A Atiyah-Singer-type index theorem for manifolds with corners

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Joint work with V. Nistor

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Introduction

In Noncommutative Geometry: pseudodifferential calculus $\longleftrightarrow$ groupoid

Background: index theorem on foliations (Connes, Skandalis)

Problem: apply Connes' approach to singular manifolds: understand the index theory on singular manifolds in terms of operators algebras, using groupoids methods.

Scheme: to define a pseudodifferential calculus, define a groupoid and use the general tools developed for the pseudodifferential calculus on a groupoid.

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What does the title mean??

Atiyah-Singer's Index Theorem for closed manifolds : embed M in $\mathbb{R}^n$, then

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{=} & \mathbb{Z} \\ {\scriptstyle ind_a^M}\uparrow & & \simeq\uparrow{\scriptstyle ind_a^{\mathbb{R}^n}} \\ K^0(TM) & \xrightarrow{i_!} & K^0(T\mathbb{R}^n) \end{array}$$

Index theorem: $ind_a^M = ind_a^{\mathbb{R}^n} \circ i_!$

For manifolds with corners: what is the *analytic index*?

Pseudodifferential calculus on groupoids

- G is a Lie groupoid (more generally a continuous family groupoid) $\rightsquigarrow$ algebra of pseudodifferential operators $\Psi^\infty(G)$

- pseudodifferential operator on G: G-equivariant continuous family of pseudodifferential operators on the fibers of G

Example:

- If M is a manifold without boundary, and $G = M \times M$, $\Psi^\infty(G)$ is the algebra of pseudodifferential operators on M.

- If M is a manifold with corners, there exists a groupoid $G(M)$ such that $\Psi^\infty(G(M))$ is the b-calculus of Melrose.

$$G(M) = \{(x, y, \lambda) \in M \times M \times \mathbb{R}_+^*, \rho(x) = \lambda\rho(y)\}$$

ρ : defining function of ∂M.

Atiyah-Singer exact sequence

$$0 \longrightarrow C^*(G) \longrightarrow \overline{\Psi^0}(G) \xrightarrow{\sigma} C(S^*(G)) \longrightarrow 0$$

($S^*(G)$: cosphere bundle of the Lie algebroid $A(G)$)

Theorem 1 *The **analytic index***

$$Ind_a : K^0(A^*(G)) \longrightarrow K_0(C^*(G))$$

is induced by the tangent groupoid $G \times]0,1] \cup A(G) \times \{0\}$.

Think of $A(G)$ as the tangent space.

Tools

$G\times]0,1]$ is open and saturated $\Rightarrow$

$$0 \to C^*(G\times]0,1]) \to C^*(G^T) \to C_0(A(G)) \to 0$$

$$\Rightarrow K_*(C^*(G^T)) \underset{\sim}{\xrightarrow{e_0}} K^*(A^*(G))$$

and $ind_a = e_1 \circ e_0^{-1} : K^*(A^*(G)) \to K_*(C^*(G))$.

Remark This is not a Fredholm index!

Define an embedding of manifolds with corners $i : M \to X$, and a commutative diagram:

$$\begin{array}{ccc} K_*(C^*(G(M))) & \xrightarrow{\simeq} & K_*(C^*(G(X))) \\ {\scriptstyle ind_a^M}\uparrow & & \simeq\uparrow{\scriptstyle ind_a^X} \\ K^*(A(G(M))) & \xrightarrow{i_!} & K^*(A(G(X))) \end{array}$$

Embedding

The embedding $i : M \to X$ has to be such that:

- it is an embedding of manifolds with corners
- $K_*(C^*(G(M))) \simeq K_*(C^*(G(X)))$: $G(X) \sim G(M)$ sufficient
- ind_a^X is an isomorphism

Construction of X such that:

- each open face of X is contractible,
- each face of M is the intersection of M and of a face of X transverse to M,
- there is a bijection between the open faces of M and those of X.

Commutative diagram

$$\begin{array}{ccc} K_*(C^*(G(M))) & \xrightarrow{\simeq} & K_*(C^*(G(X))) \\ {\scriptstyle ind_a^M}\uparrow & & \simeq\uparrow{\scriptstyle ind_a^X} \\ K^*(A(G(M))) & \xrightarrow{i_!} & K^*(A(G(X))) \end{array}$$

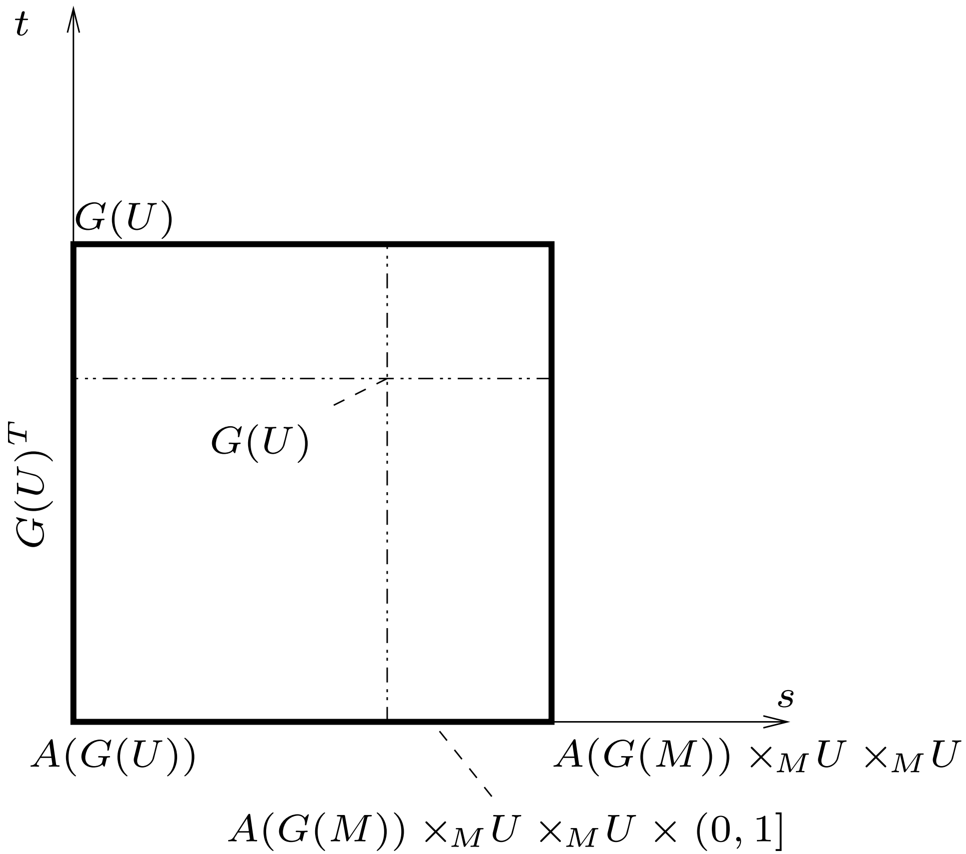
ind_a can be defined through a deformation:
$G^T(M) = A(G(M)) \times \{0\} \cup G(M) \times (0, 1]$.

Like in Atiyah-Singer: use U a tubular neighborhood of M in X.

Two steps: diagram for the submersion $U \longrightarrow M$, then diagram for $U \longrightarrow X$.

Commutative diagram

To obtain the first commutative diagram: get a *double* deformation.



Idea: $U \longrightarrow M$ fibration with smooth fibers, deform $T_{vert}U$ in $U \times_M U$ ($t = 0$), then deform $T_{hor}U = T_M$ in $G(M)$.

$\mathcal{G}_{s=1}$ is equivalent to $G(M)$.

Commutative diagram

Second step: U is open in X, so that $G^T(U) = G^T(X)_U^U$ is open in $G^T(X)$.

$$\begin{array}{ccc} C^*(G(U)) & \hookrightarrow & C^*(G(X)) \\ e_1 \uparrow & & \uparrow e_1 \\ C^*(G(U)^T) & \hookrightarrow & C^*(G(X)^T) \\ e_0 \downarrow & & \downarrow e_0 \\ C_0(A(G(U))) & \hookrightarrow & C_0(A(G(X))) \end{array}$$

Commutative diagram

Everything put together:

$$
\begin{array}{ccccc}
K_*(C^*(G(M))) & \xleftarrow{\simeq} & K_*(C^*(G(U))) & \longrightarrow & K_*(C^*(G(X))) \\
ind_a^M \uparrow & & ind_a^U \uparrow & & \uparrow ind_a^X \\
K^*(A(G(M))) & \xleftarrow[\text{Thom}]{\simeq} & K^*(A(G(U))) & \longrightarrow & K^*(A(G(X)))
\end{array}
$$