AUTOMORPHIC PSEUDO-DIFFERENTIAL OPERATORS

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Joint work with D. Zagier, Y. Manin, Max-Planck-Institut für Mathematik
Reference: in Algebraic Aspects of Integrable Systems: Progress in
Non-linear Diff. Equ. Appl., Volume 26, Birkhäuser, 1997.

Related to: Lectures of Henri Moscovici at this same NCGOA meeting

For recent developments of this work in the *classical* direction, especially to generalizing to modular groups acting on higher dimensional spaces, see papers of Min Ho Lee: http://www.math.uni.edu/ lee/pub.html. He has, for example, developed the Hilbert modular case. Also, Olav Richter's work on Rankin-Cohen brackets: http://www.math.unt.edu/ richter/. Work of Conley on 1/2-integral weight: http://www.math.unt.edu/ conley/publications.htm.

Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

The group Γ acts (projectively) on the complex upper half plane $\mathcal{H}$:

let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$, be an element of $\mathrm{SL}_2(\mathbb{R})$,

$$\gamma : z \in \mathcal{H} \mapsto \frac{az+b}{cz+d} \in \mathcal{H}$$

$$\text{since } \operatorname{Im}(\gamma(z)) = \frac{\operatorname{Im}(z)}{|cz+d|^2} > 0 \text{ as } \operatorname{Im}(z) > 0.$$

Examples: $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Every $z \in \mathcal{H}$ corresponds to a lattice $\mathcal{L}$ in $\mathbb{C}$:

$$z \in \mathcal{H} \mapsto \mathcal{L} = \mathcal{L}_z = \mathbb{Z}z + \mathbb{Z}$$

This induces a bijection

$$\Gamma \backslash \mathcal{H} \simeq \text{isomorphism classes of complex one dimensional tori}$$
$$z \mod \Gamma \mapsto \mathbb{C} \mod \mathcal{L}$$

on the left: fractional linear transformation; on the right: translation.

Let

$$g_2(z) = g_2(\mathcal{L}_z) = 60 \sum_{(m,n) \neq (0,0) \in \mathbb{Z}^2} (mz+n)^{-4}$$
$$g_3(z) = g_3(\mathcal{L}_z) = 140 \sum_{(m,n) \neq (0,0) \in \mathbb{Z}^2} (mz+n)^{-6},$$

then $\mathbb{C}/\mathcal{L}_z \simeq \mathcal{E}(\mathbb{C})$: with $\mathcal{E}$ having affine locus the solutions of:

$$y^2 = 4x^3 - g_2(z)x - g_3(z).$$

Congruence subgroups of Γ: ($\mathrm{SL}_2(\mathbb{Z})$). Let $N \geq 1$ be an integer.

$$\Gamma_0(N) = \left\{ \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$$

$$\Gamma_1(N) = \left\{ \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

$$\Gamma(N) = \left\{ \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

$*$ signifies the absence of a congruence condition.

$\Gamma(N)$ normal in Γ, but $\Gamma_0(N)$, $\Gamma_1(N)$ are not; $\Gamma_1(N)$ normal in $\Gamma_0(N)$.

Let Γ' denote any one of these groups.

Modular forms:
"functions" on lattices, so "functions" on $\Gamma' \backslash \mathcal{H}$ – the *modular points*.

Lattice: $\mathcal{L} = \mathcal{L}_\omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \qquad \omega_2/\omega_1 \in \mathcal{H}$.

Reference:
Introduction to Elliptic Curves and Modular Forms,
N. Koblitz, GTM 97, Springer (2nd Edition).

GROUP	*MODULAR POINT*
(i) $\Gamma = \mathrm{SL}_2(\mathbb{Z})$;	*Lattice* $\mathcal{L} \subseteq \mathbb{C}$
(ii) $\Gamma_1(N)$;	$(\mathcal{L}, t)$, t *a point of order exactly* N *in* $\mathbb{C}/\mathcal{L}$.
(iii) $\Gamma_0(N)$;	$(\mathcal{L}, S)$, S *a cyclic subgroup of order* N *in* $\mathbb{C}/\mathcal{L}$.
(iv) $\Gamma(N)$;	$(\mathcal{L}, \{t_1, t_2\})$, t_1, t_2 *basis of points of order exactly* N *in* $\mathbb{C}/\mathcal{L}$.

For (i) through (iv) consider

$$F : \{\textit{Modular Points}\} \to \mathbb{C}$$

of "weight" k (where k will be an integer or half-integer).

For $\lambda \in \mathbb{C}$, $\lambda \neq 0$, let $\lambda\mathcal{L} = \{\lambda\ell : \ell \in \mathcal{L}\}$, then $\lambda t \in \mathbb{C}/\lambda\mathcal{L}$; $\lambda S \subseteq \mathbb{C}/\lambda\mathcal{L}$. Require:

(i) $F(\lambda\mathcal{L}) = \lambda^{-k} F(\mathcal{L})$

(ii) $F((\lambda\mathcal{L}, \lambda t)) = \lambda^{-k} F((\mathcal{L}, t))$

(iii) $F((\lambda\mathcal{L}, \lambda S)) = \lambda^{-k} F((\mathcal{L}, S))$

(iv) $F((\lambda\mathcal{L}, \{\lambda t_1, \lambda t_2\}))) = \lambda^{-k} F((\mathcal{L}, \{t_1, t_2\}))$

Correspondence: $F \to \widetilde{F} \to f$

Lattice: $\mathcal{L} = \mathcal{L}_\omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \qquad \omega_2/\omega_1 \in \mathcal{H}$.

$\widetilde{F}$ function on vectors $\omega = {}^t(\omega_1, \omega_2)$ with $\omega_2/\omega_1 \in \mathcal{H}$.

f function on $\mathcal{H}$

Define:

(i) $\widetilde{F}(\omega) = F(\mathcal{L}_\omega)$

(ii) $\widetilde{F}(\omega) = F((\mathcal{L}_\omega, \frac{\omega_1}{N}))$

(iii) $\widetilde{F}(\omega) = F((\mathcal{L}_\omega, \mathbb{Z}\frac{\omega_1}{N}))$

(iv) $\widetilde{F}(\omega) = F((\mathcal{L}_\omega, \{\frac{\omega_1}{N}, \frac{\omega_2}{N}\}))$

In all cases, define

$$f : \mathcal{H} \to \mathbb{C}$$

by

$$f(z) = \widetilde{F}({}^t(1, z)).$$

Then: F a function of weight k on modular points implies,

$$f(\gamma(z)) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'.$$

We write this as,

$$(f \mid_k \gamma)(z) := (cz+d)^{-k} f(\gamma(z)) = f(z).$$

Fact: $k \in \mathbb{Z}$: Bijections: $F \leftrightarrow \widetilde{F} \leftrightarrow f$ with:

(1) F function of weight k on modular points;

(2) $\widetilde{F}$ function on vectors ω with $\omega_2/\omega_1 \in \mathcal{H}$ and $\widetilde{F}(\gamma(\omega)) = \widetilde{F}(\omega)$; $\widetilde{F}(\lambda\omega) = \lambda^{-k} F(\omega)$, $\lambda \in \mathbb{C}, \lambda \neq 0$;

(3) f function on $z \in \mathcal{H}$ with $(f \mid_k \gamma)(z) = f(z)$.

Why study congruence subgroups?

Lots of reasons, mainly arithmetic. Let's mention one.

Kronecker Jugendtraum and Hilbert's twelfth problem.
(Reference – for lots of good stuff!: website of J. Milne,
http://www.jmilne.org/math/index.html)

Kronecker's dream: Every abelian extension of $\mathbb{Q}$ is contained in the field obtained by adjoining to $\mathbb{Q}$ all values of the function $\exp(2\pi i z)$ for $z \in \mathbb{Q}, z \neq 0$, and every abelian extension of a quadratic imaginary field K is contained in the field obtained by adjoining to K all values of the function $j(z)$ for $z \in K$, $z \neq 0$, where $j(z)$ is the classical elliptic modular function (of weight zero) for $\mathrm{SL}_2(\mathbb{Z})$.

Actually, the above statement is not quite correct as one needs also values of elliptic functions at points of finite order of the corresponding lattice. For a precise statement, see Milne's website referenced above.

Hilbert's twelfth problem: For any number field K, find functions that play the role for K that the exponential function plays for $\mathbb{Q}$ and the modular function j plays for a quadratic imaginary fields, (see Milne website).

The appearance of points of finite order of lattices and of the values $j(\sqrt{-N})$ for $N > 0$ a non-square integer, leads naturally to the appearance of congruence subgroups. For example, there is a natural dependence relation $\Phi_N(j(Nz), j(z)) \equiv 0$ with $\Phi_N(X, Y) \in \mathbb{Z}[X, Y]$ giving the affine equation for the modular curve $\Gamma_0(N)\backslash\mathcal{H}$ and we have $\Phi_N(j(\sqrt{-N}), j(\sqrt{-N})) = 0$. We saw above relations between congruence subgroups and points of finite order.

Why study Elliptic Curves and Modular Forms?

Lots of reasons: basic in arithmetic, geometry, topology...this meeting...

Manin–Marcolli: $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathcal{H}$ has a non-commutative boundary $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{P}_1(\mathbb{R})$.

The non-commutative boundary corresponds to Morita equivalence classes of real 2-dimensional non-commutative tori.

Connes–Moscovici: Modular *Hecke* algebras: see Moscovici lectures: have associative multiplications on these algebras via Rankin–Cohen brackets. Uses P.B.C.- Manin–Zagier: multiplication from automorphic pseudo-differential operators.

Automorphic Pseudo-differential Operators

Now let Γ be any discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ acting projectively on $\mathcal{H}$.

Let z be a variable with values in $\mathbb{C}$.

Let $\partial = \frac{\partial}{\partial z}$ be differentiation with respect to z.

Co-ordinate change: $\widetilde{z} = \widetilde{z}(z)$, with $\widetilde{\partial} = \frac{\partial}{\partial \widetilde{z}}$, then $\partial = \partial(\widetilde{z})\widetilde{\partial}$.

(R, ∂): ring of functions on $\mathbb{C}$ with a (not necessarily closed) action of ∂.

Example:
R is a ring of functions $f : \mathcal{H} \to \mathbb{C}$ with $(f \mid_k \gamma)(z) = f(z)$, $\gamma \in \Gamma$, f "meromorphic at infinity":

$$f = f(z) = \sum_{n\in\mathbb{Z}} a_n \exp(2\pi i n z), \qquad a_n = 0, \quad n \geq -M, \text{ some } M \geq 0.$$

We say then that f is a *modular function of weight* k. Just *modular function* usually means that $k = 0$.

If also f "holomorphic at infinity": $a_n = 0$, $n < 0$, we say that f is a *modular form of weight* k: $f \in M_k(\Gamma)$.

Formal pseudo-differential operators, ∂^{-1} formal inverse of ∂:

$$\psi DO(R) := \left\{ \sum_{n \in \mathbb{Z}} h_n \partial^{-n} : h_n \in R, h_n = 0 \quad n \geq -M, \text{ some } M \geq 0 \right\}$$

$\psi DO(R)$: Laurent series in ∂^{-1} with coefficients in R.

Differential operators: $DO(R)$ are $\psi DO(R)$'s with $h_n = 0$ for $n > 0$.

$\psi DO(R)$'s form a ring (extend Leibniz's rule on $DO(R)$'s):

$$\left(\sum_n g_n \partial^{-n} \right) \left(\sum_m h_m \partial^{-m} \right) = \sum_{n,m} \sum_{r \geq 0} \binom{-n}{r} g_n \partial^r (h_m) \partial^{-n-m-r}.$$

Extended binomial coefficient: $w \in \mathbb{C}, \ell \in \mathbb{N}$,

$$\binom{w}{\ell} = \frac{w(w-1)\dots(w-\ell+1)}{\ell!}.$$

$\psi DO(R)$ of weight k (assumed an integer here, could be complex number):

$$\psi DO(R)_k = \left\{ \sum_{n=0}^{\infty} f_n \partial^{k-n}, \quad f_n \in R \right\}.$$

$$\psi DO(R)_{k_1} \cdot \psi DO(R)_{k_2} \subseteq \psi DO(R)_{k_1+k_2}.$$

Note:

$$\psi DO(R) = \psi DO(R)_{-1} \oplus DO(R).$$

Short exact sequence: denote by (S)

$$0 \to \psi DO(R)_{-k-1} \to \psi DO(R)_{-k} \to R \to 0 \qquad (S)$$

The last arrow is the map: $\sum_{m\geq 0} f_m \partial^{-k-m} \mapsto f_0$.

Action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$:

$$\widetilde{z} = \gamma(z) = \frac{az+b}{cz+d}, \quad \widetilde{\partial} = \frac{\partial}{\partial \widetilde{z}} = (cz+d)^2 \partial.$$

Therefore,

$$\widetilde{\partial}^k = \sum_{n=0}^{\infty} n! \binom{k}{n} \binom{k-1}{n} c^n \, (cz+d)^{2k-n} \, \partial^{k-n}$$

Have an induced action of Γ on $\psi DO = \psi DO(R)$.

The induced action of Γ *on* $f_0 \in R$ *in the above sequence (S) is:*

$$f_0 \mapsto (f_0 \mid_{2k} \gamma)(z) = (cz+d)^{-2k} \, f_0 \left(\frac{az+b}{cz+d} \right).$$

$M_k = M_k(\Gamma) = M_k(R, \Gamma)$: $f \in R$ invariant under $f \mapsto f \mid_k \gamma$, $\gamma \in \Gamma$.

Take Γ-invariants in exact sequence (S):

$$0 \to \psi DO(R)^{\Gamma}_{-k-1} \to \psi DO(R)^{\Gamma}_{-k} \to M_{2k}(\Gamma) \to 0 \qquad (S)_{\Gamma}$$

We proved the sequence $(S)_{\Gamma}$ splits canonically to give a lift:

$$M_{2k}(\Gamma) \to \psi DO^{\Gamma}_{-k}.$$

More precisely:

Theorem: (PBC – Manin – Zagier)
For $k \geq 1$ define $\mathcal{L}_k : R \to \psi DO(R)_{-k}$, by

$$\mathcal{L}_k(f) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+k)!(n+k-1)!}{n!(n+2k-1)!} f^{(n)} \partial^{-k-n},$$

and $\mathcal{L}_{-k} : R \to DO(R)_k$, by

$$\mathcal{L}_{-k}(f) = \sum_{n=0}^{k-1} \frac{(2k-n)!}{n!(k-n)!(k-n-1)!} f^{(n)} \partial^{k-n},$$

with $\mathcal{L}_0(f) = f$. Then,

$$\mathcal{L}_k\left(f \mid_{2k} \gamma\right) = \mathcal{L}_k(f) \circ \gamma, \qquad \gamma \in \Gamma, \qquad k \in \mathbb{Z},$$

and therefore if $f \in M_{2k}(\Gamma)$, then $\mathcal{L}_k(f) \in \psi DO^{\Gamma}_{-k}$.

Why? Answer: Casimir Operator.

There is an action of the Lie algebra $\mathrm{sl}_2(\mathbb{C})$ on ψDO's by commutation.

Generators: $L_j = z^{j+1}\partial$, $j=-1,0,1$ acting by ($k \in \mathbb{Z}$):

$$L_{-1}\left(f\partial^k\right) = [\partial, f\partial^k] = f'\partial^k,$$

$$L_0\left(f\partial^k\right) = \left(zf' - kf\right)\partial^k,$$

$$L_1\left(f\partial^k\right) = \left(z^2 f' - 2kzf\right)\partial^k - k(k+1)zf\partial^{k-1}.$$

Casimir operator: $C = L_0^2 - \frac{1}{2}\left(L_1L_{-1} + L_{-1}L_1\right)$, trivial on functions.

We have: $C(f\partial^k) = k(k+1)f\partial^k + k(k-1)f'\partial^{k-1}$.

Casimir acts by $k(k+1)$ on $\psi DO_k/\psi DO_{k-1} \simeq R$.

Hence lift (or splitting) is an eigenvector of C with eigenvalue $k(k+1)$.

Swap k to $-k$ and let the lift be $\psi(z) = \sum_{n=0}^{\infty} f_n(z)\partial^{-k-n}$, we have

$$[C - k(k-1)]\psi = \sum_{n=1}^{\infty}\left(n(n+2k-1)f_n + (n+k)(n+k-1)f'_{n-1}\right)\partial^{-k-n}.$$

Inductively: f_n the multiple of $f^{(n)}$ given in the Theorem
(if we take $f_0 = \binom{2k-1}{k}^{-1} f$, $k \geq 0$, and $f_0 = \frac{(2|k|)!}{|k|!(|k|-1)!}f$, $k<0$).

Non-commutative, associative multiplications on modular forms

Rankin-Cohen brackets: bilinear maps on modular forms, $k, \ell, n \in \mathbb{Z}$, $n \geq 0$,

$$[\,\cdot\,,\,\cdot\,]_n = [\,\cdot\,,\,\cdot\,]_n^{(k,\ell)} : M_{2k} \otimes M_{2\ell} \to M_{2k+2\ell+2n}$$

$$[f,g]_n^{(k,\ell)}(z) = \sum_{m=0}^{n} (-1)^m \binom{2k+n-1}{n-m} \binom{2\ell+n-1}{m} f^{(m)}(z) g^{(n-m)}(z).$$

Rankin (1956): multi-linear DO's sending modular forms to modular forms.

H. Cohen (1974): made explicit the <u>*UNIQUE*</u> such map above $[\,\cdot\,,\,\cdot\,]$ (up to a multiplicative constant).

Transfer: Non-commutative composition of ψDO's to non-commutative multiplication in modular forms.

$$\mathcal{L}_k(f)\mathcal{L}_\ell(g) = \sum_{n=0}^{\infty} \mathcal{L}_{k+\ell+n}(h_n),$$

converts to a map $f \in M_{2k}, g \in M_{2\ell} \mapsto \{h_n \in M_{2k+2\ell+2n}\}$ (see paper),

By uniqueness of $[\,\cdot\,,\,\cdot\,]$ up to a multiplicative constant,

$$h_n = h_n(z) = t_n [f,g]_n^{(k,\ell)}(z).$$

By an inductive argument (which can be checked by direct computation),

Theorem: (PBC, Manin, Zagier) *For $k, \ell, n \geq 0$, define*

$$t_n(k,\ell) = \left(\frac{-1}{4}\right)^n \sum_{j\geq 0} \binom{n}{2j} \frac{\binom{-\frac{1}{2}}{j}\binom{-\frac{3}{2}}{j}\binom{\frac{1}{2}}{j}}{\binom{-k-\frac{1}{2}}{j}\binom{-\ell-\frac{1}{2}}{j}\binom{n+k+\ell-\frac{3}{2}}{j}},$$

then the multiplication μ on $M_*(\Gamma)$ defined by

$$\mu(f,g) = \sum_{n=0}^{\infty} t_n(k,\ell)[f,g]_n^{(k,\ell)}, \qquad f \in M_{2k}, g \in M_{2\ell},$$

is associative and the lift

$$\mathcal{L} = \prod_k \mathcal{L}_k : M_*(\Gamma) \to \psi DO^{\Gamma}$$

is a ring homomorphism with respect to this multiplication.

Considering pseudo-differential operators $\psi DO^{\Gamma,k}$ transforming like

$$\psi\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa}\psi(z)(cz+d)^{-\kappa}, \qquad \kappa \in \mathbb{C}$$

we can repeat everything (formally, should choose branch if $\kappa \notin \mathbb{Z}$).

By an inductive argument (which can be checked by direct computation),

Theorem: (PBC, Manin, Zagier) *For $k, \ell, n \geq 0$, $\kappa \in \mathbb{C}$ define,*

$$t_n^\kappa(k,\ell) = \left(\frac{-1}{4}\right)^n \sum_{j\geq 0} \binom{n}{2j} \frac{\binom{-\frac{1}{2}}{j}\binom{\kappa-\frac{3}{2}}{j}\binom{\frac{1}{2}-\kappa}{j}}{\binom{-k-\frac{1}{2}}{j}\binom{-\ell-\frac{1}{2}}{j}\binom{n+k+\ell-\frac{3}{2}}{j}},$$

Then, the multiplication μ^κ on $M_*(\Gamma)$ defined by,

$$\mu^\kappa(f,g) = \sum_{n=0}^{\infty} t_n^\kappa(k,\ell)[f,g]_n^{(k,\ell)}, \qquad f \in M_{2k}, g \in M_{2\ell},$$

is associative and the lift

$$\mathcal{L}^\kappa = \prod_k \mathcal{L}_k^\kappa : M_*(\Gamma) \to \psi DO^{\Gamma,\kappa}$$

is a ring homomorphism with respect to this multiplication, where

$$\mathcal{L}_k^\kappa = \sum_{n=0}^{\infty} \frac{\binom{-k}{n}\binom{-k+\kappa-1}{n}}{\binom{-2k}{n}} f^{(n)} \partial^{-k-n}.$$

Generalized by Connes-Moscovici to modular Hecke algebras.

Residues, Duality and Symmetry

Following Manin (1991, NC Geom. book), we looked at

$$\mathrm{Res}_\partial : \sum_m h_m(z)\partial^m \mapsto h_{-1}(z)dz \in H(R),$$

where

$$H(R) = \Omega^1(R)/d\Omega^0(R),$$

with $\Omega^1(R) = Rdz$ and $d\Omega^0(R) = dR = \{f'(z)dz, f \in R\}$.

We have
(a) $\mathrm{Res}_\partial(\psi \circ g) = \mathrm{Res}_\partial(\psi) \circ g$, for g holomorphic co-ordinate change,
(b) $\mathrm{Res}_\partial(\psi_1(z)\psi_2(z)) = \mathrm{Res}_\partial(\psi_2(z)\psi_1(z))$: tracial property,

There is a bijection between $H(R)^\Gamma$ and $M_2(\Gamma)/\partial(M_0(\Gamma))$ given by

$$f(z)dz \mapsto f(z),$$

where $H(R)^\Gamma$ is the Γ-invariant elements of $H(R)$. We have the map

$$\mathrm{Res}_\partial : \psi DO^{\Gamma,\kappa} \to H(R)^\Gamma \simeq M_2(\Gamma)/\partial(M_0(\Gamma)),$$

and the projection

$$P : M_*(\Gamma) \to M_2(\Gamma)/\partial(M_0(\Gamma)) \simeq H(R)^\Gamma.$$

The following properties are proved in our paper,

Proposition: (PBC–Manin–Zagier)
(i) $P([f,g]_n) = 0$*,* $n > 0$*,*
(ii) $\{f,g,h\}_n := P([f,g]_n h)$*,* $n \geq 0$
is invariant under cyclic permutations.

As a consequence we have, for example, the following:

For $f \in M_{2k}$, $g \in M_{2\ell}$, $h \in M_{2m}$, $k + \ell + m = 1 - n$,

Corollary: (PBC–Manin–Zagier)

$$\mathrm{Res}_\partial \left(\mathcal{L}^{2k}(f)\mathcal{L}^{2\ell}(g)\mathcal{L}^{2m}(h)\right) = \mathrm{Res}_\partial \left(\mathcal{L}\left(\mu(f,g)\right)\mathcal{L}(h)\right) = t_n(k,l)\{f,g,h\}_n.$$

The expression on the left is invariant under cyclic permutations of f, g, and h by the tracial property of Res_∂, and the triple bracket $\{f,g,h\}_n$ is invariant under cyclic permutations by part (ii) of the Proposition, so the coefficient $t_n(k,\ell)$ inherits these symmetries: e.g.

$$t_n(k,\ell) = t_n(\ell,m) = t_n(\ell, 1-n-k-\ell).$$

An interpretation of the non-commutative residue in terms of cyclic cohomology has been given in the work of Connes-Moscovici on Modular Hecke Algebras.