathrm{i}\mathrm{Re}(\alpha z+\beta w)}\, \mathrm{d}\mu_{S,T}(z,w) \\ &= \int_{\mathbb{C}} \mathrm{e}^{\mathrm{i}\mathrm{Re} z}\, \mathrm{d}\mu_{\alpha S+\beta T}(z) \\ &= \int_{\mathbb{C}^2} \mathrm{e}^{\mathrm{i}\mathrm{Re}(\alpha z+\beta w)}\, \mathrm{d}\nu(z,w) \\ &= \int_{\mathbb{R}^4} \mathrm{e}^{\mathrm{i}(y,x)}\, \mathrm{d}\nu(x), \end{aligned}$$

as desired. ∎

As in the previous section, one can easily generalize the proof given above to the case of an arbitrary finite set of commuting operators, $\{T_1, \ldots, T_n\}$. That is, we actually have the following alternative description of $\mu_{T_1,\ldots,T_n}$:

5.5 Theorem. *Let $n \in \mathbb{N}$, and let $T_1, \ldots, T_n$ be mutually commuting operators in $\mathcal{M}$. Then $\mu_{T_1,\ldots,T_n}$ is the unique compactly supported Borel probability measure on $\mathbb{C}^n$ which satisfies the identity*

$$\tau(\log|\alpha_1 T_1 + \ldots \alpha_n T_n - \lambda \mathbf{1}|) = \int_{\mathbb{C}^n} \log|\alpha_1 z_1 + \ldots + \alpha_n z_n - \lambda|\, \mathrm{d}\mu_{T_1,\ldots,T_n}(z_1, \ldots, z_n). \tag{5.14}$$

for all $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and for almost every $\lambda \in \mathbb{C}$ w.r.t. Lebesgue measure.

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