Meromorphic approximation of Cauchy Integrals and inverse source or crack problems

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What we do...

The purpose of this talk is to make a connection between :

1. Certain inverse problems of the Laplacean on a plane simply connected domain, where inner boundaries or source terms are to be recovered from outer boundary data,

2. approximation on the outer boundary of the domain to a Cauchy integral analytic outside the domain by a function meromorphic in the domain,

and to see how constructive techniques from the second subject impinge on the first.

...in practice...

The inverse problems that we consider are closely connected to classical modelling and identification issues in engineering, *e.g.* they are related to non-destructive control and Electro-Encephalography, although then they usually arise in 3D except for situations with circular symmetry.

To motivate the practical side of the approach we shall discuss the use of our 2D-techniques on inverse source problems in a spherical geometry, and raise issues for higher dimensions that parallel those we will discuss in the plane.

...with potentials...

Typically, the potential of a compactly supported signed measure μ is a function of the form

$$U(x,\mu) = \int K(x,y)\, d\mu(y),$$

where $K(.,y)$ is a fondamental solution of an elliptic operator with singularity at y.

The only operator that we ever consider is the familiar Laplacean in $\mathbb{R}^n$:

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

in which case :

$$\begin{aligned} U(z,\mu) &= \int \log |z-t|^{-1}\, d\mu(t) \quad \text{if } n=2, \quad z,t \in \mathbb{C}, \\ U(x,\mu) &= \int \|x-y\|^{2-n}\, d\mu(y) \quad \text{if } n>2, \quad x,y \in \mathbb{R}^n. \end{aligned}$$

...and Green potentials...

On an open set Ω with regular boundary $\partial\Omega$, we also obtain from the same definition the Green potential of a measure supported in Ω by letting the kernel $K(.,y)$ be the so-called Green function $G_\Omega(.,y)$ of Ω, namely the fundamental solution of the Laplacean restricted to Ω with singularity at $y \in \Omega$, subject to the boundary condition :
$G_\Omega(x,y) \to 0$ when $x \to \partial\Omega$.

For instance, in the unit disk $\mathbb{D} \subset \mathbb{R}^2$, we get :

$$U_{\mathbb{D}}(z,\mu) = \int \log \frac{|1-\bar{t}z|}{|z-t|}\, d\mu(t) \quad \text{for} \quad z,t \in \mathbb{D}.$$

...whose energy....

We need a few notions from potential theory. A positive measure μ has finite energy if :

$$I(\mu) = \int U(x, \mu) \, d\mu(x) < +\infty;$$

note, since μ has compact support by our standing assumptions, that the above energy integral is never $-\infty$ either. A set E such that $I(\mu) = +\infty$ for every positive $\mu \neq 0$ supported on E is called polar.

Polar sets are very thin; for instance, if μ has finite energy, it cannot charge polar sets. In particular, polar sets are Lebesgue-neglectible. A property is said to hold quasi-everywhere if it holds except perhaps on a polar set.

...brings them large capacities....

For K a non-polar compact set, there is a unique ω_K to minimize $I(\omega)$ over all probability measures ω supported on K. The number

$$C(K) = e^{-I(\omega_K)}$$

is called the capacity of K. Then ω_k is characterized by the fact that $U(.,\omega_K)$ is constant quasi-everywhere on K, and necessarily then equal to $\log 1/C(K)$. Moreover, ω_k is supported on the outer boundary of K.

For instance, if $n = 2$, the equilibrium measure of the unit disk $\mathbb{D}$ is also that of the unit circle $\mathbb{T}$, namely $d\omega_{\mathbb{T}}(\theta) = d\theta$.

As another example, if $[a,b]$ is a real segment,

$$d\omega_{[a,b]}(t) = \frac{dt}{\pi\sqrt{(b-t)(t-a)}}.$$

Note that $\omega_{[a,b]}$ charges the endpoints.

...that condense their potential...

Similar considerations apply to the Green potential of an open set Ω : the Green energy of the positive measure ω supported in Ω is

$$I_\Omega(\omega) = \int U_\Omega(x,\mu)\, d\mu(x),$$

and again the Green equilibrium measure of the non-polar compact set $K \subset \Omega$ is the probability measure $\omega_{\Omega,K}$ minimizing the Green energy $I_\Omega(\omega)$ over those ω supported on K. Still it is characterized by the fact that $U_\Omega(.,\omega_{\Omega,K})$ is constant quasi-everywhere on K. Only this time the definition of capacity goes differently :

$$I_\Omega(\omega_{\Omega,K}) = \frac{1}{C(\Omega^c,K)},$$

where we have set $\Omega^c = \mathbb{R}^n \setminus \Omega$ for simplicity and where $C(\Omega^c,K)$ is, by definition, the capacity of the condenser (Ω^c,K).

$\boxed{\text{...is to invert them.}}$

In general terms, the inverse problem of potential theory goes as follows :

Given a bounded open subset $\Omega \subset \mathbb{R}^n$, find a measure μ supported in Ω from the knowledge of its potential $U(.,\mu)$on $\mathbb{R}^n \setminus \Omega$.

When $n > 3$, $U(.,\mu)$ is harmonic in $\mathbb{R}^n \setminus \overline{\Omega} \cup \{\infty\}$ and therefore, up to solving a Dirichlet problem there, it is equivalent to require the knowledge of $U(.,\mu)$ on $\partial\Omega$ only.
When $n = 2$ it is not so, because at infinity $U(.,\mu) \sim -\mu(\mathbb{C}) \log |z|$. But for instance if $\partial\Omega$ is rectifiable, it is equivalent to require the knowledge of $U(.,\mu)$ and $\partial U(.,\mu)/\partial n$ on $\partial\Omega$.

But if you sweep it out...

Now, if Ω_1 is an open set with compact closure in Ω and containing the support of μ :

$$\text{supp}\mu \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega,$$

then there exists a unique measure $\widehat{\mu}$ supported on $\partial\Omega_1$, with $|\mu| = |\widehat{\mu}|$, such that

$$U(.,\mu) = U(.,\widehat{\mu}) \quad \text{on} \quad \mathbb{R}^n \setminus \overline{\Omega_1}.$$

This measure is called the balayage (or sweeping out) of μ onto $\partial\Omega_1$. It is characterized by the fact that

$$U(.,\mu) - U(.,\widehat{\mu}) = U_{\Omega_1}(.,\mu).$$

...you loose your potential...

But since, by construction, $U(., \widehat{\mu})$ has the same potential as $U(., \mu)$ outside of $\overline{\Omega}_1 \subset \Omega$, it solves the problem as well as $U(., \mu)$.

Thus we see that the inverse problem of potential theory is not well-posed in the generality that we stated.

The best we can hope for is to recover a distinguished representative in the equivalence class of measures compactly supported on Ω generating the same potential as μ on $\mathbb{R}^n \setminus \Omega$, unless we have enough prior information on μ to constrain the algorithm computing such a representative so as to make it coincide with μ.

...unless it is shaped like a star.

Although it is not directly connected to the rest of our discussion, let us mention a classical result which is rather typical of known conditions on μ for the inverse problem to have a unique solution.

It deals with the situation where a density function f is globally defined with $\Omega \subset \mathrm{supp} f$, and where $\mu = \chi_{\Omega_1} f$ where χ_{Ω_1} is the characteristic function of some open set $\Omega_1 \subset \Omega$.

Theorem [Novikov,1938]
Assume that Ω_1 is star-shaped with respect to the origin, and that $\partial f / \partial r$ is continuous on $\Omega \setminus \{0\}$ where it satisfies $f \partial(r^n f)/\partial r \geq 0$. Then the restriction to $\mathbb{R}^n \setminus \Omega$ of $U(., \chi_{\Omega_1} f)$ uniquely defines Ω_1.

Differentiating from the original...

In this talk, we shall be concerned with the following differential version of the inverse problem, obtained by taking gradients:

Given a bounded open subset $\Omega \subset \mathbb{R}^n$, find a measure μ supported in Ω from the knowledge of $\nabla U(., \mu)$on $\mathbb{R}^n \setminus \Omega$.

In $\mathbb{R}^2$, we prefer to use $\partial/\partial z = \partial/\partial x - i\partial/\partial y$ rather than the vector gradient $\nabla = (\partial/\partial x, \partial/\partial y)$. Note that :

$$\frac{\partial U}{\partial z}(z,\mu) = \int \frac{d\mu(t)}{z-t} \quad \text{if } n=2, \quad z,t \in \mathbb{C},$$
$$\frac{\nabla U(x,\mu)}{2-n} = \left(\int \frac{x_j}{\|x-y\|^n} d\mu(y)\right)_{1\leq j \leq n} \quad \text{if } n>2.$$

$\boxed{\text{...we end up at home with Cauchy.}}$

For a long while now, we shall be concerned with $n = 2$. In this case we adress the following question :

Given a bounded open subset $\Omega \subset \mathbb{R}^n$, what can be constructively recovered from a measure μ supported in Ω if we know its Cauchy transform

$$\int \frac{d\mu(t)}{z - t}$$

on $\mathbb{R}^n \backslash \Omega$, and what does approximation theory have to say on this issue?

We try to be rational...

Approximation Theory can help with one thing, namely approximate the analytic function defined by the Cauchy transform of μ on $\mathbb{R}^n \setminus \Omega$.

In the present context, it is natural to consider rational approximants because they coincide with potentials of (complex) discrete measures. If we make this twist of considering complex discrete measures to approximate real ones, we face a most familiar problem, that is :

How do the poles of a rational approximant to a Cauchy transform behave in connection with the measure defining this transform?

...although it is not easy...

This is certainly not an easy question. The behaviour of poles governs the convergence of almost any rational approximation scheme, but though it has been intensively studied results are still rather specialized.

In connexion with our inverse problem, we shall first discuss rational approximation to the most classical type of Cauchy transform, namely Markov functions. These arise from positive measures supported on a real segment.

If r_m is a sequence of rational functions with poles ξ_j, m for $1 \leq j \leq d_m$, our notion of convergence for these poles is the weak* convergence of the counting probability measure $1/d_m \sum_j \delta_{\xi_{j,m}}$. Recall that the sequence of complex measures ν_m converges weak* to ν if, for each h which is continuous with compact support in $\mathbb{C}$,

$$\lim_{m\to\infty} \int h \, d\nu_m = \int h \, d\nu.$$

...and with Padé...

We begin with the familiar Padé approximation. The Padé approximant of type $(m-1, m)$ to f analytic at infinity is the rational function p_{m-1}/q_m that interpolates f with order at least $2m$ at infinity.

If f is the Cauchy transform of μ, then

$$\int q_m(t) t^k \, d\mu = 0, \quad \text{for } 0 \leq k \leq m-1$$

and, since $\mu \geq 0$, classical results on orthogonal polynomials [Szegö][Erdös-Turan][Widom][Stahl-Totik] imply that the poles of p_{m-1}/q_m lie on the smallest segment $[a, b]$ containing suppμ, and if μ is "not too thin on its support" their counting measure converges weak*, as $m \to \infty$, to $\omega_{[a,b]}$.

Since the equilibrium measure charges the endpoints, this looks like an efficient way to asymptotically recover $[a, b]$, a substantial contribution to the inverse problem already.

...being so far away...

Unfortunately this is not as promising as it looks from the effective viewpoint, because the computation of Padé approximants gets unstable :infinity is too far.

After work by [Gonchar-Lopèz], [Stahl-Totik], the next idea is to use multipoint Padé interpolants at finite distance :

if for each m we let $(\zeta_{j,m})$ with $1 \leq j \leq 2m$ be a conjugate-symmetric family of interpolating nodes,and if we assume that $1/2n \sum_j \delta_{\zeta_{j,m}}$ converges weak* to ν, the rational function p_{m-1}/q_m interpolating f at the nodes $(\zeta_{j,m})$ has counting measure of poles converging weak* to some equilibrium distribution on [a,b], although this distribution now corresponds to an equilibrium problem with external field $U(.,\nu)$ that we did not introduce.

...we better go meromorphic...

Although multipoint interpolants will do better than Padé, they do not perform so well either. Notwithstanding the fact that it is uneasy to design the sequence of nodes so that again the weak* limit of the poles charges the endpoints, computation runs quickly unstable again.

In facts, there are deep reasons why no linear algorithm can do a great job here.

This is why we turn to meromorphic approximation. When doing so, we assume for simplicity that $\Omega = \mathbb{D}$, and we let H^p be the familiar Hardy space of $L^p(\mathbb{T})$-functions whose Fourier coefficients of strictly negative index do vanish. Our approximating family H^p_m will consist of meromorphic functions h/q_m where $h \in H^p$ while $\deg q_m \leq m$ and $q_m \neq 0$ on $\mathbb{T}$.

...to get hold of the poles...

Theorem [Prokhorov-Saff- L.B.,2001]
If f is a Markov function with defining measure μ supported in $\mathbb{D}$, and if $1 \leq p \leq \infty$, then all the poles of a best approximant to f from H_m^p lie in the minimal segment $[a,b]$ containing suppμ and, if $\log d\mu/dt \in L^1([a,b])$, their counting measure converges weak*, as $m \to \infty$, to $\omega_{[a,b],\mathbb{T}}$.

Also [Stahl-Wielonsky-L.B.,2001] if $p = 2$. When $p > 1$, meromorphic approximation is concerned with orthogonality equations of the form :

$$\int \frac{q_m(t)}{\tilde{q}_m^2(t)} t^k \, w_m(t) \, d\mu = 0, \quad \text{for } 0 \leq k \leq m-1,$$

where

$$\tilde{q}_m(z) = z^m \overline{q_m(1/\bar{z})}$$

is the reciprocal of q_m and w_m an outer function. Also, $w_m q_m / \tilde{q}_m$ can be identified with the m-th singular vector of a Hankel operator with symbol f when $p \geq 2$ [Adamjan-Arov-Krein, 1972], [Seyfert-L.B.,2000].

...in an effective manner.

When $p = \infty$, the computation of best meromorphic approximants can be carried out from the Hankel matrix thanks to Adamjan-Arov-Krein theory.

When $p < \infty$, one has to resort to numerical optimization, and in this connection it is important that the preceding result actually holds true for any sequence of critical *e.g.* stationnary) points of the error function. Indeed, local minima is all that a local search can guarantee.

Altogether, this gives one computationally cheap way to recover $[a, b]$, which is of course far from solving the inverse problem but already provides valuable information.

$\boxed{\text{Though } \mu \text{ may not be real...}}$

If μ is not positive, nor even real, but still supported on $\mathbb{R}$, poles need no longer lie on $[a, b]$ but their asymptotic behaviour remains the same if we assume a little smoothness.

As to Padé approximation and orthogonal polynomials with respect to complex measures, let us quote among many works those of [Baxter] [Nutall-Singh] [Stahl] [Gonchar-Rachmanov] [Magnus] [Aptekarev][Aptekarev-Van Assche].

For instance if $d\mu(t) = g(t)dt$ where g is continuous and nonzero quasi-everywhere on $[a, b]$, the counting measure of the poles still converges weak* to $\omega_{[a,b]}$ [Stahl]. And if g is nowhere zero then all the poles go to $[a, b]$ [Magnus].

...we can carry that weight...

As to meromorphic approximation, if we write the polar decomposition $d\mu(t) = \phi(t)d|\mu|(t)$, the counting measure of the poles still converges weak* to $\omega_{[a,b],\mathbb{T}}$ provided that ϕ has bounded variation and $|\mu|$ satisfies criterion Λ of Stahl and Totik :

$$C\Bigg(\Bigg\{t \in \operatorname{supp}\mu, \quad \limsup_{r>0,\, r\to 0} \frac{\log\Big(1/\mu\big([t-r, t+r]\big)\Big)}{\log\big(1/r\big)}$$

$$< +\infty\Bigg\}\Bigg) = C(\operatorname{supp}\mu).$$

[Mandrea-Totik-L.B.,2000] [Küstner,2002]. We mention that the number of poles that may not converge to $[a,b]$ is bounded by the variation of $\arg\mu$. The result is about regular asymptotic distribution of orthogonal polynomials w.r.t. a complex measure with a varying weight which is implicitly defined by q_m.

...to try to crack...

Having generalized the regular asymptotic distribution of poles in rational and meromorphic approximation to Cauchy transforms of positive measures (Markov functions) over to Cauchy transforms of more general complex measures, we can use Möbius transformations to allow for non real segments in the Padé case or hyperbolic geodesic arcs of $\mathbb{D}$ in the meromorphic case, and if the measure defining the Cauchy transform on such an arc is analytic, we can deform this arc by the Cauchy theorem in order to apply the previous results to a larger class of functions (recall that the Recall that hyperbolic geosedic arcs of $\mathbb{D}$ are simply arcs of circle orthogonal to $\mathbb{T}$).

This is what we are going to apply in the forthcoming instance of inverse problem.

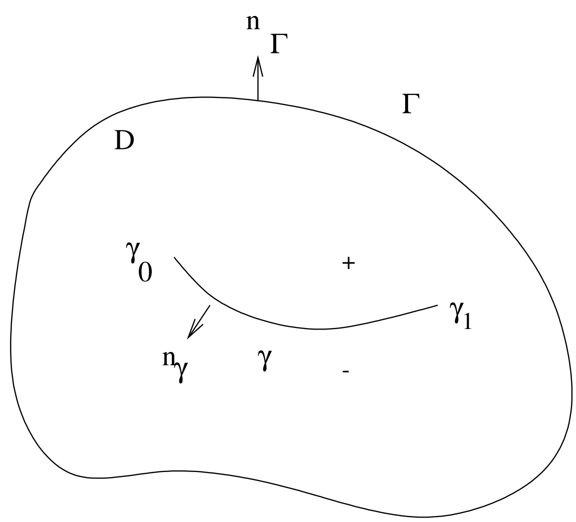
...the crack problem...

On a simply connected plane domain Ω with Hölder-smooth outer boundary Γ, containing in its interior a Hölder-smooth oriented open arc γ, let $u : \Omega \to \mathbb{R}$ satisfy:

$$\begin{cases} \Delta u &= 0 \quad \text{in } \Omega \setminus \overline{\gamma}, \\ \dfrac{\partial u}{\partial n_\Gamma} &= \Phi \quad \text{on } \Gamma, \\ \dfrac{\partial u^\pm}{\partial n_\gamma} &= 0 \quad \text{on } \gamma, \end{cases}$$

where n_Γ and n_γ denote the outer unit normal to Γ and γ, while u^+ and u^- respectively indicate the determinations of u on the positive and negative side of γ.

...in order to recover ...



Given $\Phi \in L^2(\mathbb{T})$ the function u uniquely exists, and the inverse problem is to recover γ from the knowledge of pairs (Φ, u).

Physically, u could be a temperature or a potential, and Φ a flux of heat or a current.

...from being ill-posed.

A large number of researchers have worked on this problem. To quote just a few :

[Alessandrini, 1993], [Alessandrini, Beretta, Vessella, 1996], [Alessandrini, Diaz Valenzuela, 1996], [Alessandrini, DiBenedetto, 1997], [Alessandrini, 1997], [Andrieux, Ben Abda, 1996], [Leblond, Mandréa, Saff, L.B., 1999], [Ben Abda, Kallel, Leblond, Marmorat, 2002], [Bruhl, Hanke, Pidcock, 2001], [Bryan, Vogelius, 1992], [Cimetière, Delvare, Jaoua, Kallel, Pons, 2002], [Elcrat, Isakov, Neculoiu, 1995], [Friedman, Vogelius, 1989], [Kim, Seo, 1996], [Santosa, Vogelius, 1991], [Weikl, Andra, Schnack, 2001].

Identifiable: two pairs (Φ, u) are enough;
Ill-posed: in the sense of Hadamard (which is why we take everything Hölder-smooth).

To be explicit...

Existing methods roughly fall in two classes.

Iterative methods proceed with parametrizing the crack and numerically minimize the difference between the observed u and the one deduced from the current parameter values.

Quasi-explicit methods, that compare Φ with its analog when there is no crack, and make use of the Green formula to relate the difference on Γ to some operator involving γ. This involves checking the range of this operator, which usually requires strong assumptions on γ to be effective.

The quasi-explicit methods are of course natural candidates to initialize the heavier iterative methods. The technique that we present relates to quasi-explicit methods.

...we express integrally...

To use what we did before, we express the solution as a Cauchy integral :

Theorem [Leblond, Mandrea, Saff, L.B., 1999]
With the previous notations $u = \mathrm{Re}\, f$ in $\Omega \setminus \gamma$, where

$$f(z) = g(z) + \frac{1}{2i\pi} \int_\gamma \frac{\sigma(\xi)}{\xi - z} d\xi,$$

with $g \in H^\infty(\Omega)$ and $\sigma = u^+ - u^-$.

If we meromorphically approximate f on Γ the function g plays no role, and only the Cauchy integral remains.

...although we deform...

Let us assume without loss of generality that $\Omega = \mathbb{D}$. To apply our results, we need to deform γ into a hyperbolic geodesic, which requires σ to be analytic on a sufficiently big neighborhood of γ. We give one example of a condition ensuring this :

Theorem [Leblond, Mandrea, Saff, L.B]
Let $P : V \to \mathbb{D}$ be a conformal map such that $[0,1] \subset V$ with V bounded, and suppose $P([0,1]) = \gamma$. Denote by $\mathcal{G}$ the geodesic arc in $\mathbb{D}$ joining $\gamma_0 = P(0)$ and $\gamma_1 = P(1)$. If the hyperbolic distance between every two consecutive intersection points of $\mathcal{G}$ with γ is less than

$$\frac{1}{4}\log\left(1+\sqrt{2}\right) \simeq 0.220343,$$

then γ can be deformed into $\mathcal{G}$ within the domain of analyticity of σ.

$\boxed{\text{...to lay hands on the ends.}}$

The previous condition expresses that γ is hyperbolically not too far from a geodesic arc $\mathcal{G}$ having the same endpoints. When this is the case, the poles of the best meromorphic approximants converge weak* to $\omega_{\mathcal{G},\mathbb{T}}$, and since the equilibrium measure charges the endpoints the poles should indicate the endpoints of γ.

This does not solve the inverse problem proper, but provides a very helpful information at a very low cost.

Let's go back to the source...

Let us briefly describe another inverse problem to which the technique applies. Here is a 3D-model for electro-encephalography:

$\Omega \subset \mathbb{R}^3$ is a ball, union of 3 disjoint homogeneous spherical connected layers (scalp, skull and brain) Ω_i, $i = 0, 1, 2$, conductivity $\sigma_{|\Omega_i} = \sigma_i$

$$\begin{cases} -\nabla \cdot (\sigma \nabla u) = F \text{ in } \Omega \\ \\ \dfrac{\partial u}{\partial \nu}_{|\partial\Omega} = \phi \text{ current flux } (= \ 0) \\ \\ u_{|\partial\Omega} = g \text{ electric potential} \end{cases}$$

unknown pointwise sources in Ω_0:

$$F = \sum_{j=1}^{m_1} \lambda_j \, \delta_{S_j} + \sum_{k=1}^{m_2} p_k \,.\, \nabla \, \delta_{C_k}$$

[M. Hämäläinen, R. Hari, J. Ilmoniemi, J. Knuutila, O.V. Lounasmaa, 1993], [Faugeras & al,1999], [El Badia, Ha Duong, 2000].

...where head is only brain...

We suppose there is only one layer, namely brain: the inverse problem of peeling off the outer layers is not easy, but studied independently [Kozlov,Maz'ya,Fomin, 1991]. Then the inverse problem becomes :
Given ϕ and measurements of g on $\partial\Omega$, find the number and the location of pointwise sources :

$S_j\,,\ C_k \in \Omega_0$

and their moments : $\lambda_j \in \mathbb{R}$, $p_k \in \mathbb{R}^3$

The compatibility condition is :

$$\int_{\partial\Omega} \phi\, ds = -\sum_{j=1}^{m_1} \lambda_j = 0.$$

Identifiability holds [El Badia, Ha Duong, 2000].

...with fundamental solutions.

If we have m_1 monopolar and m_2 dipolar sources, the solutions in 2D and 3D are given by :

$$u_2(X) = h_2(X) - \sum_{j=1}^{m_1} \frac{\lambda_j}{2\pi} \log \frac{1}{\|X - S_j\|}$$

$$+ \sum_{k=1}^{m_2} \frac{< p_k \, , \, X - C_k >}{2\pi \, \|X - C_k\|^2}$$

$$u_3(X) = h_3(X) - \sum_{j=1}^{m_1} \frac{\lambda_j}{4\pi \, \|X - S_j\|}$$

$$+ \sum_{k=1}^{m_2} \frac{< p_k \, , \, X - C_k >}{4\pi \, \|X - C_k\|^3}$$

where h_2 and h_3 are harmonic.

$\boxed{\text{They are complex...}}$

In 2D, we get meromorphic functions again :

(PS), for $\xi = x + i\,y \in \mathbb{D} \setminus S$, $u = \mathrm{Re} f_2$ with

$$f_2(\xi) = \mathcal{A}(\xi)$$

$$+ \sum_{j=1}^{m_1} \frac{\lambda_j}{2\pi} \log \frac{1}{\xi - S_j} + \sum_{k=1}^{m_2} \frac{p_k}{2\pi\,(\xi - C_k)}$$

for some function $\mathcal{A}$ analytic in $\mathbb{D}$.

This time we need no conditions to deform, but we must adress what happens if we have more than two monopolar sources.

...but their capacity is extreme...

We use the following theorem:

Theorem [Stahl, L.B.]
If f is analytic for $|z| \geq 1$, and can be continued analytically except for finitely many singularities and poles but is not single-valued, best meromorphic approximants have a counting measure that converges weak* to the Green equilibrium distribution of the continuum of minimal Green capacity outside of which f is single-valued.

By a theorem of Stahl, this continuum consists of finitely many analytic arcs with crossings that end up at the singularities, and the Green equilibrium measure charges these singularities. This makes it possible again to recover them, at least theoretically, in the asymptotic.

...an even extends to 3D...

In $3D$, there is not yet a theory of extremal domains that could help analysing the discretization of potentials.

However, if we slice the ball and pose a meromorphic approximation problem in each slice for $z = z_p$, we get a function with branch-points :

$$u_p(\xi) = \frac{1}{4\,\pi} \times \left[- \sum_{j=1}^{m_1} \frac{\lambda_j}{Q_{j,p}^{1/2}(\xi)} + \sum_{k=1}^{m_2} \frac{P_{k,p}(\xi)}{Q_{k,p}^{3/2}(\xi)} \right]$$

$$P_{k,p}(\xi) = \mathrm{Re}\,(p_{k,x} - i\,p_{k,y})\,(\xi - \xi_k) + p_{k,z}\,(z_p - z_k)$$

dipolar moments $p_k = (p_{k,x}, p_{k,y}, p_{k,z})$

$$Q_{l,p}(\xi) = |\xi - \xi_l|^2 + (z_p - z_l)^2$$

$$= -\frac{1}{\xi}\,\xi_l\,(\xi - \xi_{l,p}^{-})\,(\xi - \xi_{l,p}^{+})$$

...where we chase maxima.

$$[\xi_{l,p}^{\pm} = \frac{\xi_l}{2\,|\xi_l|^2}\left\{|\xi_l|^2 + r_p^2 + h^2\right.$$

$$\left.[\pm\sqrt{[(|\xi_l| + r_p)^2 + h^2]\,[(|\xi_l| - r_p)^2 + h^2]}\right\}$$

An easy computation shows that each branch-point $\xi_{l,p}^{\pm}$ has maximum modulus at the height of some source, and this way we can try to locate them.