

**Noncommutative
Spectral Invariants
and Black Hole Entropy**

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Contents

- Motivations from black hole thermodynamics
- Weyl theorem
- QFT index theorem
- Conformal nets as noncommutative manifolds with infinitely many degrees of freedom
- Abstract results on index
- Local spectral invariants
- Comments

Bekenstein formula. The entropy S of a black hole is proportional to the area A of its horizon

$$S = A/4$$

Note: S is proportional to the *area*, not to the volume as a naive microscopic interpretation of entropy would suggest (logarithmic counting of possible states).

This dimensional reduction has led to the *holographic principle* by t'Hooft, Susskind, ...

The horizon is not a physical boundary, but a submanifold where coordinates pick critical values \rightarrow *conformal symmetries*

The proportionality factor $1/4$ is fixed by Hawking temperature (*quantum* effect).

Discretization of the horizon (Bekenstein): horizon is made of cells of area ℓ^2 and k degrees of freedom ($\ell =$ Planck length):

$$A = n\ell^2,$$

$$\text{Degrees of freedom} = k^n,$$

$$S = Cn \log k = C \frac{A}{\ell^2} \log k,$$

$$dS = C \log k$$

Conclusion.

Black hole entropy



Two-dimensional conformal quantum field theory
with a “fuzzy” point of view

Legenda: Fuzzy = *noncommutative geometrical*

Weyl's theorem and ellipticity. M compact oriented Riemann manifold, Δ Laplace operator on $L^2(M)$. The eigenvalues of M can be thought as “resonant frequencies” of M and capture most of the geometry of M (M. Kac).

Weyl theorem: heat kernel expansion as $t \rightarrow 0^+$

$$\text{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \dots)$$

or, by Tauberian theorems, asymptotic density distribution of eigenvalues of Δ as $\lambda \rightarrow +\infty$,

$$N(\lambda) \sim \frac{\text{vol}(M)}{(4\pi)^{n/2} \Gamma((n/2) + 1)} \lambda^{n/2}$$

$N(\lambda)$ eigenvalues $\leq \lambda$, Γ Euler Gamma-function.

The *spectral invariants* n and a_0, a_1, \dots encode geometric information and in particular

$$a_0 = \text{vol}(M), \quad a_1 = \frac{1}{6} \int_M \kappa(m) d\text{vol}(m),$$

κ scalar curvature. $n = 2$: a_1 is proportional to the Euler characteristic $= \frac{1}{2\pi} \int_M \kappa(m) d\text{vol}(m)$ by Gauss-Bonnet theorem.

Infinite dim. quantum systems; log-ellipticity.

h positive selfadjoint operator on Hilbert space \mathcal{H} and H the Fermi second quantization of h on the exponential of \mathcal{H} . Then as $t \rightarrow 0^+$:

$$\frac{\log \text{Tr}(e^{-tH})}{\text{Tr}(e^{-th})} = \frac{\text{Tr} \log(1 + e^{-th})}{\text{Tr}(e^{-th})} = O(t)$$

A positive linear operator H on a Hilbert space is *log-elliptic* (or QFT elliptic) if there exists $n > 0$ and $a_i \in \mathbb{R}$, $a_0 \neq 0$, such that

$$\log \text{Tr}(e^{-tH}) \sim \frac{1}{t^{n/2}}(a_0 + a_1 t + \dots) \quad \text{as } t \rightarrow 0^+$$

$n \equiv$ dimension of H , $a_i \equiv i^{\text{th}}$ spectral invariant of H . Thus

finitely many degrees of freedom \rightarrow ellipticity
 infinitely many degrees of freedom \rightarrow log-ellipticity

and log-ellipticity captures the spectral invariants of the existing one-particle Hamiltonian.

Conformal nets on S^1 . A (local) *Möbius covariant net* \mathcal{A} on S^1 is a map

$$I \in \mathcal{I} \rightarrow \mathcal{A}(I) \subset B(\mathcal{H})$$

$\mathcal{I} \equiv$ family of proper intervals of S^1 , that satisfies:

A. Isotony. $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$

B. Locality. $I_1 \cap I_2 = \emptyset \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$

C. Möbius covariance. \exists unitary rep. U of the Möbius group Möb on \mathcal{H} such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \text{Möb}, \quad I \in \mathcal{I}.$$

D. Positivity of the energy. Generator L_0 of rotation subgroup of U (conformal Hamiltonian) is positive.

E. Existence of the vacuum. $\exists!$ U -invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for the von Neumann algebra $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$ and unique U -invariant.

First consequences

Irreducibility: $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = B(H)$.

Reeh-Schlieder theorem: Ω is cyclic and separating for each $\mathcal{A}(I)$.

Bisognano-Wichmann property: Tomita-Takesaki modular operator Δ_I and conjugation J_I of $(\mathcal{A}(I), \Omega)$, are

$$\begin{aligned} U(\Lambda_I(2\pi t)) &= \Delta_I^{it}, \quad t \in \mathbb{R}, && \text{dilations} \\ U(r_I) &= J_I && \text{reflection} \end{aligned}$$

(Guido-L., Frölich-Gabbiani)

Haag duality: $\mathcal{A}(I)' = \mathcal{A}(I')$

Factoriality: $\mathcal{A}(I)$ is III_1 -factor (or $\mathcal{A}(I) = \mathbb{C}$).

Additivity: $I \subset \cup_i I_i \implies \mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i)$ (Freedenhagen, Jorss).

$\text{Diff}(S^1)$ group of orientation-preserving smooth diffeomorphisms of S^1 .

A (local) *conformal net* \mathcal{A} is a Möbius covariant net s.t.

F. Conformal covariance. \exists a projective unitary representation U of $\text{Diff}(S^1)$ on \mathcal{H} extending the unitary representation of Möb s.t.

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \text{Diff}(S^1), \\ U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I), \quad g \in \text{Diff}(I'), \end{aligned}$$

$$\text{Diff}(I) = \{g \in \text{Diff}(S^1) : g(t) = t \quad \forall t \in I'\}.$$

Split property. \mathcal{A} satisfies the *split* property if the von Neumann algebra

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \simeq \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$$

(natural isomorphism) if $\bar{I}_1 \cap \bar{I}_2 = \emptyset$.

$$\text{Tr}(e^{-tL_0}) < \infty, \quad \forall t > 0 \implies \text{split} .$$

Representations. A representation π of \mathcal{A} on a Hilbert space \mathcal{H} is a map $I \in \mathcal{I} \mapsto \pi_I$, with π_I a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_{\tilde{I}}|_{\mathcal{A}(I)} = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \subset \mathcal{I} .$$

π is Möbius (resp. diffeomorphism) *covariant* if there is a projective unitary representation U_π of Möb (resp. $\text{Diff}^{(\infty)}(S^1)$) on \mathcal{H} such that

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$ and $g \in \text{Möb}$ (resp. $g \in \text{Diff}^{(\infty)}(S^1)$). Note: π is irreducible $\implies U$ is projective unitary rep. of $\text{Diff}(S^1)$.

Doplicher-Haag-Roberts argument: given an interval I and a representation π of \mathcal{A} , there is

an endomorphism of \mathcal{A} localized in I equivalent to π ; namely ρ is a representation of \mathcal{A} on the vacuum Hilbert space \mathcal{H} , unitarily equivalent to π , such that $\rho_{I'} = \text{id} \upharpoonright_{\mathcal{A}(I')}$.

Index-statistics thm.

$$\text{DHR dim. } d(\rho) = \sqrt{\text{Jones index } \text{Ind}(\rho)}$$

tensor category $\xrightarrow[\text{restriction}]{\text{full functor}}$ tensor category
 End. local. in I \rightarrow End. of $\mathcal{A}(I)$

QFT index theorem. Let \mathcal{V} be a $d + 1$ dimensional globally hyperbolic spacetime manifold with a bifurcate Killing horizon and \mathcal{R} a “wedge” of \mathcal{V} (Kay-Wald setting). Typical examples: Schwarzschild \subset Schwarzschild-Kruskal, Rindler \subset Minkowski.

$\kappa = \kappa(\mathcal{R})$ be the *surface gravity*

$\mathcal{A}(\mathcal{O})$ local von Neumann algebras with usual properties. Restriction to the horizon is a Möb covariant (Wiesbrock, Summers, Verch, Guido, Roberts, R.L.) and expected uniquely conformal (Weiner, Carpi)

φ (KMS) Hartle-Hawking state (often unique KMS for Killing evolution). Hawking temperature $\beta^{-1} = \kappa(\mathcal{R})/2\pi$

ρ and σ endomorphisms localized on the horizon with KMS states $\varphi_\rho, \varphi_\sigma$

$dF \equiv \frac{1}{2}(F(\varphi_\rho|\varphi_\sigma) + F(\varphi_{\bar{\rho}}|\varphi_{\bar{\sigma}}))$ incremental free energy (can be defined!), then

$$\log d(\rho) - \log d(\sigma) = \frac{2\pi}{\kappa(\mathcal{R})} dF$$

analytical index = geometric index

cf. discretization of the horizon: $d(\rho) \in \mathbb{N}$ by Doplicher-Haag-Roberts.

Complete rationality. (Y. Kawahigashi, M. Müger, R.L.) I_1, I_2 intervals $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, $E \equiv I_1 \cup I_2$.

$$\mu\text{-index } \mu_{\mathcal{A}} \equiv [\mathcal{A}(E')' : \mathcal{A}(E)]$$

(Jones index). \mathcal{A} conformal:

\mathcal{A} completely rational $\stackrel{\text{def}}{=} \mathcal{A}$ split & $\mu_{\mathcal{A}} < \infty$

Thm. \mathcal{A} completely rational: then

$$\mu_{\mathcal{A}} = \sum_i d(\rho_i)^2$$

sum of all irred. sectors. (F. Xu in $SU(N)$ models);

- $\mathcal{A}(E) \subset \mathcal{A}(E')' \sim$ LR inclusion (quantum double);
- Representations form a modular tensor category (i.e. non-degenerate braiding).

Modularity and log-ellipticity of \mathcal{A} . A conformal net \mathcal{A} is two-dimensional *log-elliptic* if its conformal Hamiltonian L_0 is log-elliptic with dimension 2, i.e.

$$\log \operatorname{Tr}(e^{-2\pi t L_0}) \sim \frac{1}{t}(a_0 + a_1 t + \dots) \quad \text{as } t \rightarrow 0^+$$

log-ellipticity is essentially the *nuclearity condition* of Buchholz and Wichmann (and we fix the dimension).

With ρ rep. of \mathcal{A} , set $L_{0,\rho}$ conf. Hamiltonian of ρ ,

$$\chi_\rho(\tau) = \operatorname{Tr} \left(e^{2\pi i \tau (L_{0,\rho} - c/24)} \right) \quad \operatorname{Im} \tau > 0.$$

specialized character, c the central charge.

\mathcal{A} is *modular* if $\mu_{\mathcal{A}} < \infty$ and

$$\begin{aligned} \chi_\rho(-1/\tau) &= \sum_{\nu} S_{\rho,\nu} \chi_\nu(\tau), \\ \chi_\rho(\tau + 1) &= \sum_{\nu} T_{\rho,\nu} \chi_\nu(\tau). \end{aligned}$$

with S, T the (algebraically defined) Verlinde-Rehren matrices generating a representation of $SL(2, \mathbb{Z})$. One has:

- Modularity \implies complete rationality
- Modularity holds in all computed rational case, e.g. $SU(N)_k$ -models
- \mathcal{A} modular, $\mathcal{B} \supset \mathcal{A}$ irreducible extension $\implies \mathcal{B}$ modular.
- All conformal nets with central charge $c < 1$ are modular.

Modular nets as NC manifolds (∞ degrees of freedom)

2-dim. cpt manifold M	conformal net \mathcal{A}
$\text{supp}(f) \subset I$	$x \in \mathcal{A}(I)$
Laplacian Δ	conf. Hamiltonian L_0
Δ elliptic	L_0 log-elliptic
area $\text{vol}(M)$	NC area $a_0(2\pi L_0)$
Euler charact. $\chi(M)$	NC Euler char. $12a_1$

Thm. \mathcal{A} is modular. The following asymptotic formula holds as $t \rightarrow 0^+$:

$$\log \operatorname{Tr}(e^{-2\pi t L_0}) \sim \frac{\pi c}{12} \frac{1}{t} - \frac{1}{2} \log \mu_{\mathcal{A}} - \frac{\pi c}{12} t$$

Thus \mathcal{A} is two-dimensional log-elliptic with non-commutative area $a_0 = 2\pi c/24$

In any representation ρ , as $t \rightarrow 0^+$:

$$\log \operatorname{Tr}(e^{-2\pi t L_{0,\rho}}) \sim \frac{\pi c}{12} \frac{1}{t} + \frac{1}{2} \log \frac{d(\rho)^2}{\mu_{\mathcal{A}}} - \frac{\pi c}{12} t$$

Note: spectral density $L_0 \rightarrow$ normalized index

$$\log d(\rho) - \frac{1}{2} \log \mu_{\mathcal{A}} = \lim_{t \rightarrow 0^+} \frac{d}{dt} t \log \operatorname{Tr}(e^{-t L_{0,\rho}}) .$$

Conjecture 1:

$$\mathcal{A} \text{ c. rational} \Leftrightarrow \lim_{t \rightarrow 0^+} \frac{d}{dt} t \log \operatorname{Tr}(e^{-t L_0}) > -\infty$$

Conjecture 2:

$$\mathcal{A} \text{ modular} \Leftrightarrow \mathcal{A} \text{ completely rational}$$

By Kohlbecker's Tauberian theorem as $\lambda \rightarrow \infty$

$$\log N(\lambda) \sim 2\pi\sqrt{\frac{c}{6}}\lambda$$

where $N(\lambda)$ is the number of eigenvalues (with multiplicity) of $L_{0,\rho}$ that are $\leq \lambda$. (Partial version of Cardy's formula on the 2-dim. Minkowski space).

Entropy. From the physics viewpoint it is natural to define $S_{\mathcal{A}}$, the *entropy of \mathcal{A}* , as the leading coefficient of the expansion of $\log \text{Tr}(e^{-2\pi t L_0})$, thus

$$a_0 = S_{\mathcal{A}} ,$$

$$a_1, a_2, \dots = \text{higher order corrections to } S_{\mathcal{A}} .$$

By definition, the entropy is proportional to the noncommutative area: it is just a matter of reading the same formula from different point of views. Meaning of spectral invariants:

<i>Inv.</i>	<i>Value</i>	<i>Geometry</i>	<i>Physics</i>
a_0	$\pi c/12$	NC area	Entropy
a_1	$-\frac{1}{2} \log \mu_{\mathcal{A}}$	NC Euler charact.	1 st order entr.
a_2	$-\pi c/12$	2 nd spectral invariant	2 nd order entr.

$a_2 = -a_0$, consequence of modular symmetry.

Incremental free energy. \mathcal{A} modular implies a strong Kac-Wakimoto formula

$$\begin{aligned} \log \text{Tr}(e^{-2\pi t L_{0,\rho}}) - \log \text{Tr}(e^{-2\pi t L_{0,\sigma}}) \\ = \log d(\rho) - \log d(\sigma) + o(t) \end{aligned}$$

cf. QFT index theorem: here true difference of free energy

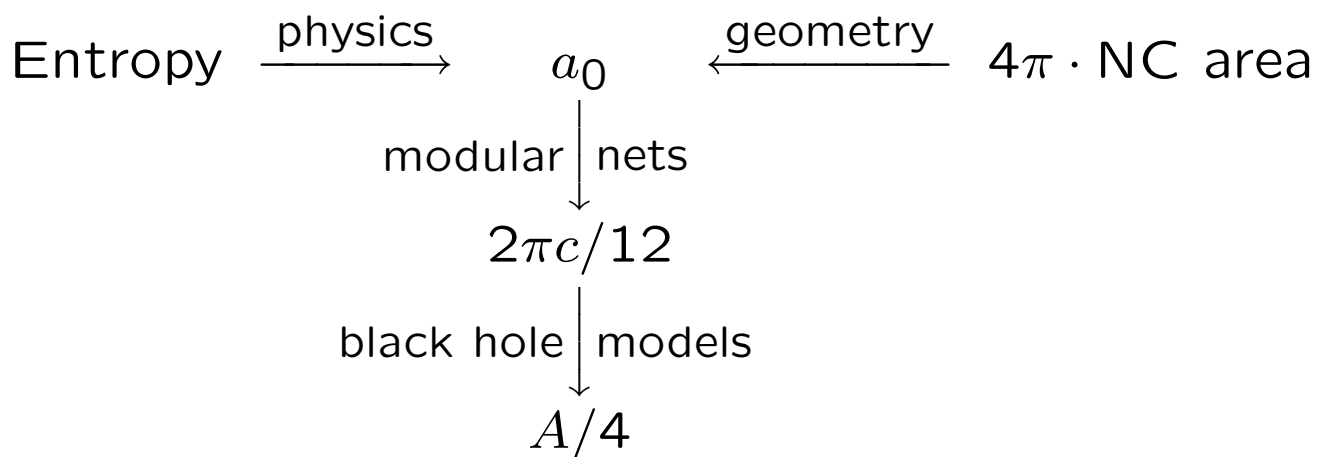
$$dF = \frac{2\pi}{\kappa} (\log d(\rho) - \log d(\sigma)) = \frac{\pi}{6\kappa} (\chi_\sigma - \chi_\rho)$$

Relation to black hole entropy. I. Microscopic derivation of black hole entropy and its relation to conformal symmetries and central charge is discussed Strominger, Vafa and others. We illustrate our discussion by the work of Carlip. Yet we use here only the value of the central charge and not Cardy's formula nor the boundary term of the energy.

For a black hole in the above class considered by Carlip we have

$$S_{\mathcal{A}} = A/4$$

where A is the area of the black hole horizon. Thus



Diff(S^1) and its covers. $\text{Diff}(S^1) =$ smooth oriented diffeomorphisms of S^1 . The Lie algebra of $\text{Diff}(S^1)$ is $\text{Vect}(S^1)$ The Virasoro algebra is the (complexification of) the unique, non-trivial one-dim. central extension of $\text{Vect}(S^1)$.
Generators:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

and $[L_n, c] = 0$.

$\text{Diff}^{(n)}(S^1) = n\text{-cover of } \text{Diff}(S^1)$. Set for a fixed $n > 0$

$$L_{\pm m}^{(n)} \equiv \frac{1}{n} L_{\pm mn} ,$$

$$L_0^{(n)} \equiv \frac{1}{n} L_0 + \frac{c}{24} \frac{(n^2 - 1)}{n} .$$

The map

$$\begin{cases} L_m \mapsto L_m^{(n)} \\ c \mapsto nc , \end{cases}$$

gives an embedding of the Virasoro algebra into itself. There corresponds an embedding of $\text{Diff}^{(n)}(S^1)$: There is a unique continuous isomorphism $M^{(n)}$ of $\text{Diff}^{(n)}(S^1)$ into $\text{Diff}(S^1)$ such that for all $g \in \text{Diff}^{(n)}(S^1)$ the following diagram commutes

$$\begin{array}{ccc} S^1 & \xrightarrow{M_g^{(n)}} & S^1 \\ z^n \downarrow & & \downarrow z^n \\ S^1 & \xrightarrow{\underline{g}} & S^1 \end{array}$$

i.e. $M_g^{(n)}(z)^n = \underline{g}(z^n)$ for all $z \in S^1$. $\underline{g} \in \text{Diff}(S^1)$ corresponds to g .

The mean free energy (topological increment of the second spectral invariant). A conformal net in any representation. We divide S^1 into n equally spaced cells, namely we consider the n -interval $E_n \equiv \sqrt[n]{S^+}$, S^+ upper semicircle. Each interval I_k contains minimal information (as the cells of Planck length).

Two canonical evolution associated with E_n corresponding to the rotations on the full S^1 :

First: rescaled rotations $R(\frac{1}{n}\vartheta)$, rescaled conformal Hamiltonian $\hat{L}_0^{(n)} \equiv \frac{1}{n}L_0$

Second: associated with $U^{(n)}$ (rep. of $\text{Diff}^{(n)}(S^1)$), Hamiltonian $L_0^{(n)} = \frac{1}{n}L_0 + \frac{c}{24} \frac{(n^2-1)}{n}$, takes care of “boundary effects”. The geometrical complexity should be encoded in the difference between the two terms.

Associated free energy: difference of the free energy given the corresponding partition functions at infinite temperature:

$$F_n \equiv t^{-1} \log \text{Tr}(e^{-t2\pi L_0^{(n)}}) - t^{-1} \log \text{Tr}(e^{-t2\pi \hat{L}_0^{(n)}})$$

thus

$$F_n = \frac{c}{24} \frac{(n^2 - 1)}{n} 2\pi$$

hence: model independent formula for the mean free energy associated to the “discretization of S^1 ”.

$$F_{\text{mean}} = 2\pi \frac{c}{24}$$

Note:

$$a_2(2\pi L_0^{(n)}) - a_2(2\pi \hat{L}_0^{(n)}) = F_n$$

NC geometrical meaning of F_{mean} .

Two-dimensional conformal QFT, both chiral components contribute to the topological en-

tropy and physical topological entropy duplicates:

$$F_{\text{mean}} = 2\pi \frac{c}{12}$$

Relation to black hole entropy. II As above

$$F_{\text{mean}} = A/4$$

model independent, no modularity assumption.

The modular group of a n -interval von Neumann algebra. Model independent, in arbitrary representation, of Schroer and Wiesbrock formula for $U(1)$ -current algebra.

$E \equiv \sqrt[n]{I}$ symmetric n -interval of S^1 , $E = \{z \in S^1 : z^n \in I\}$. I_0, I_1, \dots, I_{n-1} n connected components of E , $I_k = R(2\pi k/n)I_0$. A split conformal net on S^1 , in a irreducible representation. Split isomorphism:

$$\chi_E : \mathcal{A}(I_0) \vee \dots \vee \mathcal{A}(I_{n-1}) \rightarrow \mathcal{A}(I_0) \otimes \dots \otimes \mathcal{A}(I_{n-1}) .$$

Rotation invariant product state φ on $\mathcal{A}(E)$:

$$\varphi \equiv (\varphi_0 \otimes \varphi_1 \otimes \cdots \otimes \varphi_{n-1}) \cdot \chi_E ,$$

φ_k normal faithful state on $\mathcal{A}(I_k)$ and $\varphi_k = \varphi_0 \cdot \text{Ad}U(R(2k\pi/n))$.

$\Phi_k : \mathcal{A}(I_k) \rightarrow \mathcal{A}(I)$ isomorphism associated with z^n , namely

$$\Phi_k(x) \equiv U(h_k)xU(h_k)^*, \quad x \in \mathcal{A}(I_k)$$

where $h_k \in \text{Diff}(S^1)$ s.t. $h_k(z) = z^n, z \in I_k$.

$\varphi_k \equiv \omega_I \cdot \Phi_k$, where ω vacuum state (or KMS state), φ_E the associated rotation invariant product state on $\mathcal{A}(E)$. Then the modular group σ^{φ_E} is given by

$$\sigma_t^{\varphi_E} = \text{Ad}U^{(n)}(\Lambda_I(-2\pi t)) \upharpoonright_{\mathcal{A}(E)}$$

Λ_I lift to $\text{Möb}^{(n)}$ of “dilation” of I .

indeed, with $V(t) \equiv U^{(n)}(\Lambda_I(-2\pi t))$,

$\text{Ad}V(t) \upharpoonright_{\mathcal{A}(E)} = \sigma_t^{\varphi_E}, \quad \text{Ad}V(-t) \upharpoonright_{\mathcal{A}(E')} = \sigma_t^{\varphi_{E'}}$

Index and entropy. Abstract mathematical results concerning Jones index in the Kosaki framework and Connes-Haagerup noncommutative measure theory.

N_1, N_2 commuting factors on a Hilbert space \mathcal{H} , $N_1 \vee N_2 = B(\mathcal{H})$, i.e. $M_1 \equiv N_2' \supset N_1$, $M_2 \equiv N_1' \supset N_2$ irreducible subfactors

$\varphi_i = (\cdot \xi_i, \xi_i)$ state on N_i , ξ_i cycl. separ. for N_i

$V(t) = e^{-itK}$ a one-parameter unitary group on \mathcal{H} s.t.

$$\text{Ad}V(t) \upharpoonright_{N_1} = \sigma_t^{\varphi_1}, \quad \text{Ad}V(-t) \upharpoonright_{N_2} = \sigma_t^{\varphi_2},$$

where σ^{φ_i} is the modular group of (N_i, φ_i) .

Thm. $[M_1 : N_1] = (e^K \xi_1, \xi_1)(e^{-K} \xi_2, \xi_2)$

If \exists unitary U s.t. $UN_1U^* = N_2$, $\varphi_2 = \varphi_1 \cdot \text{Ad}U$ and $UV(t)U^* = V(-t)$, then

$$(e^K \xi_1, \xi_1) = (e^{-K} \xi_2, \xi_2) = [M_1 : N_1]^{\frac{1}{2}},$$

thus

$$K = -\log \frac{d\varphi_1 \cdot \varepsilon_1}{d\varphi_2} + \frac{1}{2} \log[M_1 : N_1]$$

where $\varepsilon_1 : M_1 \rightarrow N_1$ expectation (finite-index case). Thus

$$(K\xi_2, \xi_2) = -(\log \frac{d\varphi_1 \cdot \varepsilon_1}{d\varphi_2} \xi_2, \xi_2) + \frac{1}{2} \log[M_1 : N_1]$$

= Araki entropy + Pimsner-Popa entropy.

Entropy and spectral invariants with the proper Hamiltonian. We replace the conformal Hamiltonian L_0 with the “local” Hamiltonian

$$K_1 \equiv i(L_1 - L_{-1}) ,$$

the generator of the one-parameter dilatation unitary group associated with the upper semi-circle S^+ .

Dynamics	rotations	dilations
Hamiltonian	L_0	K_1
State	ω	$\omega _{\mathcal{A}(I)}$
Pos. energy	$L_0 \geq 0$	KMS condition

Dilations satisfy the equilibrium condition at Hawking temperature and are natural to be considered.

We will now consider the “ n -cell” dynamics

dilations \longrightarrow n -dilations

in analogy with the passage rotation \rightarrow n -rotation with the action of $\text{Diff}^{(n)}(S^1)$ and compute noncommutative spectral invariants in complete generality.

A split local conformal net on S^1 , $E \equiv E_n = \sqrt[n]{I}$ and K_n the infinitesimal generator of $V^{(n)}(t) = U^{(n)}(\Lambda_I(-2\pi t))$

$$K_n \equiv i(L_1^{(n)} - L_{-1}^{(n)}) = \frac{i}{n}(L_n - L_{-n}),$$

$E'_n = \sqrt[n]{I'}$. $\varphi_{E_n} = (\cdot\xi_n, \xi_n)$ canonical rotation-invariant product state on $\mathcal{A}(E_n)$. We have:

$$\boxed{(e^{-2\pi K_n} \xi_n, \xi_n) = d(\rho) \mu_{\mathcal{A}}^{\frac{n-1}{2}}}$$

thus

$$\begin{aligned} \log(e^{-\frac{2\pi i}{n}(L_n - L_{-n})} \xi_n, \xi_n) \\ = \frac{n-1}{2} \log\left(\sum_i d(\rho_i)^2\right) + \log d(\rho) \end{aligned}$$

Let $\hat{\varphi}_{E_n} = \varphi_{E_n} \cdot \varepsilon_{E_n}$ the state on $\hat{\mathcal{A}}(E_n)$ extended by the expectation $\varepsilon_{E_n} : \hat{\mathcal{A}}(E_n) \rightarrow \mathcal{A}(E_n)$.

Then

$$K_n = -\frac{1}{2\pi} \left(\log \left(\frac{d\hat{\varphi}_{E_n}}{d\varphi_{E'_n}} \right) + \frac{n-1}{2} \log \mu_{\mathcal{A}} + \log d(\rho) \right)$$

The quantity

$$Z_n(t) \equiv (e^{-tK_n} \xi_n, \xi_n)$$

is the geometric partition function associated to the symmetric n -interval partition of S^1 , thus

$$\begin{aligned} F_{n,\mu} &\equiv -t^{-1} \log Z_n(t)|_{t=2\pi} \\ &= -\frac{n-1}{4\pi} \log \mu_{\mathcal{A}} - \frac{1}{2\pi} \log d(\rho) \end{aligned}$$

is the associated n - μ -free energy. Dividing by the numbers of cells (intervals) we get *mean*

μ -free energy.

$$F_{\text{mean},\mu} = -\frac{1}{4\pi} \log \mu_{\mathcal{A}}$$

The 0th and 1st spectral invariants are then defined by

$$a_{0,\mu} \equiv \lim_{n \rightarrow \infty} \frac{t \log Z_n(t)}{n} \Big|_{t=2\pi}$$

$$a_{1,\mu} \equiv \lim_{n \rightarrow \infty} \frac{d}{dt} \frac{t \log Z_n(t)}{n} \Big|_{t=2\pi}$$

Note that $-\frac{d}{dt} \log Z_n(t)$ is the n - μ -energy $H_{n,\mu}$ associated with $Z_n(t)$. Due to the thermodynamical relation

$$\text{Free energy} = T \cdot \text{Entropy} - \text{Energy}$$

where T is the temperature, we thus define the mean $n - \mu$ -entropy by $S_{n,\mu} = t(F_{n,\mu} + H_{n,\mu})$. We have:

$$S_{n,\mu} = S(\hat{\varphi}_{E_n} | \varphi_{E'_n})$$

Araki relative entropy.

We have the “local” spectral invariants

$$\begin{cases} a_{0,\mu} = \frac{1}{2} \log \mu_{\mathcal{A}} , \\ a_{1,\mu} = -S_{\text{mean},\mu} = \log \mu_{\mathcal{A}} - \lim_{n \rightarrow \infty} \frac{1}{n} S(\hat{\varphi}_{E_n} | \varphi_{E'_n}) \end{cases}$$

Final comments. Adding a massive charge to a black hole should increase the total mass of the black hole, hence make a change of the spacetime itself and of the entropy. In a theory of quantum gravity, the spacetime itself should be noncommutative from the start (Doplicher-Fredenhagen-Roberts). In the setting of QFT on a curved spacetime the backreaction from the gravitational field is ignored and the spacetime is classical. In our previous work one considered the addition of a single charge: the increment of entropy is there an “higher order effect” and become visible in the associated noncommutative geometry, while the classical spacetime remains fixed. The entropy in the present work also has a noncommutative geometrical nature, but rather reflects the global

noncommutative geometrical complexity of the system.

It would be interesting to relate our setting with Connes' Noncommutative Geometry. A link should be possible in a supersymmetric context, where cyclic cohomology appears. In this respect model analysis with our point of view, in particular in the supersymmetric frame, may be of interest. Note also that Connes' spectral action concerns the Hamiltonian spectral density behavior.