L^2-invariants and operator Algebras

Lecture 1: The space of non-commutative laws.

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Based on the following papers:

- D. S., “Some estimates on the non-microstates free entropy dimension with applications to q-semicircular families”, IMRN **51** 1757–2772, 2004

- A. Connes + D.S., “L^2-homology for von Neumann algberas”, to appear in J. Reine Angew. Math.

- I. Mineyev + D.S., “Non-microstates free entropy dimension for groups”, to appear in GAFA.

- D.S., “Remarks on Free entropy dimension”, preprint, 2005.

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$\mathcal{L}$ is foliated by leaves $\mathcal{M}(\mu)$ through $\mu\in\mathcal{L}$

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For $\vec{T} = (T_1, \ldots, T_n) \in (L^2(\mu)\bar{\otimes}L^2(\mu))^n$, consider

$$\mu \mapsto \mu_{X_1+\sqrt{t}T_1*S,\ldots,X_n+\sqrt{t}T_n*S}$$

free Brownian motion of “variance” $\vec{T}$.

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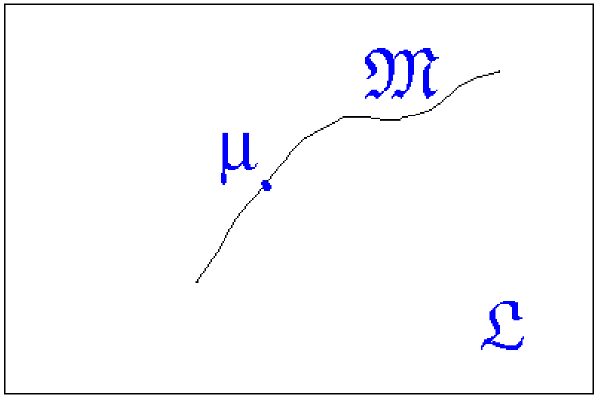

Tangent space to $\mathcal{M}$.

Idea (Voiculescu): $\vec{T} \in T_\mu \mathcal{M}$ if $\exists \vec{\xi}_{\vec{T}} \in W^*(\mu)$ so that

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Fact. $\xi_{\vec{T}}$ exists iff $1 \otimes 1 \in \mathrm{dom}\partial^*_{\vec{T}}$, where $\partial_{\vec{T}} : \mathbb{C}[X_1, \ldots, X_n] \to (L^2(\mu)\bar{\otimes}L^2(\mu))^n$ is given by

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Moreover, $\xi_{\vec{T}} = \partial^*_{\vec{T}}(1 \otimes 1)$.

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Since $\vec{T} \mapsto \partial_{\vec{T}}$ is clearly linear, we have that $\vec{T} \mapsto \xi_{\vec{T}}$ is linear.

Singular Riemannian structure on $T\mathcal{L}$.

We thus define

$$T_\mu\mathcal{M} = \overline{\{\vec{T} \in T_\mu\mathcal{L} : \xi_{\vec{T}} \text{ exists}\}}^{T_\mu\mathcal{L}}.$$

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Note: $\sum_j \|0, \ldots, 1 \otimes 1_j, \ldots, 0\|_\Phi^2$ is exactly the free Fisher information $\Phi^*(\mu)$.

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Corr. $T_\mu \mathcal{M}$ is a $W^*(\mu) \bar{\otimes} W^*(\mu)^o$-bimodule.

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$$\text{where } \|D\|^2_{X_1,\dots,X_n} = \sum_j \|[D,X_j]_{HS}\|_2^2$$

and these norms for different choices of generators are equivalent.

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$$\begin{aligned} T_\mu \mathcal{M} &= HS/\ker\theta, \quad \theta(H) = [H, X] \\ \underline{\delta} &= \dim_N(HS/\ker\theta) = 1 - \dim_N \ker\theta, \quad N = L^\infty(\mu \times \mu). \end{aligned}$$

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- Random matrix models with a convex polynomial potential. Can find D bounded with $[D, X_j] = \delta_{ij} P_1$.

Examples.

- $\gamma_j =$generators of a group Γ, c a cocycle with values in ℓ^2, $t_j = c(\gamma_j)$. Let $X_j = \gamma_j \in \mathbb{C}\Gamma, \tau_\Gamma$ Let $T_j = \mathrm{diag}(t_j) \in HS$. Then $\partial_{\vec{T}}$ is well-defined and in fact $\vec{T} \in T_\mu \mathcal{M}$.

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$$\ell^2\text{-cocycles} \quad \subset \quad T_\mu \mathcal{M}$$

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Free entropy dimension for groups.

Theorem. Let $X_1, \ldots, X_n \in \mathbb{C}\Gamma$ be generators of the group algebra. Then

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$\beta^{(2)}_k(M,\Gamma) = \dim_{L(\Gamma)} H^k_{(2)}(M)$

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Can also do L^2 homology instead.

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$N = L(\Gamma)$

$$\beta_k^{(2)}(\Gamma) \quad = \quad \dim_N(\ker d : \Omega_{(2)}^k \to \Omega_{(2)}^{k+1}) - \dim_N(\overline{\mathrm{Im}\ d} : \Omega_{(2)}^{k-1} \to \Omega_{(2)}^k)$$

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Thus L^2 cohomology counts L^2 vs compactly-supported solutions to cohomology problems.

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$\Gamma = \mathbb{F}_2$: no compactly supported cycles, but there exists ℓ^2 cycles.

Algebra analogs

L^2 cohomology counts L^2 vs compactly-supported solutions to cohomology problems.
$f \in C_c(\Gamma) \iff f \in \mathbb{C}\Gamma;$