AMALGAMATED FREE PRODUCTS OF w-RIGID FACTORS AND CALCULATION OF THEIR SYMMETRY GROUPS, PART I

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Definition. Let (M, τ) be a II_1 factor and $B \subset M$ a von Neumann subalgebra. $\{\eta_n\}_n \subset M$ is a *bounded, homogeneous, orthonormal basis* (BHOB) of M over B if

(0). $\eta_0 = 1$

(1). $E_B(\eta_i^* \eta_j) = \delta_{ij}, \forall i, j$

(2). $\overline{\Sigma \eta_n B} = L^2(M, \tau)$.

$B \subset M$ is *homogeneous* if there exists a BHOB of M over B.

Lemma. *Let (M, τ) be a II_1 factor and $B \subset M$ a von Neumann subalgebra. Consider the following cases:*

(a). $B = \mathbb{C}1$.

(b). $B = A \subset M$ is Cartan.

(c). $B = N \subset M$ is an irreducible inclusion of II_1 *factors and B is regular in M.*

Then $B \subset M$ is homogeneous.

In both cases (b) and (c), M has a BHOB *made of unitary elements in $\mathcal{N}_M(B)$.*

Definition (S. Popa 2001) Let M be a separable II_1 factor and Q a von Neumann subalgebra, $Q \subset M$ is rigid if $\forall \phi_n : M \to M$ c.p. s.t. $\phi_n(1) \leq 1$, $\tau \circ \phi_n \leq \tau$, $\|\phi_n(x) - x\|_2 \to 0$, $\forall x \in M$, then $\|\phi_n(b) - b\|_2 \to 0$ uniformly for $b \in (Q)_1$.

Theorem. *Let M_0 be a finite von Neumann algebra and (M_i, τ_i), $i = 1, 2$, be II_1 factors, all with a common von Neumann subalgebra $B \subset M_i$, s.t. $\tau_{0|B} = \tau_{1|B} = \tau_{2|B}$, and s.t. $B \subset M_i$ are homogeneous. Denote $M = M_0 *_B M_1 *_B M_2$. Let $Q \subset M$ be a rigid diffuse von Neumann subalgebra. Assume no corner of Q can be embedded into M_0 inside M.*

$1°$. *$\exists!\ q_1, q_2 \in \mathcal{Z}(Q' \cap M)$ s.t. $q_1 + q_2 = 1$ and $u_i(Qq_i)u_i^* \subset M_i$ for some $u_i \in \mathcal{U}(M)$, $i = 1, 2$.*

$2°$. *If $\mathcal{N}_M(Q)''$ is a factor then $\exists!\ i \in \{1, 2\}$ s.t. $uQu^* \subset M_i$ for some $u \in \mathcal{U}(M)$.*

Theorem (N. Ozawa 2004). *Suppose that* $M_1, \dots, M_m$ *and* $N_1, \dots, N_n$ *are semiexact, non-prime, non-injective finite factors and let* M_0*, and* N_0 *be semiexact, semisolid finite factors.*

If $\theta : M \simeq N$ *then* $m = n$ *and, after some permutation of indices,* $\theta(M_i)$ *and* N_i *are unitary conjugate in* N, $\forall i \geq 1$.

Definition. A II_1 factor N is w-rigid if $\exists Q \subset N$ s.t.

(1). Q is diffuse.

(2). $\mathcal{N}_N(Q)'' = N$.

(3). $Q \subset N$ is rigid.

Theorem. *Let* $M_1, \dots, M_m$ *and* $N_1, \dots, N_n$ *be w-rigid* II_1 *factors,* $1 \leq n, m \leq \infty$*. Let* M_0*,* N_0 *have Haagerup's compact approximation property (possibly* $M_0, N_0 = \mathbb{C}$*). Let* $M = *_{i=0}^m M_i$*,* $N = *_{j=0}^n N_j$.

If $\theta : M \simeq N^t$ *then* $m = n$ *and, after some permutation of indices,* $\theta(M_i)$ *and* N_i^t *are unitary conjugate in* N^t, $\forall i \geq 1$.

Corollary. *Let* $m \in \mathbb{N}$ *and let* $M_1, \ldots, M_m$ *be w-rigid* II_1 *factors. Let* M_0 *have Haagerup's compact approximation property. Denote* $M = *_{i=0}^{m} M_i$.

Then $\mathscr{F}(M) \subset \cap_{i=1}^{m} \mathcal{F}(M_i)^{1/m!}$.

If $\mathscr{F}(M_1) = \{1\}$ *(e.g.* $M = L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$*) then* $\mathscr{F}(M) = \{1\}$.

Theorem (K. Dykema, F. Radulescu 2000). *Let* M_i *be* II_1 *factors* $\forall i \in \mathbb{N}$ *and let* $M = *_{i\in\mathbb{N}} M_i$, *then* $\forall t \in \mathbb{R}_+^*$ $M^t \simeq *_{i\in\mathbb{N}} M_i^t$.

Theorem. *Let* M *be a w-rigid* II_1 *factor with* $\mathscr{F}(M) = \{1\}$.

If $S \subset \mathbb{R}_+^*$ *is an arbitrary infinite (possibly uncountable) subgroup then the* II_1 *factor* $M^S = *_{s\in S} M^s$ *has* $\mathscr{F}(M) = S$.

Let R and R_0 be hyperfinite II_1 factors.

Let $\Gamma = \Gamma_0 * \Gamma_1 * \cdots * \Gamma_n$, $\Lambda = \Lambda_0 * \Lambda_1 * \cdots * \Lambda_m$, with Γ_i, Λ_j non-trivial groups.

Let $\sigma : \Gamma \to \mathrm{Aut}(R)$ and $\alpha : \Lambda \to \mathrm{Aut}(R_0)$ be free, ergodic action.

Denote $M = R \rtimes_\sigma \Gamma$, $N = R_0 \rtimes_\alpha \Lambda$.

Theorem. *Assume $R \subset M$ and $R_0 \subset N$ are rigid inclusions. Let $\theta : M \simeq N^t$, for some $t > 0$. Then we have:*

1°. *$\exists u \in \mathcal{U}(N^t)$ s.t. $\mathrm{Ad}(u)(\theta(R)) = R_0^t$.*

Thus, $\Gamma \simeq \Lambda$, and σ and α^t are cocycle conjugate actions with respect to the identification $\Gamma \simeq \Lambda$.

2°. *If in addition Γ_0 and Λ_0 are free groups and Γ_i, Λ_j are free indecomposable, $\neq \mathbb{Z}$, $\forall 1 \leq i \leq n$ $1 \leq j \leq m$,*

then $\Gamma_0 \simeq \Lambda_0$, $n = m$ and $\exists \pi$ a permutation of the indices $1 \leq j \leq m$ and $u_j \in \mathcal{U}(N)^t$ s.t. $\mathrm{Ad}(u_j)(\theta(M_j)) = N^t_{\pi(j)}$, $\mathrm{Ad}(u_j)(\theta(R)) = R_0^t$, $\forall j \geq 1$.

In particular, $\Gamma_j \simeq \Lambda_{\pi(j)}$, and $\sigma_{|\Gamma_j}$ and $\alpha^t_{|\Lambda_{\pi(j)}}$ are cocycle conjugate with respect to this identification of groups, $\forall j \geq 1$.

Theorem. *Let $\sigma_n : \Gamma_n \to \mathrm{Aut}(R)$ be a properly outer action of a discrete countable group Γ_n, $n \geq 1$.*

Then there exists a properly outer action σ of the group $\Gamma = *_n \Gamma_n$ on R such that $\sigma_{|\Gamma_n}$ is conjugate to σ_n, $\forall n \geq 0$.

Notation. We denote by $f\mathcal{T}$ the class of properly outer actions $\sigma : \Gamma_0 * \Gamma_1 \to \mathrm{Aut}(R)$ on the hyperfinite II_1 factor R, with the properties:

(1). Γ_0 is free indecomposable; Γ_1 is w-rigid.

(2). $\sigma_0 = \sigma_{|\Gamma_0}$ has the relative property (T), i.e. $R \subset R \rtimes_{\sigma_0} \Gamma_0$ is a rigid inclusion; $\sigma_1 = \sigma_{|\Gamma_1}$ is a non-commutative Bernoulli shift action of Γ_1 on $R = \overline{\otimes}_g (N_0, \tau_0)_g$, where $N_0 = R$ or $N_0 = M_{n\times n}(\mathbb{C})$ for some $n \geq 2$.

(3). $\sigma(\Gamma_1)$ and the normalizer of $\sigma(\Gamma_0)$ in $\mathrm{Out}(R)$ (which is countable) are free independent.

Example. Given any compact abelian group K, let $\Gamma_0 = SL(n, \mathbb{Z})$, $\Gamma_1 = SL(m, \mathbb{Z}) \times \hat{K}$, for $n, m \geq 3$. Then $\exists \sigma : \Gamma_0 * \Gamma_1 \to R$ s.t. $(\sigma, \Gamma_0 * \Gamma_1)$ is in the class $f\mathcal{T}$.

Corollary. *Let* $\sigma : \Gamma_0 * \Gamma_1 \to \mathrm{Out}(R)$ *be a* $f\mathcal{T}$ *action and denote* $M = R \rtimes_\sigma (\Gamma_0 * \Gamma_1)$. *Then:*

1°. $\mathscr{F}(M) = \{1\}$ *and* $\mathrm{Out}(M) = \mathrm{Char}(\Gamma_0) \times \mathrm{Char}(\Gamma_1)$.

2°. *Given any compact abelian group* K, $\exists$ $(\sigma, \Gamma_0 * \Gamma_1)$ *such that the the corresponding* $f\mathcal{T}$ *factor* M *satisfies* $\mathrm{Out}(M) = K$.

Remark. Let $f\tilde{\mathcal{T}}$ be the class of properly outer actions $\sigma : \Gamma_0 * \Gamma_1 \to \text{Aut}(R)$ on the hyperfinite II_1 factor R, satisfying the properties:

(a). Γ_0, Γ_1 free indecomposable, $\neq \mathbb{Z}$;

(b). $R \subset R \rtimes_{\sigma_0} \Gamma_0$ is a rigid inclusion and $\sigma_1 = \sigma_{|\Gamma_1}$ is non-cocylce conjugate to σ_0 (for instance, σ_1 a non-commutative Bernoulli shift);

(c). $\sigma(\Gamma_1)$ is free independent with respect to the set $\mathcal{N}(\sigma_0(\Gamma_0)) \cup \mathcal{N}^{op}(\sigma_0(\Gamma_0))$, consisting of all automorphisms and anti-automorphisms of R^∞ that normalize $\sigma_0(\Gamma_0)$.

Then $M = R \rtimes_\sigma (\Gamma_0 * \Gamma_1)$ satisfy $\text{Out}(M) = \text{Char}(\Gamma_0 * \Gamma_1)$. Moreover $\mathscr{F}(M) = \{1\}$.

If α is an anti-automorphism of M then it must normalize $\sigma_0(\Gamma_0)$, in contradiction with the choice (c).

Thus, if we choose the groups Γ_0, Γ_1 without characters, e.g. $\Gamma_i = SL(n, \mathbb{Z}), n \geq 3$, then the resulting factors M have no outer symmetries at all !