Noncommutative Geometry and Geometric Analysis on Heisenberg Manifolds

Raphaël Ponge
(Ohio State University)

I. Heisenberg Manifolds

Definition. *A Heisenberg manifold is a manifold M together with a distinguished hyperplane $H \subset TM$.*

The most important examples of Heisenberg manifolds are:

- Foliations;
- Contact manifolds;
- Cauchy-Riemann manifolds.

• <u>CR manifolds</u>

Definition. *A CR structure on an orientable manifold M^{2n+1} is given by a rank n complex subbundle $T_{1,0} \subset T_{\mathbb{C}}M$ so that:*

- $T_{1,0}$ is integrable, i.e. $[T_{1,0}, T_{1,0}] \subset T_{1,0}$.

- $T_{1,0} \cap T_{0,1} = \{0\}$, where $T_{1,0} = \overline{T_{1,0}}$.

Equivalently, $H = \Re(T_{1,0} \oplus T_{0,1})$ has the structure of a complex vector bundle of rank n.

Examples. 1. Heisenberg group $\mathbb{H}^{2n+1}$.

2. Hypersurface $M \subset \mathbb{C}^n$.

3. Circle bundles over complex manifolds.

• Motivations for studying CR manifolds:

1. Important geometric objects (thanks to work of Cartan, Chern-Moser, Kohn, Fefferman, Tanaka, Webster ...).

2. Step towards a geometric index formula for complex manifold with boundary (i.e. boundary version of the Riemann-Roch theorem).

3. Step towards the study of the NCG of Lorentzian manifolds with Fefferman metric.

• Contact Manifolds:

Definition. *A contact structure on an orientable manifold M^{2n+1} is given by a nonvanishing 1-form θ such that $d\theta$ is everywhere nondegenerate on $H = \ker \theta$.*

Examples. 1. Heisenberg group $\mathbb{H}^{2n+1}$.

2. Nondegenerate hypersurface $M \subset \mathbb{C}^n$.

3. Boundary of symplectic manifold, e.g. cosphere bundle S^*M of a manifold M.

• Motivations for studying contact manifolds:

1. Important geometric structure (thanks to work of Elyashberg, Gromov, Rumin ...).

2. Step towards a geometric index formula for symplectic manifold with boundary.

• Tangent Lie Group Bundle:

Lemma. *There exists a 2-form* $\mathcal{L} : H \times H \to TM/H$ *so that*

$$\mathcal{L}_m(X(m), Y(m)) = [X, Y](m) \bmod H_m$$

for sections X, Y *of* H *near* $m \in M$.

Definition. *The tangent Lie group bundle* GM *is obtained by endowing the bundle,*

$$(TM/H) \oplus H$$

with the grading and product such that

$$t.(X_0 + X') = t^2 X_0 + tX', \quad t \in \mathbb{R},$$
$$(X_0 + X').(Y_0 + Y') =$$
$$X_0 + Y_0 + \frac{1}{2}\mathcal{L}(X', Y') + X' + Y',$$

for sections X_0, Y_0 *of* TM/H *and* X', Y' *of* H.

Proposition. *We have:*

$$\operatorname{rk}\mathcal{L}_m = 2n \Longleftrightarrow G_mM \simeq \mathbb{H}^{2n+1} \times \mathbb{R}^{d-2n},$$

where $\mathbb{H}^{2n+1}$ *the* $(2n+1)$*-dimensional Heisenberg manifold.*

This result justifies the terminology Heisenberg manifold.

Heisenberg Chart:

Definition. *1) A* H*-frame is a local frame* $X_0, X_1, \ldots, X_d$ *of* TM *such that* $X_1, \ldots, X_d$ *span* H*.*

2) A Heisenberg chart is a local chart for M *together with a* H*-frame on its domain.*

Remarks (RP, arXiv '04). 1) We can characterize the type of a Heisenberg manifold (foliation, contact, CR) by means of the structure of its tangent Lie group bundle.

2) Using some privileged coordinates at every point $x \in M$ we can interpret GM as the Lie group of a Lie algebras span by jets of the vector fields $X_0, X_1, \ldots, X_d$ of a H-fame around x.

3) The privileged coordinates allows us to get a straightforward construction of the tangent groupoid, very much close to Connes' original construction of the tangent groupoid of a manifold.

• <u>List of Main Geometric Operators:</u>

- $\overline{\partial}_b$-complex and its associated Laplacian, the Kohn Laplacian, on a CR manifold.

- Rumin's contact complex and its associated Laplacian on a contact manifold;

- Horizontal (sub-)Laplacian Δ_H on a general Heisenberg manifold;

- Hörmander (sub-)Laplacian $\Delta = -(X_1^2 + \ldots + X_p^2) + X_0$, where $X_1, \ldots, X_d$ span H.

With the exception of the contact Laplacian these Laplacians are sublaplacians, i.e. locally of the form

$$\Delta = -(X_1^2 + \ldots + X_d^2) + \lambda X_0 + \mu_1 X_1 + \ldots + \mu_d X_d + \nu,$$

for some H-frame $X_0, X_1, \ldots X_d$.

II. Heisenberg Calculus

The Heisenberg calculus is the right pseudodifferential tool to study the main differential operators on Heisenberg manifolds.

It was independently introduced by Beals-Greiner ('84) and M. Taylor ('84), extending previous works by Dynin, Folland-Stein and Boutet de Monvel.

• Differential Operator of Order m

Continuous operator $P : C^\infty(M) \to C^\infty(M)$ locally of the form

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \qquad D_x = \frac{1}{i}\frac{\partial}{\partial x}.$$

Associated to P is its symbol

$$p(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha,$$

so that $P = p(x, D_x)$ and we have

$$Pu(x) = (2\pi)^{-n} \int e^{ix.\xi} p(x,\xi) \hat{u}(\xi) d\xi.$$

• Classical Symbol of Order m:

Using this last formula we can define $p(x, D_x)$ for more general symbols such that

$$p \sim \sum_{j \geq 0} p_{m-j}, \quad p_{m-j}(x, \lambda\xi) = \lambda^{m-j} p(x, \xi),$$

i.e. for any $N \geq 0$ as $|\xi| \to \infty$ we have

$$|D_x^\alpha D_\xi^\beta (p - \sum_{j \leq J} p_{m-j})(x, \xi)| = \mathrm{O}(|\xi|^{-N})$$

for $J \geq J_{\alpha\beta}$ large enough.

• <u>ΨDO of Order m:</u>

Continuous operator $P : C_c^\infty(M) \to C^\infty(M)$ locally of the form

$$P = p(x, D_x) + R,$$

where p is a classical symbol of order m and R is a smoothing operator.

• Heisenberg Calculus (Local Theory)

Let $U \subset \mathbb{R}^{d+1}$ be a Heisenberg chart with H-frame $X_0, \ldots, X_d$. The Heisenberg calculus is such that:

- X_0 has order 2 and $X_1, \ldots, X_d$ order 1;

- At any $x \in U$ the calculus is modelized by that of homogeneous left-invariant convolution operators on the tangent group G_xU.

• Heisenberg Symbols of order m

Smooth functions on $U \times \mathbb{R}^{d+1}$ such that

$$q \sim \sum_{j \geq 0} q_{m-j}, \quad q_{m-j}(x, \lambda.\xi) = \lambda^{m-j} q_{m-j}(x, \xi),$$

where $\lambda.\xi = (\lambda^2 \xi_0, \lambda\xi_1, \ldots, \lambda\xi_d)$

Since the vector fields X_j's are approximated by left-invariant vector fields X_j^x's at any point $x \in U$ we use them to quantize the Heisenberg symbols. Let $\sigma_j(x, \xi)$ be the classical symbol of $\frac{1}{i} X_j$ and set

$$\sigma = (\sigma_0, \ldots, \sigma_d) \quad \text{and} \quad X = (X_1, \ldots, X_d).$$

Then $-iX = \sigma(x, D_x)$ and we quantize a Heisenberg symbol $q(x, \xi)$ by letting

$$q(x, -iX) = q(x, \sigma(x, D_x)).$$

Definition. *An operator* $Q : C_c^\infty(U) \to C^\infty(U)$ *is a* $\Psi_H DO$ *of order* m *when it is of the form*

$$Q = q(x, -iX) + R,$$

where q *is a Heisenberg symbol of order* m *and* R *is smoothing.*

We let $\Psi_H^m(U)$ denote the space of Ψ_HDO's of order m.

• Model Operator and Symbolic Calculus

Definition. *Let* $Q \in \Psi_H^m(U)$*. Then the model operator of* Q *at* x *is the left-invariant* Ψ_H *on* $G_x U$ *given by*

$$Q^x = q_m^x(-iX^x),$$

where $q_m^x = q_m(x,.)$ *and* $q_m(x,\xi)$ *is the principal symbol of* Q*.*

Proposition. *For* $j = 1,2$ *let* q_j *be a homogeneous Heisenberg symbol of degree* m_j*. Then there exists a homogeneous Heisenberg symbol* $q = q_1 * q_2$ *of degree* $m_1 + m_2$ *so that*

$$(q_1 * q_2)^x(-iX^x) = q_1^x(-iX^x)q_2^x(-iX^x).$$

Remark. This convolution is local w.r.t. x, but is neither commutative, nor microlocal (i.e. local w.r.t. ξ).

Using the convolution for Heisenberg symbols we get:

Proposition. *Let* $Q_j \in \Psi_H^{m_j}(U)$, $j = 1, 2$. *Then* $Q = Q_1 Q_2$ *is a* $\Psi_H DO$ *of order* $m_1 + m_2$ *and we have*

$$q_{m_1+m_2} = q_{m_1} * q_{m_2} \text{ and } (Q_1 Q_2)^x = Q_1^x Q_2^x.$$

Heisenberg Calculus (Global Theory):

Proposition (Beals-Greiner, Taylor). *The class of $\Psi_H DO's$ of order m is invariant by the action of Heisenberg diffeomorphisms.*

This allows us to define Ψ_HDO's of order m on any Heisenberg manifold (M, H).

Proposition (RP '04). *To any $P \in \Psi_H^m(M)$ we can associate:*

- A model operator P^x at any point x as a left-invariant convolution operator on $G_x M$;

- A principal symbol as a homogeneous smooth function on $\mathfrak{g}^* M \setminus 0$.

Proposition. *For* $P \in \Psi_H^m(M)$ *TFAE:*

(i) The principal symbol of P *is invertible (w.r.t. convolution for Heisenberg symbols).*

(ii) P *admits a parametrix* $Q \in \Psi_H^{-m}(M)$ *(i.e. we have* $QP = PQ = 1 \bmod \Psi^{-\infty}(M)$*).*

Moreover, if (i) and (ii) hold then P *is hypoelliptic, i.e. we have Sobolev estimates*

$$\|u\|_{s+m/2} \leq C(\|Pu\|_s + \|u\|_s) \ \forall u \in C_c^\infty(M).$$

The main geometric operators on Heisenberg manifolds are hypoelliptic in the above sense, except the Kohn Laplacian which has an invertible principal symbol only outside forms of some given bidegrees.

III. Hypoelliptic Functional Calculus

In order to be able to make use of the NCG framework we shall now construct complex powers of hypoelliptic Ψ_HDO's.

Here we're facing a big technical hurdle: due to the non-microlocality of the Heisenberg calculus we cannot carry through Seeley's approach to the complex powers.

Instead we will rely on two new approaches:

1. Heat kernel approach: good for differential operators which are positive;

2. Almost homogeneous calculus: good for general hypoelliptic Ψ_HDO's.

• Holomorphic Families of Ψ_HDO's

Let $\Omega \subset \mathbb{C}$ be open.

Definition. *A family $(q_z)_{z\in\Omega}$ of symbols is holomorphic when:*

- The order m_z is an analytic function of z;

- (q_z) is a hol. family of smooth functions;

- The bounds of $q \sim \sum_{j\geq 0} q_{z,m_z-j}$ are locally uniform in z.

Definition. *A family (Q_z) of Ψ_HDO's is holomorphic if it is locally of the form*

$$Q_z = q_z(x, -iX) + R_z,$$

where (q_z) is a holomorphic family of symbols and (R_z) is a holomorphic family of smoothing operators.

• Heat Kernel Approach to Complex Powers

Let P be a differential operator of even Heisenberg order m such that:

- P is (semi-)positive, i.e. $\langle Pu, u\rangle \geq 0$;
- P has an invertible principal symbol.

Then for any $s \in \mathbb{C}$ we can define P^s by standard L^2-functional calculus.

Theorem (RP '04). *The above complex powers form a holomorphic family of Ψ_HDO's.*

The proof is quite easy (and again new). It relies on combining the Mellin formula,

$$P^{-s} = \Gamma(s)^{-1} \int_0^\infty t^s e^{-tP} \frac{dt}{t}, \quad \Re s > 0,$$

together with an extension of the pseudodifferential representation of the heat kernel of P due to Beals-Greiner-Stanton (JDG '84).

This result applies to all our semi-positive examples (Kohn Laplacian, contact Laplacian) and it has several interesting consequences:

- Fill a gap in the proof of Julg-Kasparov of BC conjecture for $SU(n,1)$ (complex powers of the contact Laplacian).

- Heat kernel asymptotics for hypoelliptic differential operators, extending results of Beals-Greiner-Stanton for (positive) sub-laplacians.

- Construction of weighted Sobolev spaces $W^s_H(M)$, $s \in \mathbb{R}$, interpolating Folland-Stein spaces at positive integers and providing us with sharp regularity estimates for Ψ_HDO's.

• $\Psi_H DO$ Representation of the Resolvent

We can give a pseudodifferential representation of the resolvent using an "almost homogeneous" $\Psi_H DO$ calculus with parameter.

In this context the parameter set is a *pseudocone*, i.e a subset $\Lambda \subset \mathbb{C} \setminus 0$ of the form $\Lambda = \Theta \cup B$ with Θ conical and B bounded.

Let Λ be an open pseudocone and let $p \in \mathbb{Z}$.

Definition. $\mathrm{Hol}^p(\Lambda)$ *is the space of functions* $f \in \mathrm{Hol}(\Lambda)$ *so that for any closed pseudocone* $\Lambda' \subset \Lambda$ *we have*

$$|f(\lambda)| \leq C_{\Lambda'}(1 + |\lambda|)^p, \quad \lambda \in \Lambda'.$$

Let $w \in \mathbb{N}^*$ and let U be a Heisenberg chart with H-frame $X_0, \ldots, X_d$. We define parametric Ψ_HDO's on U as follows.

• <u>Parametric symbols of order m:</u>

Symbols $q_{(\lambda)} \in C^\infty(U \times \mathbb{R}^{d+1}) \otimes \mathrm{Hol}^p(\Lambda)$ with an asymptotic exapansion,

$$q_{(\lambda)} \sim \sum_{j \geq 0} q_{(\lambda),m-j},$$

where:

- $q_{(\lambda),l} \in C^\infty(U \times (\mathbb{R}^{d+1} \setminus 0)) \otimes \mathrm{Hol}^p(\Lambda)$ is almost homogeneous of degree l, i.e. for any $t > 0$ we have

$$q_{(t^w\lambda),l}(x, t.\xi) - t^l q_{(\lambda),l}(x, \xi) \in \mathcal{S}(\mathbb{R}^{d+1}) \otimes \mathrm{Hol}^p(\Lambda);$$

- the sign $\sim$ means a symbol asymptotics whose bounds grow as $\mathrm{O}(|\lambda|^p)$ with λ.

• Parametric Ψ_HDO's of order m:

The class $\Psi_{H,w}^{m,p}(U)$ consists of families $Q_{(\lambda)}$ of the form,

$$Q_{(\lambda)} = q_{(\lambda)}(x, -iX) + R_{(\lambda)},$$

where $q_{(\lambda)}$ is a parametric symbol of order m and $R_{(\lambda)}$ belongs to $\Psi^{-\infty}(U) \otimes \mathrm{Hol}^p(\Lambda)$.

Proposition. *1) If $P_j \in \Psi_{H,w}^{m_j,p_j}(U)$, $j = 1, 2$, then $P_1 P_2 \in \Psi_{H,w}^{m_1+m_2,p_1+p_2}(U)$.*

2) The class $\Psi_{H,w}^{m,p}(U)$ is invariant by Heisenberg diffeomorphisms.

Thanks to the 2nd part we can define parametric Ψ_HDO's on any Heisenberg manifold.

Let P be a $\Psi_H DO$ of order m and let L be ray in $\mathbb{C} \setminus 0$.

Definition. *L is a principal cut for P if near any point $x \in M$ there are:*

- a Heisenberg chart U,

- an open pseudocone Λ containing L,

such that on $U \times \mathbb{R}^{d+1} \times \Lambda$ the principal symbol of $P - \lambda$ is invertible.

Remark. The definition depends only on the principal symbol of P and implies the invertibility of the principal symbol P.

Definition. $\Theta(P)$ *is the union set of all the principal cuts of* P.

The principal set $\Theta(P)$ is a cone which is open when M is compact.

Assume $\Theta(P)$ is not empty. Then:

Proposition (RP '04). *1) The spectrum of* P *consists in a discrete and unbounded set of eigenvalues.*
2) The intersection of $\operatorname{Sp} P$ *with any closed cone* $\Theta \subset \Theta(P)$ *is finite.*

Let $\bar{\Theta}(P)$ be the cone obtained from $\Theta(P)$ by removing from it all its rays that are through an eigenvalue of P and set

$$\Lambda(P) = \bar{\Theta}(P) \cup [D(0, R_0) \setminus 0].$$

where $R_P = \text{dist}(0, \text{Sp}\, P \setminus 0)$.

Theorem. *1)* $\Lambda(P)$ *is an open pseudocone.*
2) $(P - \lambda)^{-1}$ *belongs to* $\Psi_{H,m}^{-m,-1}(M; \Lambda(P))$*.*

Cayley Hamilton Decomposition:

Assume $\Theta(P)$ nonempty. The characteristic suspace and projector associated to an eigenvalue $\lambda \in \mathrm{Sp}\, P$ are

$$E_\lambda(P) = \cup_{k\geq 0} \ker(P-\lambda)^k,$$
$$\Pi_\lambda(P) = \int_{|\mu-\lambda|=r} (P-\mu)^{-1} d\mu,$$

where r is small enough to isolate λ from the rest of the spectrum.

Proposition (RP '04). *1) $\Pi_\lambda(P)$ is a smoothing operator which projects onto $E_\lambda(P)$ along $E_{\bar{\lambda}}(P^*)^\perp$.*

2) $E_\lambda(P)$ is a finite dimensional subspace of $C^\infty(M)$.

We extend the previous definitions to the eigenvalue $\lambda = \infty$ by letting

$$\Pi_\infty(P) = \lim_{R\to\infty} \int_{|\mu-\lambda|=R} \mu^{-1} P^{-1} (P-\mu)^{-1} d\mu,$$
$$E_\infty(P) = \operatorname{im} \Pi_\infty(P).$$

Theorem (RP '04). *For any $s \in \mathbb{R}$ we have*

$$L^2(M) = \overline{\dot{+}_{\lambda \in \mathrm{Sp} \cup \{\infty\}} E_\lambda(P)},$$
$$\sum_{\lambda \in \mathrm{Sp} \cup \{\infty\}} \Pi_\lambda(P) = 1.$$

This follows from a general result about the Cayley-Hamilton decomposition of compact operators and closed operators with compact resolvent on Hilbert spaces (RP' 04).

• <u>Partial Inverse:</u>

Consider the characteristic space,

$$E_{\bar{\mathbb{C}}\setminus 0}(P) = \overline{\dot{+}_{\lambda\in(\operatorname{Sp} T\cup\{\infty\})\setminus 0} E_\lambda(P)}.$$

Definition. *The partial of P, denoted P^{-1}, is the bounded operator on $L^2(M)$ that vanishes on $E_0(P)$ and inverts P on $E_{\bar{\mathbb{C}}\setminus 0}(P)$.*

Theorem (RP '04). *1) The partial inverse P^{-1} is a $\Psi_H DO$ of order $-m$.*

2) We have

$$PP^{-1} = P^{-1}P = 1 - \Pi_0(P),$$
$$(P^k)^{-1} = (P^{-1})^k, \quad k = 1, 2, \ldots.$$

In particular, P^{-1} is a parametrix for P.

• Complex Powers:

Let $L_\theta = \{\arg\lambda = \theta\}$ be a ray contained in $\Lambda(P)$. Then we can define a bounded operator on $L^2(M)$ by letting

$$P_\theta^s = \frac{1}{2i\pi}\int_{\Gamma_\theta} \lambda^s (P-\lambda)^{-1} d\lambda, \quad \Re s < 0.$$

Proposition. *We have*

$$P_\theta^{s_1+s_2} = P_\theta^{s_1} P_\theta^{s_2},$$
$$P_\theta^{-k}) = P^{-k}, \quad k = 1, 2, \ldots,$$

where P^{-k} denotes the partial inverse of P^k.

Proposition (RP '04). *The family* $(P_\theta^s)_{\Re s<0}$ *is a holomorphic family of* Ψ_H*DO's so that* $\text{ord} P_\theta^s = s.\text{ord} P$.

Thanks to this for $\Re s \geq 0$ we can directly defined P_θ^s as a $\Psi_H DO$ by letting

$$P_\theta^s = P^k P_\theta^{s-k},$$

where k is any integer $> \Re s$. Then we get:

Theorem (RP '04). *The family* $(P_\theta^s)_{s\in\mathbb{C}}$ *is an analytic 1-parameter group of* Ψ_H*DO's so that* $\text{ord} P_\theta^s = s.\text{ord} P$ *and*

$$P_\theta^1 = P, \quad P_\theta^{-1} = P^{-1}, \quad P_\theta^0 = 1 - \Pi_0(P).$$

• Consequences of this Approach:

- For $\theta = \pi$ yields uniform boundedness results along vertical stripes $a \leq \Re z b$ in terms of W_H^s-Sobolev spaces and the aperture of $\Lambda(P)$ around L_π.

- Existence and unicity result for the heat equation,

$$\begin{cases} \partial_t u(x,t) + P_x u(x,t) = v(x,t), \\ u(x,0) = u_0. \end{cases}$$

associated to W_H^s-data (v, u_0).

IV. Noncommutative Residue Trace

Let (M^{d+1}, H) be a compact Heisenberg manifold.

We can construct a noncommutative residue trace on Ψ_HDO's of integer order which is a complete analogue of the celebrated Wodzicki-Guillemin noncommutative residue trace.

• <u>Logarithmic Singularity</u>

Proposition (RP '01). *Let* $P \in \Psi_H^m(M)$*,* $m \in \mathbb{Z}$*. Then:*

1) Near the diagonal the kernel $k_P(x,y)$ *of* P *has a behavior near the diagonal of the form*

$$k_P(x,y) = \sum_{j=-(m+d+2)}^{-1} a_j(x,\psi_x(y)) \\ - c_P(x) \log \|\psi_x(y)\| + \mathrm{O}(1),$$

where $a_j(x,\lambda.z) = \lambda^j a_j(x,z)$*.*

2) The coefficient $c_P(x)$ *makes sense globally as a density on* M*.*

• <u>Analytic Extension of the Trace:</u>

If ord$P < -(d+2)$ then P is traceable and we have

$$\text{Tr}\, P = \int_M k_P(x,x).$$

Using an analytic continuation of the map $P \to k_P(x,x)$ we get:

Theorem (RP '01). *1) The trace has a unique analytic continuation* $P \to \text{TR}\, P$ *to* $\Psi_H^{\mathbb{C}\setminus\mathbb{Z}}(M)$.

2) $\text{TR}[P_1, P_2] = 0$ *when* $\text{ord}P_1 + \text{ord}P_2 \notin \mathbb{Z}$.

Let P be a $\Psi_H DO$ of integer order m. Then TR has essentially a pole at P:

Theorem (RP '01). *Let (P_z) be a holomorphic family of Ψ_HDO's such that:*

- $P_0 = P$;

- $\text{ord} P_z = z + m$.

Then the map $z \rightarrow \text{TR}\, P_z$ has a simple pole at $z = 0$ in such way that

$$\text{res}_{z=0}\, \text{TR}\, P_z = -\int_M c_P(x).$$

Definition. *The noncommutative residue of* $P \in \Psi_H^{\mathbb{Z}}(M)$ *is*

$$\text{Res}\, P = \int_M c_P(x).$$

From the construction of the functional Res we obtain:

Proposition (RP '01, '04). *1)* Res *is a local functional, i.e. is given by integration of a density.*

2) We have

$$\text{ord}Q.\,\text{Res}\, P = \text{res}_{z=0}\, \text{TR}\, PQ_\theta^{-z}$$

for any positive order $\Psi_H DO$ Q *with principal cut* L_θ *.*

3) Res *is trace, i.e.* $\text{Res}[P_1, P_2] = 0$*.*

On the other hand, a famous result of Wodzicki ('84) and Guillemin ('93) is:

Theorem. *If M is connected then any trace on $\Psi(M)$ is proportional to the Wodzicki-Guillemin noncommutative residue* Res.

In the Heisenberg setting we get:

Theorem (RP '04). *If M is connected then any trace on $\Psi_H^{\mathbb{Z}}(M)$ is proportional to the (Heisenberg) noncommutative residue* Res.

• Some Consequences of this Construction:

- Short proof of a result of Hirachi ('04) on the logarithmic singularity of the Szegö kernel.

- Zeta functions and spectral asymmetry of hypoelliptic ΨDO's and connection with a conjecture of Fefferman-Hirachi.

- Heat kernel asymptotics for general hypoelliptic Ψ_HDO's (via inverse Mellin transform).

- Conformal variations of spectral invariants associated to the conformal powers of the Δ_b-sublaplacian on a CR manifold (recently constructed by Gover-Graham ('03)).

- Area of 3-dimensional CR manifold.

• Quantized Calculus (Connes)

Let $\mathcal{H}$ be a Hilbert space. Then the following table of equivalences holds.

Classical Infinitesimal	Quantized Calculus
Complex Variable	Operator on $\mathcal{H}$
Real Variable	Selfadjoint Operator
Infinitesimal Variable	Compact Operator
Infinites. of order α	Compact Operator s.t. $\mu_k(T) = \mathrm{O}(k^{-\alpha})$
Integral $\int f(x)dx$	Dixmier Trace $⨍T$

Here:

- $\mu_k(T)$ is the $(k+1)$'th characteristic value of the compact operator T, that is

$\mu_k(T) = (k+1)$'th eigenvalue of $|T| = \sqrt{T^*T}$.

- The Dixmier trace is defined on infinitesimal operators of order ≥ 1 and such that, for $T \geq 0$,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k\leq N} \mu_k(T) = L \Rightarrow ⨍ T = L.$$

Thus ⨍ vanishes on trace-class operators, hence on infinitesimal operators of order > 1 (like the integral).

Theorem (Connes '88). *Let M be a compact manifold and let P be a ΨDO on M of negative order $-m$.*

1) P is an infinitesimal operator of order $\frac{m}{\dim M}$.

2) If $\operatorname{ord} P = -\dim M$, then

$$⨍ P = \frac{1}{\dim M} \operatorname{Res} P,$$

where Res *denotes the Wodzicki-Guillemin noncommutative residue trace.*

Theorem (RP '01). *Let* (M, H) *be a compact Heisenberg manifold and let* P *be a* $\Psi_H DO$ *on* M *of negative order* $-m$.

1) P *is an infinitesimal operator of order* $\frac{m}{\dim M+1}$.

2) If $\text{ord} P = -(\dim M + 1)$, *then*

$$⨍ P = \frac{1}{\dim M + 1} \operatorname{Res} P.$$

Consequence:

We can integrate any $\Psi_H DO$, even when it is not an infinitesimal operator of order ≥ 1, just by letting

$$⨍ P = \frac{1}{\dim M + 1} \operatorname{Res} P.$$

• <u>Area of a CR manifold:</u>

Let (M^{2n+1}, θ) be a pseudohermitian manifold and let Δ_b be its sublaplacian.

For any $f \in C^\infty(M)$ we have

$$⨍ f \Delta_b^{-(n+1)} = \int_M f(x)(d\theta)^n \wedge \theta.$$

Thus $ds = \sqrt{\Delta_b}$ recaptures the contact volume. This leads us to define

$$\text{Area}_\theta\, M = ⨍ ds^2.$$

Theorem (RP '01). *If* $\dim M = 3$ *then*

$$\text{Area}_\theta M = \int\!\!\!\!\!- ds^2 = \int_M r_M(x) d\theta \wedge \theta,$$

where r_M *denotes the Tanaka-Webster scalar curvature of* M.

Example. For $S^3 \subset \mathbb{C}^2$ we get

$$\text{Area}_\theta S^3 = \frac{\pi^2}{2\sqrt{2}}$$

Potential Application: Should yield a spectral interpretation of the Einstein-Hilbert action of a Lorentzian manifold with Fefferman metric.

VI. Local Index Formula for Heisenberg Manifold

• The Local Index Formula in NCG

The local index formula ultimately holds in a purely operator theoretic setting (Connes-Moscovici '95).

This allows us to recover the Atiyah-Singer index formula for Dirac operators, as well as many other examples.

This framework uses two main tools:

- Spectral triples;
- Cyclic cohomology.

• Spectral Triples:

A spectral triple is a triple $(\mathcal{A}, \mathcal{H}, D)$ where:

- $\mathcal{H}$ is a Hilbert space together with a $\mathbb{Z}_2$-grading $\gamma : \mathcal{H}_+ \oplus \mathcal{H}_- \to \mathcal{H}_- \oplus \mathcal{H}_+$;

- $\mathcal{A}$ is an involutive unital algebra represented in $\mathcal{H}$ and commuting with the $\mathbb{Z}_2$-grading γ;

- D is a selfadjoint unbounded operator on $\mathcal{H}$ s.t. $[D, a]$ is bounded $\forall a \in \mathcal{A}$ and of the form,

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \qquad D_\pm : \mathcal{H}^\mp \to \mathcal{H}^\pm.$$

Spectral Triple

The datum of D above defines an index map $\text{ind}_D : K_0(\mathcal{A}) \to \mathbb{Z}$ so that

$$\text{ind}_D[e] = \text{ind}\, eD^+e,$$

for anyselfadjoint idempotent $e \in M_q(\mathcal{A})$.

- p-summability:

We say that D is p-summable when we have

$$\mu_k(D^{-1}) = \text{O}(k^{-1/p}) \quad \text{as } k \to +\infty,$$

- Dimension spectrum:

Let $\Psi^0_D(\mathcal{A})$ be the algebra generated by the $\mathbb{Z}_2$-grading γ and the $\delta^k(a)$'s, $a \in \mathcal{A}$, where δ is the derivation $\delta(T) = [|D|, T]$ (assuming $\mathcal{A}$ is contained in $\cap_{k\geq 0} \operatorname{dom} \delta^k$).

Definition. *The dimension spectrum of* $(\mathcal{A}, \mathcal{H}, D)$ *is the union set of the singularities of all the zeta functions ,*

$$\zeta(P; z) = \operatorname{Tr} P|D|^{-z}, P \in \Psi^0_D(\mathcal{A}).$$

- Noncommutative residue trace:

Assuming p-summability and simple and discrete dimension spectrum we define a noncommutative residue trace on $\Psi_D^0(\mathcal{A})$ by letting:

$$\fint P = \mathrm{Res}_{z=0} \,\mathrm{Tr}\, P|D|^{-z} \quad \text{for } P \in \Psi_D^0(\mathcal{A}).$$

This functional is local in the sense of noncommutative geometry since it vanishes on the elements of $\Psi_D^0(\mathcal{A})$ that are traceable.

Theorem (Connes-Moscovici '95). *Suppose that $(\mathcal{A}, \mathcal{H}, D)$ is p-summable and has a discrete and simple dimension spectrum.*

1) The following formulas define an even cocycle $\varphi_{\text{CM}} = (\varphi_{2k})$ in the (b, B)-complex of the algebra $\mathcal{A}$.

- For $k = 0$,

$$\varphi_0(a^0) = \text{finite part of } \operatorname{Tr} \gamma a^0 e^{-tD^2} \text{ as } t \to 0^+,$$

- For $k \neq 0$,

$$\varphi_{2k}(a^0, \ldots, a^{2k}) = \sum_{\alpha} c_{k,\alpha} ⨍ \gamma P_{k,\alpha} |D|^{-2(|\alpha|+k)},$$

$$P_{k,\alpha} = a^0 [D, a^1]^{[\alpha_1]} \ldots [D, a^{2k}]^{[\alpha_{2k}]},$$

where the $c_{k,\alpha}$'s are universal rational constants and the symbol $T^{[j]}$ denotes the j'th iterated commutator with D^2.

2) We have:

$$\operatorname{ind}_D[e] = \langle [\varphi_{\text{CM}}], e \rangle \quad \forall e \in K_0(\mathcal{A}).$$

where $\langle ., . \rangle$ is the pairing of cyclic cohomology with K-theory.

• Index Formula for Heisenberg manifolds:

Here we consider:

- A (compact) Heisenberg manifold (M, H);

- A $\mathbb{Z}_2$-graded bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ over M with grading γ;

- An operator $D \in \Psi_H^1(M, \mathcal{S})$ which anti-commutes with γ and such that the complex powers of D^2 form a holomorphic family of Ψ_HDO's;

Using the complex powers of D^2 and the non-comutative residue for the Heisenberg calculus we get:

Proposition (RP '01). *The spectral triple,*

$$(C^\infty(M), L^2(M, \mathcal{S}), D)$$

is $(d+2)$*-summable and has a simple dimension spectrum contained in*

$$\{k \in \mathbb{Z};\ k \leq \dim M + 1\}.$$

Therefore, the theorem of Connes and Moscovici applies.

In this setting the index formula has a geometric description as follows.

Let $\mathcal{E}$ be a Hermitian vector bundle together with a unitary connection ∇. Then:

a) Since here $\mathcal{A} = C^\infty(M)$ the CM cocycle is actually a current C_D whose pairing with the K-theory class of $\mathcal{E}$ is given by

$$\langle [C_D], [\mathcal{E}] \rangle = \langle C_D, \mathrm{Ch}\, F^{\mathcal{E}} \rangle,$$

where $\mathrm{Ch}\, F^{\mathcal{E}} = \mathrm{Tr}\, e^{-F^{\mathcal{E}}}$ is the total Chern form of the curvature $F^{\mathcal{E}}$ of $\mathcal{E}$.

b) Using ∇ we can twist D into $D_{\nabla,\mathcal{E}}$ given by the composition

$$\Gamma(\mathcal{S}\otimes\mathcal{E}) \overset{1_{\mathcal{E}}\otimes\nabla}{\longrightarrow} \Gamma(\mathcal{S}\otimes T^*M\otimes\mathcal{E}) \overset{\pi_D\otimes 1_{\mathcal{E}}}{\longrightarrow} \Gamma(\mathcal{S}\otimes\mathcal{E}),$$

$$\pi_D[(f^0 df^1)\otimes\sigma] = f^0[D,f^1]\sigma.$$

This operator anticommutes with the grading of $\mathcal{S}\otimes\mathcal{E}$ and we have

$$\mathrm{ind}_D[\mathcal{E}] = \mathrm{ind}\, D^+_{\nabla,\mathcal{E}}.$$

Therefore, we obtain:

Theorem (RP '01). *1) There exists an even de Rham current C_D on M such that for any Hermitian vector bundle $\mathcal{E}$ over M with unitary connection ∇ we have*

$$\operatorname{ind} D^{+}_{\nabla,\mathcal{E}} = \langle C_D, \operatorname{Ch} F^{\mathcal{E}} \rangle.$$

2) The components C_{2k}, $k = 0, 2, \ldots$, of C_D are given by the following formulas.

- For $k = 0$,

$$\langle C_0, f^0 \rangle = \int_M f_0(x) \operatorname{Str}_{\mathcal{S}} a_0(D^2, x),$$

- For $k \neq 0$,

$$\langle C_{2k}, f^0 df^1 \wedge \ldots \wedge df^{2k} \rangle = \sum_{\alpha} c_{k,\alpha} ⨍ \gamma P_{k,\alpha} |D|^{-2(|\alpha|+k)},$$

$$P_{k,\alpha} = f^0 [D, f^1]^{[\alpha_1]} \ldots [D, f^{2k}]^{[\alpha_{2k}]},$$

where $⨍$ denotes the noncommutative residue for the Heisenberg calculus.

• Geometric Operators

There is an obstruction to constructing significant geometric operators for which the previous index formula applies. Namely, we cannot transplant into the CR and contact settings the Dirac construction used by Aitiyah-Singer, for it yields non-hypoelliptic operators. However, using a construction inspired by that of the mixed transverse signature operator of Connes-Moscovici (GAFA '95), we can build hypoelliptic signature-type operators out of:

- the Kohn-Rossi complex (RP' 01);

- Rumin's contact complex (RP unpublished).

The actual computation of the corresponding currents is still in progress.